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SPLINES AND THE LOGARITHMIC FUNCTION

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The paper studies a spline function $S_n(x)$ ($0 < x < \infty$) of degree n , with knots at the points of the geometric progression $x_k = q^k$ (q is fixed > 1 , $k = 0, \pm 1, \pm 2, \dots$), which is shown to be uniquely defined by the following two properties: 1°. $S_n(x)$ interpolates the function $f(x) = \log x / \log q$ at all the knots q^k , 2°. $S_n(x)$ satisfies the functional equation $S_n(qx) = S_n(x) + 1$ for $x > 0$. $S_n(x)$ is explicitly determined and shown to share with $f(x)$ some of its global properties. The main point is the detailed study of the somewhat surprising behavior of $S_n(x)$ as $n \rightarrow \infty$.

Introduction. Spline interpolants of exponential functions were discussed in [3], Lecture 3, and were found to be useful in a study of cardinal spline interpolation ([3], Lecture 4). Here we attempt to interpolate by splines the logarithmic function $\log x$ ($0 < x < \infty$). As the domain of definition of $\log x$ is the positive halfline we do not use the usual cardinal polynomial splines, but rather so-called *cardinal q -splines* defined as follows.

Let n be a natural number and q be real, $q > 1$. We denote by

$$(1) \quad \sum_{n,q} = \{S(x)\}$$

the class of functions $S(x)$, defined on the half-axis $0 < x < \infty$, and satisfying the following two conditions:

1. The restriction of $S(x)$ to every interval $(q^\nu, q^{\nu+1})$ (ν any integer) is a polynomial of degree at most n .
- 2.

$$(2) \quad S(x) \in C^{n-1}(0, \infty).$$

Such functions $S(x)$ are commonly called *spline functions* (or splines) of degree n having as knots the points q^ν ($-\infty < \nu < \infty$). We also call them *q -splines* to remind us of the nature of their knots. Finally, the term *cardinal q -splines* is to indicate that the domain of $S(x)$ is $(0, \infty)$ and that *all* knots q^ν many occur. We now propose the following

Problem 1. To find

$$(3) \quad S_n(x) \in \sum_{n,q}$$

such as to interpolate the function

$$(4) \quad f(x) = \frac{\log x}{\log q}$$

at the points (q^ν) , hence such that

$$(5) \quad S_n(q^\nu) = \nu \quad \text{for all integers } \nu.$$

As stated this problem admits infinitely many solutions. Indeed, a moment's reflexion (see [3], Lemma 1.1 on p. 33) will show that we obtain all solutions of Problem 1 as follows. The symbol π_n denoting the class of polynomials of degree $\leq n$, let $P(x)$ be an arbitrary element of π_n such that

$$(6) \quad P(1) = 0, \quad P(q) = 1,$$

and let

$$(7) \quad \begin{aligned} S(x) = & P(x) + a_1(x - q)_+^n + a_2(x - q^2)_+^n + \dots \\ & + a_0(1 - x)_+^n + a_{-1}(q^{-1} - x)_+^n + \dots, \quad (0 < x < \infty), \end{aligned}$$

where we use the notation $u_+ = \max(u, 0)$. In the interval $[1, q]$ all truncated powers vanish and so $S(x) = P(x)$. In the interval $[q, q^2]$, besides $P(x)$, only the term $a_1(x - q)_+^n$ does not vanish and we can determine a_1 uniquely by requiring that $S(q^2) = 2$. Next we determine a_2 uniquely from $S(q^3) = 3$, a.s.f. Likewise the coefficients a_0, a_{-1}, \dots are successively and uniquely determined by the interpolatory condition $S(q^{-1}) = -1, S(q^{-2}) = -2, \dots$. We see that the interpolating q -spline $S(x)$ is uniquely defined once we have chosen $P(x) \in \pi_n$ such as to satisfy (6).

Clearly Problem 1 will become meaningful only if we impose on $S_n(x)$ further conditions. This we do as follows. We observe that the function (4) satisfies the functional equation $f(qx) = f(x) + 1$. This suggests that *we require the q -spline $S_n(x)$ to satisfy the functional equation*

$$(8) \quad S_n(qx) = S_n(x) + 1, \quad (0 < x < \infty).$$

It should be noticed that if (8) is satisfied and also

$$(9) \quad S_n(1) = 0,$$

then all relations (5) evidently follow.

The contents of the seven sections of this paper are briefly as follows. In § 1 we construct the unique $S_n(x)$ satisfying the conditions (5) and (8), and call it *the logarithmic q -spline of degree n* . Unexpectedly, its representation (7) can be explicitly described (Theorem 1). In § 2 we show that the global behavior of $S_n(x)$ is much like that of the function $\log x / \log q$: $S'_n(x)$ is multiply monotone (Theorem 2).

This gives a direct construction of $S_n(x)$ by successive integrations of the explicitly described step-function $S_n^{(n)}(x)$ (Corollary 1 of § 2).

These results led us naturally to expect that the logarithmic q -spline $S_n(x)$ will converge to $\log x/\log q$ as $n \rightarrow \infty$. This, however, is not the case and the §§ 4, 5, 6, 7 describe the peculiar behavior of $S_n(x)$ as $n \rightarrow \infty$. Our main results are as follows. We define

$$(10) \quad F(x, \theta) = \sum_{s=-\infty}^{\infty} (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}), \quad (x > 0).$$

This is a periodic function of θ of period 1, whose Fourier expansion (5.2) is described in § 5. In terms of $F(x, \theta)$ we find that

$$(11) \quad S_n(x) = F\left(x, \frac{\log n}{\log q}\right) + o(1) \text{ as } n \rightarrow \infty,$$

where the error term $o(1)$ is uniform in x for $x > 0$ (Theorem 4 of § 4). The average value of $F(x, \theta)$ over a period in θ is $a_0(x) = \log x/\log q$, while its total variation in a period is of the order of

$$(12) \quad \left| \Gamma\left(-\frac{2\pi i}{\log q}\right) \right| = \frac{\log q}{e^{2\pi^2/\log q} - e^{-2\pi^2/\log q}}^{1/2}$$

which is small if q is not too large (for $q = 2$ the quantity (12) is of the order of 10^{-6}). Figure 1 of Section 6 shows graphically all the terms of the relation (11) for $q = 2$, $x = \sqrt{2}$, and $n = 16, 17, \dots, 64$.

In the last section 7 it is shown that the logarithmic means of the sequence $S_n(x)$ do converge to the "correct" limit $\log x/\log q$ (Theorem 5). In a way this paper may be regarded as a contribution to the study of the functional equation $f(qx) = f(x) + 1$.

1. Construction of the cardinal q -spline $S_n(x)$ satisfying (5) and (8). The representation (7) of our solution is explicitly described by the following

THEOREM 1 *There is a unique $S_n(x)$ satisfying the conditions (5) and (8) and it is given by*

$$(1.1) \quad S_n(x) = P_n(x) + (-1)^n \sum_{\nu=1}^{\infty} q^{-n\nu} (x - q^\nu)_+^n - \sum_{\nu=0}^{\infty} q^{n\nu} (q^{-\nu} - x)_+^n, \\ (0 < x < \infty),$$

where

$$(1.2) \quad P_n(x) = \binom{n}{1} \frac{x-1}{q-1} - \binom{n}{2} \frac{x^2-1}{q^2-1} + \dots + (-1)^{n-1} \frac{x^n-1}{q^n-1}.$$

We call $S_n(x)$ the logarithmic q -spline of degree n .

Proof. We begin by determining the polynomial $P_n(x) = P(x)$ representing $S_n(x) = S(x)$ in the interval $(1, q)$. From the conditions (2) and (8) we obtain by differentiation

$$(1.3) \quad q^\nu S^{(\nu)}(qx) = S^{(\nu)}(x), \quad (\nu = 1, \dots, n-1),$$

and setting $x = 1$ we obtain

$$(1.4) \quad q^\nu P^{(\nu)}(q) - P^{(\nu)}(1) = 0, \quad (\nu = 1, \dots, n-1).$$

Clearly $P(1) = 0$, $P(q) = 1$, and therefore

$$(1.5) \quad P(q) - P(1) = 1.$$

We may summarize the relations (1.4) and (1.5) by saying that the polynomial $P(qx) - P(x)$ assumes for $x = 1$ the value 1 to the n th order. This means that for an appropriate constant c we have the identity

$$(1.6) \quad P(qx) - P(x) = 1 + c(x-1)^n.$$

The left side vanishing if $x = 0$, we must have $c = (-1)^{n-1}$ and (1.6) becomes

$$(1.7) \quad P(qx) - P(x) = 1 - (1-x)^n.$$

All $P(x) \in \pi_n$ satisfying (1.7) are easily determined. Writing

$$(1.8) \quad P(x) = a_0 + a_1x + \dots + a_nx^n,$$

substituting into (1.7) and comparing coefficients on both sides, we obtain that

$$a_1(q-1) = \binom{n}{1}, a_2(q^2-1) = -\binom{n}{2}, \dots, a_n(q^n-1) = (-1)^{n-1}.$$

Therefore all solutions of (1.7) are of the form.

$$P(x) = a_0 + \binom{n}{1} \frac{x}{q-1} - \binom{n}{2} \frac{x^2}{q^2-1} + \dots + (-1)^{n-1} \frac{x^n}{q^n-1}.$$

Finally, the condition $P(1) = 0$ shows that $P_n(x)$ indeed has the form (1.2). We may retrace our steps and see that the polynomial (1.2) satisfies (1.7), hence (1.4) and that $P(1) = 0$, $P(q) = 1$.

The polynomial $P_n(x)$ representing $S_n(x)$ in the interval $(1, q)$ having been determined, we could now extend the definition of $S_n(x)$ to all of $(0, \infty)$ by the representation (7) and the procedure described for determining the coefficients a_ν . However, it is better to proceed as follows: *We extend the definition of $S_n(x)$ from the interval $[1, q]$, where $S_n(x) = P_n(x)$, to all positive values of x by means of*

the functional equation

$$(1.9) \quad S_n(qx) = S_n(x) + 1.$$

However, now we must verify that the condition $S(x) \in C^{n-1}(0, \infty)$ is satisfied. By iteration of (1.9), i.e. replacing x by qx, q^2x, \dots and adding the results, we find that

$$(1.10) \quad S(x) = P\left(\frac{x}{q^{r-1}}\right) + r - 1 \quad \text{if } q^{r-1} < x < q^r,$$

$$(1.11) \quad S(x) = P\left(\frac{x}{q^r}\right) + r \quad \text{if } q^r < x < q^{r+1}.$$

From (1.11) and letting x decrease to q^r we obtain that $S(q^r + 0) = P(1) + r = r$, while (1.10) on letting x increase to q^r we find $S(q^r - 0) = P(q) + r - 1 = 1 + r - 1 = r$. Therefore $S(q^r + 0) = S(q^r - 0)$ and $S(x) \in C(0, \infty)$. There remains to show that

$$(1.12) \quad S^{(\nu)}(q^r + 0) = S^{(\nu)}(q^r - 0), \quad (\nu = 1, \dots, n-1).$$

However, on differentiating (1.10) and (1.11), we obtain

$$(1.13) \quad S^{(\nu)}(q^r - 0) = \frac{1}{q^{\nu(r-1)}} P^{(\nu)}(q) \quad \text{and} \quad S^{(\nu)}(q^r + 0) = \frac{1}{q^{\nu r}} P^{(\nu)}(1),$$

respectively. Now the relations (1.12) amount to $q^\nu P^{(\nu)}(q) = P^{(\nu)}(1)$, and these are precisely the boundary conditions (1.4) satisfied by our polynomial (1.2).

We are still to determine the explicit values of the coefficient a_r of (7), as described by (1.1). If $x \neq q^r$, for all integers r , then we may differentiate the relation (1.9) even n times when (1.13) becomes, in view of (1.2),

$$(1.14) \quad S^{(n)}(q^r - 0) = \frac{1}{q^{n(r-1)}} P^{(n)}(q) = \frac{q^n}{q^{nr}} (-1)^{n-1} \frac{n!}{q^n - 1},$$

$$(1.15) \quad S^{(n)}(q^r + 0) = \frac{1}{q^{nr}} P^{(n)}(1) = \frac{1}{q^{nr}} (-1)^{n-1} \frac{n!}{q^n - 1}.$$

These values show that the coefficients a_r of (7) are equal to

$$a_r = \frac{1}{n!} \{S^{(n)}(q^r + 0) - S^{(n)}(q^r - 0)\} = \frac{1}{n! q^{nr}} (-1)^n n! \quad \text{if } r \geq 1,$$

and

$$a_r = (-1)^{n-1} \frac{1}{n!} \{S^{(n)}(q^r + 0) - S^{(n)}(q^r - 0)\} = \frac{(-1)^{n-1}}{n! q^{nr}} (-1)^n n! \\ \text{if } r \leq 0.$$

Therefore

$$a_r = \begin{cases} (-1)^n q^{-nr} & \text{if } r \geq 1, \\ -q^{-nr} & \text{if } r \leq 0. \end{cases}$$

This establishes (1.1) and also Theorem 1.

2. The nice global behavior of $S_n(x)$. The function

$$(2.1) \quad f(x) = \frac{\log x}{\log q}$$

that we are interpolating by $S_n(x)$ has the derivative $(x \log q)^{-1}$ which is *completely monotone* in $(0, \infty)$, because

$$(2.2) \quad f^{(\nu)}(x) = \frac{(-1)^{\nu-1}}{\log q} \frac{(\nu-1)!}{x^\nu}, \quad (\nu = 1, 2, \dots).$$

To what extent does $S_n(x)$ imitate this regular behavior of $f(x)$? That it does so as best as it can (remember that $S_n(x)$ has only n derivatives, the n th being a step-function) is shown by

THEOREM 2. *The logarithmic q -spline $S_n(x)$ has the properties*

$$(2.3) \quad (-1)^{\nu-1} S_n^{(\nu)}(x) > 0 \text{ in } 0 < x < \infty,$$

$$(2.4) \quad S_n^{(\nu)}(x) = O(x^{-\nu}) \text{ for } \nu = 1, 2, \dots, n.$$

Proof. We first establish (2.4). Differentiation of (1.9) gives $q^\nu S^{(\nu)}(qx) = S^{(\nu)}(x)$, where $x \neq q^r$ for all r , if $\nu = n$. Iterating this we obtain that

$$(2.5) \quad S^{(\nu)}(q^r x) = q^{-\nu r} S^{(\nu)}(x) \quad (r \text{ integer}).$$

Let us write

$$(2.6) \quad M_\nu = \sup_{1 < x < q} |S^{(\nu)}(x)|.$$

Assuming that

$$(2.7) \quad q^r < x < q^{r+1},$$

then (2.5) and (2.6) show, on replacing x by xq^{-r} , that

$$|S^{(\nu)}(x)| \leq M_\nu q^{-\nu r} = M_\nu q^{-\nu(r+1)} q^\nu < M_\nu x^{-\nu} q^\nu$$

and therefore

$$(2.8) \quad |S^{(\nu)}(x)| < (M_\nu q^\nu) x^{-\nu}.$$

As the right side does not depend on r , the condition (2.7) becomes irrelevant and (2.8) holds for all positive x . This establishes (2.4).

We now establish (2.3) by induction for decreasing values of ν . (2.3) is true for $\nu = n$: This is shown by (1.15) which gives

$$(2.9) \quad S_n^{(n)}(x) = (-1)^{n-1} \frac{n!}{q^n - 1} (q^r)^{-n} \quad \text{if } q^r < x < q^{r+1}.$$

Therefore (2.3) holds for $\nu = n$. Assuming that (2.3) holds for a value of ν ($\nu > 1$), let us establish (2.3) for $\nu - 1$ in place of ν . By (2.4) we can write

$$(2.10) \quad S_n^{(\nu-1)}(x) = - \int_x^\infty S_n^{(\nu)}(t) dt \quad (x > 0),$$

and this already shows that $S_n^{(\nu-1)}(x)$ has the correct sign. This completes a proof of Theorem 2.

We may summarize our findings by stating

COROLLARY 1. *We may think of the logarithmic q -spline $S_n(x)$ as generated as follows: We define $S_n^{(n)}(x)$ as a step-function on $(0, \infty)$ by the relations (2.9), for all integral values of r . We integrate successively $S_n^{(n)}(x)$ by means of the relations (2.10) for $\nu = n, n - 1, \dots, 3, 2$, obtaining finally the decreasing positive function $S'_n(x)$. Finally, we define $S_n(x)$ by*

$$(2.11) \quad S_n(x) = \int_1^x S'_n(t) dt, \quad (x > 0).$$

That the resulting function $S_n(x)$ is an element of $\sum_{n,q}$, i.e. a cardinal q -spline, is clear because we integrate a step-function $S_n^{(n)}(x)$ n times. What does seem remarkable is that the resulting function should solve the interpolation problem

$$(2.12) \quad S_n(q^\nu) = \nu, \quad \text{for all } \nu,$$

and, what is more, even satisfy the functional equation

$$(2.13) \quad S_n(qx) = S_n(x) + 1.$$

3. Direct reconstruction of $S_n(x)$ by Corollary 1. Actually the reconstruction of $S_n(x)$ as described by Corollary 1 is easily carried out explicitly yielding our previous results. We start from (2.9) and observe that this step function may certainly be written in the form

$$(3.1) \quad S_n^{(n)}(x) = \sum_{s=-\infty}^{\infty} c_s (q^s - x)_+^0, \quad (x > 0).$$

If $q^r < x < q^{r+1}$ then (3.1) becomes $S_n^{(n)}(x) = \sum_{s=r} c_s$ and this must equal the right side of (2.9). By differencing we obtain

$$c_r = (-1)^{n-1} \frac{n!}{q^n - 1} q^{-rn} (q^n - 1) = (-1)^{n-1} n! q^{-rn},$$

and substituting into (3.1) we obtain that

$$(3.2) \quad S_n^{(n)}(x) = (-1)^{n-1} n! \sum_{r=-\infty}^{\infty} q^{-rn} (q^r - x)_+^0$$

On performing ν times the operation (2.10) we obtain

$$S_n^{(n-\nu)}(x) = (-1)^\nu (-1)^{n-1} n! \sum_n q^{-rn} (q^r - x)_+^\nu / \nu!$$

and in particular that

$$(3.3) \quad S'_n(x) = \sum_{-\infty}^{\infty} q^{-r} n (1 - xq^{-r})_+^{n-1}.$$

The last integration (2.11) yields

$$(3.4) \quad S_n(x) = \sum_{-\infty}^{\infty} \{(1 - q^{-r})_+^n - (1 - xy^{-r})_+^n\}, \quad (x > 0).$$

If we assume that $1 < x < q$, then (3.4) reduces to

$$\begin{aligned} S_n(x) &= \sum_{r=1}^{\infty} \{(1 - q^{-r})^n - (1 - xq^{-r})^n\} \\ &= \sum_{r=1}^{\infty} \left\{ 1 - \binom{n}{1} q^{-r} + \dots + (-1)^n q^{-nr} \right. \\ &\quad \left. - \left(1 - \binom{n}{1} xq^{-r} + \dots + (-1)^n x^n q^{-rn} \right) \right\}. \end{aligned}$$

If we cancel within the brackets the two unit terms, then the resulting series may be summed termwise and we obtain that

$$\begin{aligned} S_n(x) &= \binom{n}{1} (x-1) \frac{q^{-1}}{1-q^{-1}} - \binom{n}{2} (x^2-1) \frac{q^{-2}}{1-q^{-2}} \\ &\quad + \dots + (-1)^{n-1} (x^n-1) \frac{q^{-n}}{1-q^{-n}} \\ &= \binom{n}{1} \frac{x-1}{q-1} - \binom{n}{2} \frac{x^2-1}{q^2-1} + \dots + (-1)^{n-1} \frac{x^n-1}{q^n-1} \\ &\quad \text{in } 1 < x < q, \end{aligned}$$

which agrees with the result of Theorem 1. The expression (3.4) will be found useful in our next section.

4. The peculiar behavior of $S_n(x)$ as $n \rightarrow \infty$. We have derived in (3.4) the expression

$$(4.1) \quad S_n(x) = \sum_{r=-\infty}^{\infty} \{(1 - q^{-r})_+^n - (1 - xq^{-r})_+^n\}, \quad (0 < x < \infty).$$

This expression will show against all expectations that the q -spline $S_n(x)$ does not converge to $f(x) = \log x / \log q$ as $n \rightarrow \infty$. The key to our discussion is the following way of writing the series (4.1) Using the square brackets to denote the integral part we define

$$(4.2) \quad \theta_n = \frac{\log n}{\log q} - \left[\frac{\log n}{\log q} \right] = \frac{\log n}{\log q} - k_n,$$

hence $n = q^{\theta_n + k_n}$ and therefore $1 = q^{\theta_n + k_n} / n$. We may therefore rewrite (4.1) as

$$S_n(x) = \sum_r \left\{ \left(1 - \frac{1}{n} q^{\theta_n + k_n - r} \right)_+^n - \left(1 - \frac{1}{n} x q^{\theta_n + k_n - r} \right)_+^n \right\}$$

and finally as

$$(4.3) \quad S_n(x) = \sum_{s=-\infty}^{\infty} \left\{ \left(1 - \frac{1}{n} q^{\theta_n + s} \right)_+^n - \left(1 - \frac{1}{n} x q^{\theta_n + s} \right)_+^n \right\}.$$

We also define the function

$$(4.4) \quad F(x, \theta) = \sum_{s=-\infty}^{\infty} \{(e^{-q^{\theta+s}} - e^{-xq^{\theta+s}})\}, \quad (x > 0, -\infty < \theta < \infty)$$

which is evidently periodic in θ of period 1. For each fixed positive x the function $F(x, \theta)$ of θ has an interval of variability

$$(4.5) \quad I(x) = \left[\min_{\theta} F(x, \theta), \max_{\theta} F(x, \theta) \right]$$

in terms of which we can state the following

THEOREM 3. *For each fixed positive x the set of limit points of the sequence $(S_n(x))_{n=1,2,\dots}$ is identical with the interval $I(x)$ defined by (4.5).*

Our discussion will simplify if we consider the derivative with respect to x of the functions (4.3) and (4.4), hence

$$(4.6) \quad S'_n(x) = \sum_s q^{\theta_n + s} \left(1 - \frac{1}{n} x q^{\theta_n + s} \right)_+^{n-1}$$

and

$$(4.7) \quad F'_x(x, \theta) = \sum_s q^{\theta+s} e^{-xq^{\theta+s}}.$$

At this point we need some lemmas.

LEMMA 1. *If a_n and a are real, $a_n \rightarrow a$, then*

$$(4.8) \quad \lim_{n \rightarrow \infty} \left(1 - \frac{a_n}{n}\right)_+^{n-1} = e^{-a}.$$

We may omit the elementary proof.

LEMMA 2. *The inequality*

$$(4.9) \quad \left(1 - \frac{x}{n}\right)_+^{n-1} < 2e^{-x} \text{ holds for } x \geq 0, n \geq 2.$$

Proof. (4.9) is trivial if $x > n$. If $0 \leq x \leq n$ we consider

$$g(x) = \left(1 - \frac{x}{n}\right)_+^{n-1} e^x$$

and observe that

$$\begin{aligned} g'(x) &= \left(1 - \frac{x}{n}\right)_+^{n-1} e^x - \frac{n-1}{n} \left(1 - \frac{x}{n}\right)_+^{n-2} e^x \\ &= e^x \left(1 - \frac{x}{n}\right)_+^{n-2} \left\{1 - \frac{x}{n} - \frac{n-1}{n}\right\} \\ &= \frac{1}{n} e^x \left(1 - \frac{x}{n}\right)_+^{n-2} (1-x). \end{aligned}$$

This shows that in $[0, n]$ we have

$$\max g(x) = g(1) = \left(1 - \frac{1}{n}\right)_+^{n-1} \cdot e \rightarrow e^{-1} \cdot e = 1 \text{ as } n \rightarrow \infty.$$

Actually $(1 - n^{-1})^{n-1} e < 2$ if $n \geq 2$ which proves the lemma.

LEMMA 3. *The function $F(x, \theta)$, defined by (4.4), satisfies the functional equation*

$$(4.10) \quad F(qx, \theta) = F(x, \theta) + 1, \quad (x > 0).$$

Proof. By (4.4)

$$\begin{aligned} F(qx, \theta) &= \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s+1}}) \\ &= \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}} + e^{-xq^{\theta+s}} - e^{-xq^{\theta+s+1}}), \end{aligned}$$

hence

$$F(qx, \theta) = F(x, \theta) + \sum_s (e^{-xq^{\theta+s}} - e^{-xq^{\theta+s+1}}).$$

Writing $xq^\theta = \alpha$, the last series becomes $\sum_s (e^{-\alpha q^s} - e^{-\alpha q^{s+1}})$ and this is easily shown to converge to the sum 1 as follows: Letting $A \rightarrow -\infty$ and $B \rightarrow +\infty$ we have

$$\begin{aligned} \sum_s (e^{-\alpha q^s} - e^{-\alpha q^{s+1}}) &= \lim_{s=A}^{B-1} \sum_{s=A}^{B-1} (e^{-\alpha q^s} - e^{-\alpha q^{s+1}}) \\ &= \lim (e^{-\alpha q^A} - e^{-\alpha q^B}) = 1 - 0 = 1. \end{aligned}$$

Observe that the functional equations (8) and (4.10) are identical. It follows that the relation

$$(4.11) \quad S_n(qx) - F(qx, \theta) = S_n(x) - F(x, \theta)$$

holds for all positive x and all real θ . Of special interest for us is the following

COROLLARY 2. *In studying the behavior of the difference $S_n(x) - F(x, \theta_n)$, we loose no generality in assuming that x is restricted to the interval $1 \leq x \leq q$.*

LEMMA 4. *Let*

$$(4.12) \quad 2 \leq n_1 < n_2 < \dots \text{ be an increasing sequence of integers,}$$

$$(4.13) \quad \eta_1, \eta_2, \dots \text{ be a sequence of reals, } 0 \leq \eta_\nu \leq 1,$$

and let x be restricted to the interval

$$(4.14) \quad 1 \leq x \leq q.$$

Then the series

$$(4.15) \quad \sum_{s=-\infty}^{\infty} q^{\eta_\nu+s} \left(1 - x \frac{q^{\eta_\nu+s}}{n_\nu}\right)_+^{n_\nu-1}$$

converges uniformly with respect to all data n_ν, η_ν and x satisfying the conditions (4.12), (4.13) and (4.14).

Proof. By Lemma 2 the series (4.15) is termwise dominated by the series

$$\sum_s q^{\eta_\nu+s} 2e^{-xq^{\eta_\nu+s}},$$

and this last series is termwise dominated by the series

$$(4.16) \quad \sum_{s=-\infty}^{\infty} q^{1+s} 2e^{-q^s}$$

whose terms are independent of all the data. Since (4.16) converges, the lemma is established.

LEMMA 5. *If we add to (4.13) the assumption*

$$(4.17) \quad \lim_{\nu \rightarrow \infty} \eta_\nu = \eta$$

then

$$(4.18) \quad \lim_{\nu \rightarrow \infty} \sum_s q^{\eta_\nu + s} \left(1 - x \frac{q^{\eta_\nu + s}}{n_\nu}\right)_+^{n_\nu - 1} = \sum_s q^{\eta + s} e^{-x q^{\eta + s}}$$

uniformly in x satisfying (4.14).

Proof. By Lemma 1

$$\lim_{\nu \rightarrow \infty} q^{\eta_\nu + s} \left(1 - x \frac{q^{\eta_\nu + s}}{n_\nu}\right)_+^{n_\nu + 1} = q^{\eta + s} e^{-x q^{\eta + s}},$$

and the uniform convergence of the series on the left side of (4.18) (Lemma 4), implies the relation (4.18).

In view of the uniform convergence in x we can integrate term-wise the limit relation (4.18) with respect to x between the limits of integration 1 and x , where we assume (4.14). These integrations are immediately performed because the expressions to be integrated were obtained by (4.6) and (4.7) by differentiation. We state the result as

LEMMA 6. *If we assume (4.12), (4.13), (4.14), and (4.17) to hold, then*

$$(4.19) \quad \lim_{\nu \rightarrow \infty} \sum_s \left\{ \left(1 - \frac{1}{n_\nu} q^{\eta_\nu + s}\right)_+^{n_\nu} - \left(1 - \frac{x}{n_\nu} q^{\eta_\nu + s}\right)_+^{n_\nu} \right\} \\ = \sum_s (e^{-q^{\eta + s}} - e^{-x q^{\eta + s}}),$$

uniformly in η_ν , η , and in x .

An approximation of the q -spline $S_n(x)$ is now obtained as follows. We apply Lemma 6 for the special case that $n_\nu = n$, and that η_ν does not depend on ν , $\eta_\nu = \eta$ for all ν . We obtain from (4.19) that

$$(4.20) \quad \lim_{n \rightarrow \infty} \sum_s \left\{ \left(1 - \frac{1}{n} q^{\eta + s}\right)_+^n - \left(1 - \frac{x}{n} q^{\eta + s}\right)_+^n \right\} = \sum_s (e^{-q^{\eta + s}} - e^{-x q^{\eta + s}}),$$

uniformly in η ($0 \leq \eta \leq 1$) and in x ($1 \leq x \leq q$).

This means the following: To every positive ε , there corresponds

an N_ϵ such that the error term in (4.20) is in absolute value $< \epsilon$ if $n > N_\epsilon$, and this for all η and x in their respective intervals. *But then it is clear that we may let η also depend on n .* Choosing in particular

$$\eta = \theta_n = \log n / \log q - [\log n / \log q]$$

we obtain by (4.3) and (4.4) the following

THEOREM 4. *We have the relation*

$$(4.21) \quad S_n(x) = F\left(x, \frac{\log n}{\log q}\right) + o(1) \text{ as } n \rightarrow \infty,$$

the error term $o(1)$ being uniform for all positive values of x .

Notice that indeed $F(x, \log n / \log q) = F(x, \theta_n)$, due to the periodicity of $F(x, \theta)$ as a function of θ . In extending the uniformity of $o(1)$ from $[1, q]$ to $(0, \infty)$ we have used Corollary 2.

The numbers θ_n being $\equiv \log n / \log q \pmod{1}$, it is clear that the sequence (θ_n) is everywhere dense in $[0, 1]$. Given θ in $[0, 1]$ we can therefore choose an increasing sequence (n_ν) such that $\theta_{n_\nu} \rightarrow \theta$ as $\nu \rightarrow \infty$. It now follows from (4.21) and the continuity of $F(x, \theta)$ that

$$(4.22) \quad \lim_{\nu \rightarrow \infty} S_{n_\nu}(x) = F(x, \theta).$$

In view of these remarks it is clear that the relation (4.21) implies the truth of Theorem 3.

5. The Fourier series of $F(x, \theta)$. Surely Theorem 4 shows the role played by the periodic function

$$(5.1) \quad F(x, \theta) = \sum_{s=-\infty}^{\infty} (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}})$$

in the behavior of $S_n(x)$ for large values of n . Let us derive its Fourier expansion

$$(5.2) \quad F(x, \theta) = \sum_{\nu=-\infty}^{\infty} a_\nu(x) e^{2\pi i \nu \theta}.$$

We find that

$$a_0(x) = \int_0^1 \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}) d\theta = \int_{-\infty}^{\infty} (e^{-q^t} - e^{-xq^t}) dt$$

and setting $q^t = u$, hence $q^t \log q dt = du$, we find that

$$\begin{aligned} a_0(x) &= \frac{1}{\log q} \int_0^\infty (e^{-u} - e^{-xu}) u^{-1} du = \frac{1}{\log q} \int_0^\infty \left(\int_1^x e^{-uv} dv \right) du \\ &= \frac{1}{\log q} \int_1^x \left(\int_0^\infty e^{-uv} du \right) dv = \frac{1}{\log q} \int_1^x \frac{dv}{v} \end{aligned}$$

and finally

$$(5.3) \quad a_0(x) = \frac{\log x}{\log q} .$$

Notice that the average of $F(x, \theta)$ over a period is equal to the presumed limit of $S_n(x)$. At this point we observe that $F(x, \theta)$ being continuous in θ , there certainly exists a θ_x such that

$$F(x, \theta_x) = \log x / \log q .$$

The last paragraph of § 4 now implies the

COROLLARY 2. *To every positive x corresponds a sequence (n_ν) such that*

$$(5.4) \quad \lim_{\nu \rightarrow \infty} S_{n_\nu}(x) = \frac{\log x}{\log q} .$$

If $\nu \neq 0$, we find that

$$\begin{aligned} a_\nu(x) &= \int_0^1 e^{-2\pi i \nu \theta} F(x, \theta) d\theta = \int_0^1 e^{-2\pi i \nu \theta} \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}) d\theta \\ &= \sum_s \int_0^1 e^{-2\pi i \nu \theta} (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}) d\theta \\ &= \int_{-\infty}^\infty e^{-2\pi i \nu t} (e^{-q^t} - e^{-xq^t}) dt . \end{aligned}$$

Setting again $q^t = u$ we obtain

$$(5.5) \quad a_\nu(x) = \frac{1}{\log q} \int_0^\infty u^{-(2\pi i \nu / \log q) - 1} (e^{-u} - e^{-xu}) du .$$

This is easily evaluated in terms of the Γ -function. If we add ε ($\varepsilon > 0$) to the exponent of u in the integrand, we may then write the integral as a difference of two Γ -integrals which amounts to

$$\frac{1}{\log q} \Gamma\left(\varepsilon - \frac{2\pi i \nu}{\log q}\right) (1 - x^{(2\pi i \nu / \log q) - \varepsilon}) .$$

On letting $\varepsilon \rightarrow 0$ we obtain that

$$(5.6) \quad a_\nu(x) = \frac{1}{\log q} \Gamma\left(-\frac{2\pi i \nu}{\log q}\right) (1 - x^{2\pi i \nu / \log q}) , \quad (\nu \neq 0) .$$

From (4.4) we see that $F(1, \theta) = 0$ for all θ . Moreover (4.10) shows that $F(q^r, \theta) = r$ for all θ . We therefore see that the interval of variability $I(x)$, defined by (4.5), reduces of the point r if $x = q^r$ (r integer). The relations (5.6) allow us to prove the converse, to the effect that $I(x)$ reduces to a single point, namely $\log x / \log q$, only if x is a power of q . For if $I(x)$ is a single point, then the unicity theorem of Fourier series shows that all coefficients (5.6) vanish. The Γ -function being $\neq 0$, we conclude that

$$1 = x^{2\pi i\nu/\log q} = e^{2\pi i\nu(\log x/\log q)} \text{ if } \nu \neq 0$$

and this indeed implies that $\log x / \log q = r$ is an integer, hence $x = q^r$.

In view of Theorem 3 we have just established

COROLLARY 3. *The relation*

$$(5.7) \quad \lim_{n \rightarrow \infty} S_n(x) = \frac{\log x}{\log q}$$

is valid if and only if $x = q^r$ for an integer r .

The Fourier series (5.2), (5.3), (5.6) allows us to write the approximation formula (4.21) in explicit terms. Observing that

$$e^{2\pi i\nu(\log n/\log q)} = n^{2\pi i\nu/\log q}$$

and using the Fourier series we obtain the following

COROLLARY 4. *For every fixed positive x we have*

$$(5.8) \quad S_n(x) = \frac{\log x}{\log q} + \frac{1}{\log q} \sum_{\nu \neq 0} \Gamma\left(-\frac{2\pi i\nu}{\log q}\right) (1 - x^{2\pi i\nu/\log q}) n^{2\pi i\nu/\log q} \\ + o(1), \text{ as } n \rightarrow \infty,$$

the error term $o(1)$ being uniform in $x > 0$.

This will be used in § 7.

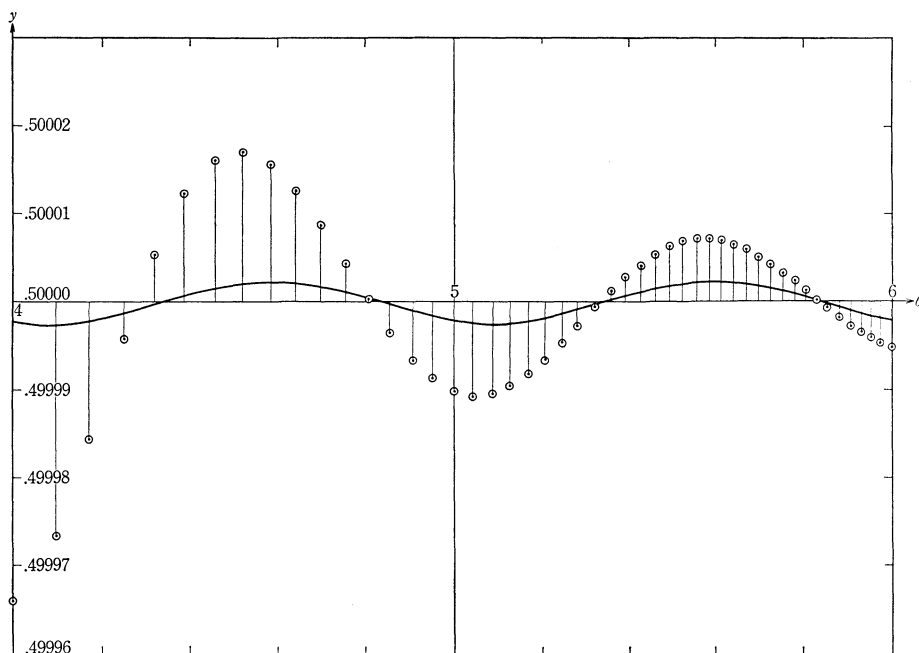
6. A graph for the case $q = 2$ and $x = \sqrt{2}$. We select

$$(6.1) \quad q = 2, \quad x = \sqrt{2}, \quad \text{hence } \frac{\log x}{\log q} = \frac{1}{2}.$$

Figure 1 shows the graph of the function

$$(6.2) \quad y = F(\sqrt{2}, \theta) \text{ in the interval } 4 \leq \theta \leq 6$$

and also the 49 points



This sheet represents in the (θ, y) -plane the rectangle defined by

$$4 \leq \theta \leq 6, \quad .49996 \leq y \leq .50003.$$

The curve is $y = F(\sqrt{2}, \theta)$, $4 \leq \theta \leq 6$.

The 49 dots represent the points $(\log n / \log 2, S_n(\sqrt{2}))$, $n = 16, 17, \dots, 64$.

Figure 1

$$(6.3) \quad \left(\frac{\log n}{\log 2}, S_n(\sqrt{2}) \right) \text{ for } n = 16, 17, \dots, 64.$$

Notice the tremendously magnified scale in the y -direction, and that the abscissae of the points (6.3) also vary from 4 to 6. Thus Figure 1 exhibits graphically all terms of the relation (4.21) or

$$(6.4) \quad S_n(\sqrt{2}) = F\left(\sqrt{2}, \frac{\log n}{\log 2}\right) + o(1) \text{ for } n = 16, 17, \dots, 64.$$

If we were to extend the graph of Figure 1 from $4 \leq \theta \leq 6$ to $4 \leq \theta \leq +\infty$, then the abscissae of consecutive dots would get closer together indefinitely, while the oscillations of their ordinates would decrease so that the dots would tend to the curve $y = F(\sqrt{2}, \theta)$, according to the relation (6.4).

Figure 1 reveals the following apparent situation: The "curve" through the points (6.3) seems to intersect the horizontal line $y = 1/2$ in practically the same points as the curve $y = F(\sqrt{2}, \theta)$. *It therefore seems that $S_n(\sqrt{2})$ is very close to its "correct" limit $1/2$ whenever n is such that $\log n / \log 2$ is close to one of the points θ*

where $F(\sqrt{2}, \theta)$ assumes its average value $1/2$. We have no explanation for this phenomenon, if true.

A word on the computations underlying Figure 1. In the first place we needed the ordinates of the points (6.3). Since $x = \sqrt{2}$ is in the interval $[1, q]$ we have $S_n(x) = P_n(x)$, by Theorem 1. Moreover the relation (1.2) shows that

$$(6.5) \quad \frac{\log x}{\log q} - P_n(x) = (-1)^n \Delta^n \frac{x^t - 1}{q^t - 1} \Big|_{t=0}.$$

Here Δ^n denotes the ordinary n th order forward difference (with step 1) with respect to t . Our colleague C. de Boor kindly computed these differences for the data (6.1) and for $n = 1, 2, 3, \dots, 70$. Beyond this point round-off difficulties became apparent. This gave us the points (6.3).

The function (6.2) was obtained from its Fourier series (5.2). It so happens that for $x = \sqrt{2}$ the coefficients (5.6) vanish for all even values of $\nu \neq 0$. For this reason by taking only the fundamental ($\nu = \pm 1$) in the expansion (5.2) we already obtain better than 15-place accuracy. The coefficients $a_1(\sqrt{2})$ and $a_{-1}(\sqrt{2})$ were computed by means of the fine tables in [1], (p. 277), and we found with 11-place accuracy that

$$(6.6) \quad F(\sqrt{2}, \theta) = \frac{1}{2} + A \sin 2\pi(\theta - \alpha),$$

where

$$(6.7) \quad A = .238061 \times 10^{-5}, \quad \alpha = .340775.$$

7. The summability of the sequence $S_n(x)$. We rewrite (5.8) as

$$(7.1) \quad S_n(x) = \frac{\log x}{\log q} + \sum_{\nu \neq 0} a_\nu(x) n^{2\pi i \nu / \log q} + o(1)$$

and are looking for a summability method that will produce the limit $\log x / \log q$. For this it is necessary that the method should assign for the sequence

$$(7.2) \quad (n^{i\lambda})_{n=1,2,\dots} \quad (\text{for constant } \lambda \neq 0)$$

the limit zero. Neither the Cesàro nor the Abel method will do that. However, the *logarithmic means* (see [2], §§ 3.8 and 4.16) will be found to work yielding the following

THEOREM 5. *We have that*

$$(7.3) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} \left\{ S_1(x) + \frac{1}{2} S_2(x) + \dots + \frac{1}{n} S_n(x) \right\} = \frac{\log x}{\log q}$$

uniformly in $x, x > 0$.

Proof. Referring to (7.1) and using the fact that

$$\sum_{\nu \neq 0} |a_\nu(x)| < \frac{2}{\log q} \sum_{\nu \neq 0} \left| \Gamma \left(-\frac{2\pi i \nu}{\log q} \right) \right| < \infty,$$

it is clear that (7.1) will imply (7.3) as soon as we use the following

LEMMA 6. For real λ we have

$$(7.4) \quad |1^{i\lambda-1} + 2^{i\lambda-1} + \dots + n^{i\lambda-1}| \leq 1 + \log n \text{ for all } n,$$

and

$$(7.5) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} (1^{i\lambda-1} + 2^{i\lambda-1} + \dots + n^{i\lambda-1}) = 0 \text{ if } \lambda \neq 0.$$

Proof. 1. The left side of (7.4) is less than

$$1 + \frac{1}{2} + \dots + \frac{1}{n} \leq 1 + \int_1^n \frac{dx}{x} = 1 + \log n.$$

2. Writing $f(x) = x^{i\lambda-1}$ we use the Euler-MacLaurin summation formula

$$\sum_1^n f(k) = \int_1^n f(x) dx + \frac{1}{2} (f(1) + f(n)) + \int_1^n \left(x - [x] - \frac{1}{2} \right) f'(x) dx$$

which shows that

$$\begin{aligned} \left| \sum_1^n f(k) \right| &< \left| \int_1^n f(x) dx \right| + \frac{1}{2} (1 + n^{-1}) + \left| \int_1^n f'(x) dx \right| \\ &= |n^{i\lambda} - 1| \lambda^{-1} + \frac{1}{2} (1 + n^{-1}) + |i\lambda - 1| \left| \int_1^n x^{-2} dx \right|. \end{aligned}$$

The last expression is $O(1)$ and therefore $o(\log n)$.

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