

# Pacific Journal of Mathematics

**LOCALITY OF THE NUMBER OF PARTICLES OPERATOR**

M. ANN PIECH

## LOCALITY OF THE NUMBER OF PARTICLES OPERATOR

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**We view the number of particles operator  $N$  as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup in an abstract Wiener setting. It is shown that if two functions  $f, g$  in the domain of  $N$  agree a.e. on an open set  $\mathcal{O}$ , then  $Nf = Ng$  on  $\mathcal{O}$ . The restriction of  $N$  to a large core acts as an infinite dimensional partial differential operator  $L$ , and it is shown that  $L$  may be defined locally in an  $L^2_{loc}$  setting.**

One of the mathematical concepts which has been the subject of considerable recent interest in constructive quantum field theory is the identification of the Bose Fock space  $\mathcal{F}$  with a space  $\mathcal{L}$  of  $L^2$  functions over some Gaussian measure space  $(\mathcal{Q}, d\mu)$ . When  $\mathcal{F} = \sum_{n=0}^{\infty} \otimes_n^s \mathcal{H}$ , the sum of the  $n$ -fold Hilbert space symmetric tensor product of the complexification  $\mathcal{H}$  of a real separable Hilbert space  $H$ , possible choices of  $(\mathcal{Q}, d\mu)$  include any measure space on which the isonormal distribution over  $H$  may be realized. This identification is nicely described by Nelson [3]. The isometric isomorphism of  $\mathcal{F}$  with  $\mathcal{L}$  preserves the canonical direct sum decomposition of  $\mathcal{F}$ ; that is, we have a corresponding decomposition  $\mathcal{L} = \sum_{n=0}^{\infty} \mathcal{L}_n$ .  $\mathcal{L}_n$  has a natural interpretation as the  $L^2$  space spanned by the Hermite functions on  $\mathcal{Q}$  of rank  $\leq n$  [3, 8].

One way of realizing the isonormal distribution on  $H$  is to complete  $H$  with respect to a weaker norm (a "measurable" norm in the sense of Gross [2]) obtaining a Banach space  $B$  in which  $H$  is densely embedded. The pair  $(H, B)$  is known as an abstract Wiener pair, and  $\mu$  is taken to be the Wiener measure  $p_1$  on the Borel sets of  $B$ , generated by the canonical Gauss cylinder set measure on  $H$  [2]. Under the identification of  $\mathcal{F}$  and  $L^2(p_1)$ , we further identify the number of particles operator on  $\mathcal{F}$  (i.e. the second quantization of the identity operator on  $H$ ) with the infinitesimal generator  $N$  of the Ornstein-Uhlenbeck velocity semigroup  $\{e^{-tN}\}$  for the Brownian motion on  $B$  [3, 4]. For  $H$  finite dimensional  $Nf(x) = -\Delta f(x) + x \cdot \text{grad } f(x)$  on smooth  $f$ . However,  $\mathcal{F}$  is usually constructed over an infinite dimensional  $H$ , and the expression of  $N$  as a differential operator must be suitably reinterpreted.

As a differential operator,  $N$  only incorporates derivatives in the directions of vectors of  $H$ . We define the  $H$ -derivative  $Dg(x)$  of a function  $g$  defined on a neighborhood of  $x$  in  $B$  and taking values in a Banach space  $W$  as follows. Let  $\hat{g}(h) = g(x + h)$  for all  $h$  in  $H$

such that  $x + h$  is in the domain of  $g$ . Then  $\hat{g}$  maps a neighborhood of the origin of  $H$  into  $W$ .  $Dg(x) \equiv \hat{g}'(0)$ , the Fréchet derivative of  $\hat{g}$  at 0. When  $g$  is real valued,  $Dg(x)$  is an element of  $H^*$  which is customarily identified via the Riesz representation with an element of  $H$ .  $D^2g(x)$  is defined by iteration, and will be identified with an element of  $\mathcal{L}(H, H)$ . Since  $B^*$  is dense in  $H^* \approx H$ , we can always find orthonormal bases  $\{e_i\}$  for  $H$  consisting of elements of  $B^*$ .

In [4] it is shown that the set

$$\mathcal{C} = \{ \text{real valued } f \in L^2(p_1) \text{ such that } |Df(x)|_H \text{ exists} \\ \text{a.e. and is in } L^2(p_1) \text{ and also } D^2f(x) \text{ exists a.e.} \\ \text{and is a Hilbert-Schmidt operator on } H \text{ with} \\ |D^2f(x)|_{\mathcal{H}-\mathcal{S}} \in L^2(p_1) \}$$

is a core for  $N$ . The action of  $N$  on an  $f$  in  $\mathcal{C}$  is as follows. If  $\{e_i\}$  is any orthonormal basis in  $H$  with  $e_i \in B^*$ , and  $P_i$  is the orthogonal projection of  $H$  onto  $\{e_i, \dots, e_i\}$ , then

$$(1) \quad \{ \langle x, P_i Df(x) \rangle - \text{trace}(P_i D^2f(x)) \}_{i=1,2,\dots}$$

is a Cauchy sequence in  $L^2(p_1)$ .  $Nf$  is the limit of this sequence, and is independent of the choice of  $\{e_i\}$ .

Other smaller cores for  $N$  are well-known. They generally consist of smooth polynomial cylinder functions. For the purposes of this note, however,  $\mathcal{C}$  possesses a property that polynomial cores fail to have. Namely, the elements of  $\mathcal{C}$  suffice to generate partitions of unity on  $B$  [1, 6]. That is, given any two concentric balls  $b_1 \subseteq b_2$  in  $B$ , we can find  $\varphi \in \mathcal{C}$  such that  $0 \leq \varphi(x) \leq 1$ ,  $\varphi(x) = 1$  on  $b_1$  and  $\varphi(x) = 0$  on  $B - b_2$ . Moreover,  $\varphi(x)$ ,  $|D\varphi(x)|_H$  and  $|D^2\varphi(x)|_{\mathcal{H}-\mathcal{S}}$  can be assumed continuous and bounded on  $B$ . We call such a  $\varphi$  a partition function for  $b_1, b_2$ . We point out that if  $H$ -differentiability were replaced with the usual Fréchet differentiability on  $B$ , it would not always be possible to find a nontrivial  $C^1$  function  $\varphi$  vanishing off  $b_2$ .

Locality of  $N$  can be stated in several ways. If two functions  $f, g$  in the domain of  $N$  have disjoint supports, then  $Nf$  and  $Ng$  have disjoint supports. Or, a stronger statement, that if  $f$  and  $g$  coincide a.e. on an open set, then  $Nf = Ng$  a.e. on that set. Or, equivalently,

**PROPOSITION 1.** *If  $f$  is in the domain of  $N$  and if  $f$  vanishes a.e. on an open subset  $\mathcal{O}$  of  $B$ , then  $Nf$  vanishes a.e. on  $\mathcal{O}$ .*

*Proof.* Since  $\mathcal{C}$  is a core for  $f$ , we can find  $f_n \in \mathcal{C}$  with  $f_n \rightarrow f(L^2)$  and  $Nf_n \rightarrow Nf(L^2)$ . Fix  $y \in \mathcal{O}$ , and choose two open balls

$b_1, b_2$  centered at  $y$ , with  $b_1 \subset b_2$  and  $\bar{b}_2 \subset \mathcal{O}$ . Choose  $\varphi_y \in \mathcal{C}$  with  $0 \leq \varphi_y(x) \leq 1$ ,  $\varphi_y(x) = 0$  on  $b_1$ ,  $\varphi_y(x) = 1$  on  $B - b_2$  and with  $\varphi_y(x), |D\varphi_y(x)|_H$  and  $|D^2\varphi_y(x)|_{\mathcal{X}-\mathcal{Y}}$  all continuous and bounded on  $B$ . Since  $\partial b_2$  has  $p_1$  measure zero, we may without loss of generality assume each  $f_n$  vanishes on  $b_2$ . Now  $\varphi_y f_n \rightarrow \varphi_y f = f$  in  $L^2$ . Also  $\varphi_y f_n \in \mathcal{C}$ , and

$$\begin{aligned} N\varphi_y f_n &= \lim_i \varphi_y(x) \{ \langle x, P_i Df_n(x) \rangle - \text{trace} (P_i D^2 f_n(x)) \} \\ &\quad + \lim_i f_n(x) \{ \langle x, P_i D\varphi_y(x) \rangle - \text{trace} (P_i D^2 \varphi_y(x)) \} \\ &\quad - 2 \lim_i \text{trace} P_i (D\varphi_y(x) \otimes Df_n(x)) . \end{aligned}$$

Dominated convergence ensures that the first limit exists, and the choice of support for  $f_n$  ensures that the subsequent terms are zero a.e. Hence  $N\varphi_y f_n = \varphi_y \cdot Nf_n$ , and so  $N\varphi_y f_n \rightarrow \varphi_y \cdot Nf$  in  $L^2$ . Since  $N$  is closed,  $\varphi_y \cdot Nf = Nf$  follows. Thus  $Nf$  vanishes a.e. on  $b_1$ . Since  $B$  is separable, it follows that  $Nf$  vanishes a.e. on  $\mathcal{O}$ .

It is expected that  $N$  should serve as the model for the Laplace-Beltrami operator on manifolds modelled on  $B$ . We will now show that we can easily locally define an operator  $L$  which extends the restriction of  $N$  to  $\mathcal{C}$ . For any open subset  $\mathcal{O}$  of  $B$ , we define

$$\begin{aligned} \mathcal{C}_\mathcal{O} = \{ \text{real valued } f \text{ defined on } \mathcal{O}, \text{ with } |Df(x)|_H \\ \text{and } |D^2f(x)|_{\mathcal{X}-\mathcal{Y}} \text{ existing a.e. on } \mathcal{O}, \text{ such that} \\ f, |Df|_H \text{ and } |D^2f|_{\mathcal{X}-\mathcal{Y}} \text{ are locally in } L^2(p_1) \text{ on} \\ \mathcal{O} \} . \end{aligned}$$

Then we may define  $L$  on  $\mathcal{C}_\mathcal{O}$  by

**PROPOSITION 2.** *Given  $f$  in  $\mathcal{C}_\mathcal{O}$ , let  $\{\mathcal{O}_n\}$  be any countable cover of  $\mathcal{O}$  by open balls such that for each  $\mathcal{O}_n$  there is a concentric  $\mathcal{O}'_n$  with  $\mathcal{O}_n \subseteq \mathcal{O}'_n \subset \mathcal{O}$  and such that  $f, |Df|_H$  and  $|D^2f|_{\mathcal{X}-\mathcal{Y}}$  are in  $L^2$  on each  $\mathcal{O}'_n$ . Let  $\varphi_n$  be a partition function for  $\{\mathcal{O}_n, \mathcal{O}'_n\}$ . Extend  $\varphi_n f$  to be zero outside  $\mathcal{O}$ . Then  $\varphi_n f \in \mathcal{C}$ , and we define  $Lf = N\varphi_n f$  on  $\mathcal{O}_n$ . Then  $Lf$  is well defined, is locally in  $L^2(p_1)$  on  $\mathcal{O}$ , and is independent of the choice of  $\mathcal{O}_n$  and  $\varphi_n$ .*

*Proof.* If  $x$  belongs to two members of the covering, say to  $\mathcal{O}_n$  and  $\mathcal{O}_m$ , then  $\varphi_n f$  and  $\varphi_m f$  agree on  $\mathcal{O}_n \cap \mathcal{O}_m$  and  $Lf$  is well-defined by Proposition 1. Hence since  $B$  is separable,  $Lf$  is independent of the choice of  $\mathcal{O}_n$  and  $\varphi_n$ .

In Reference [4] it is shown that for  $f \in \mathcal{C}$ ,

$$(2) \quad |Nf|_{L^2(p_1)}^2 \leq ||Df|_H|_{L^2(p_1)}^2 + ||D^2f|_{\mathcal{X}-\mathcal{Y}}|_{L^2(p_1)}^2 .$$

Thus for for  $f$  in  $\mathcal{C}_\mathcal{O}$ , it follows that  $Lf$  is square integrable on  $\mathcal{O}_n$ .

REMARK. A popular choice of  $(\mathcal{Q}, d\mu)$  is the underlying probability space of the realization on  $\mathcal{S}'(\mathbf{R}^d)$  of a Gaussian process over Schwartz space  $\mathcal{S}(\mathbf{R}^d)$ . That is,  $\mathcal{Q} = \mathcal{S}'$  and  $d\mu$  is a Gaussian Borel measure on  $\mathcal{S}'$ . Such measures  $d\mu$  have as supporting sets Hilbert spaces  $B \subset \mathcal{S}'$ , such that there is an  $H \subset B$  with  $(H, B)$  an abstract Wiener pair.  $d\mu|_B = p_1$ , the Wiener measure for  $(H, B)$  [7, 5]. Our Proposition 1 then may be applied in  $L^2(B, p_1)$ .

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Received July 11, 1975. Research supported by NSF grant PO-28934.

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AMERICAN MATHEMATICAL SOCIETY  
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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

# Pacific Journal of Mathematics

Vol. 61, No. 1

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