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LOCALITY OF THE NUMBER OF PARTICLES OPERATOR

M. ANN PIECH

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We view the number of particles operator N as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup in an abstract Wiener setting. It is shown that if two functions f, g in the domain of N agree a.e. on an open set \mathcal{O} , then $Nf = Ng$ on \mathcal{O} . The restriction of N to a large core acts as an infinite dimensional partial differential operator L , and it is shown that L may be defined locally in an L^2_{loc} setting.

One of the mathematical concepts which has been the subject of considerable recent interest in constructive quantum field theory is the identification of the Bose Fock space \mathcal{F} with a space \mathcal{L} of L^2 functions over some Gaussian measure space $(\mathcal{O}, d\mu)$. When $\mathcal{F} = \sum_{n=0}^{\infty} \otimes_n^s \mathcal{H}$, the sum of the n -fold Hilbert space symmetric tensor product of the complexification \mathcal{H} of a real separable Hilbert space H , possible choices of $(\mathcal{O}, d\mu)$ include any measure space on which the isonormal distribution over H may be realized. This identification is nicely described by Nelson [3]. The isometric isomorphism of \mathcal{F} with \mathcal{L} preserves the canonical direct sum decomposition of \mathcal{F} ; that is, we have a corresponding decomposition $\mathcal{L} = \sum_{n=0}^{\infty} \mathcal{L}_n$. \mathcal{L}_n has a natural interpretation as the L^2 space spanned by the Hermite functions on \mathcal{O} of rank $\leq n$ [3, 8].

One way of realizing the isonormal distribution on H is to complete H with respect to a weaker norm (a "measurable" norm in the sense of Gross [2]) obtaining a Banach space B in which H is densely embedded. The pair (H, B) is known as an abstract Wiener pair, and μ is taken to be the Wiener measure p_1 on the Borel sets of B , generated by the canonical Gauss cylinder set measure on H [2]. Under the identification of \mathcal{F} and $L^2(p_1)$, we further identify the number of particles operator on \mathcal{F} (i.e. the second quantization of the identity operator on H) with the infinitesimal generator N of the Ornstein-Uhlenbeck velocity semigroup $\{e^{-tN}\}$ for the Brownian motion on B [3, 4]. For H finite dimensional $Nf(x) = -\Delta f(x) + x \cdot \text{grad } f(x)$ on smooth f . However, \mathcal{F} is usually constructed over an infinite dimensional H , and the expression of N as a differential operator must be suitably reinterpreted.

As a differential operator, N only incorporates derivatives in the directions of vectors of H . We define the H -derivative $Dg(x)$ of a function g defined on a neighborhood of x in B and taking values in a Banach space W as follows. Let $\hat{g}(h) = g(x + h)$ for all h in H

such that $x + h$ is in the domain of g . Then \hat{g} maps a neighborhood of the origin of H into W . $Dg(x) \equiv \hat{g}'(0)$, the Fréchet derivative of \hat{g} at 0. When g is real valued, $Dg(x)$ is an element of H^* which is customarily identified via the Riesz representation with an element of H . $D^2g(x)$ is defined by iteration, and will be identified with an element of $\mathcal{L}(H, H)$. Since B^* is dense in $H^* \approx H$, we can always find orthonormal bases $\{e_i\}$ for H consisting of elements of B^* .

In [4] it is shown that the set

$$\mathcal{E} = \{ \text{real valued } f \in L^2(p_1) \text{ such that } |Df(x)|_H \text{ exists a.e. and is in } L^2(p_1) \text{ and also } D^2f(x) \text{ exists a.e. and is a Hilbert-Schmidt operator on } H \text{ with } |D^2f(x)|_{\mathcal{H}-\mathcal{H}} \in L^2(p_1) \}$$

is a core for N . The action of N on an f in \mathcal{E} is as follows. If $\{e_i\}$ is any orthonormal basis in H with $e_i \in B^*$, and P_i is the orthogonal projection of H onto $\{e_i, \dots, e_i\}$, then

$$(1) \quad \{ \langle x, P_i Df(x) \rangle - \text{trace}(P_i D^2f(x)) \}_{i=1,2,\dots}$$

is a Cauchy sequence in $L^2(p_1)$. Nf is the limit of this sequence, and is independent of the choice of $\{e_i\}$.

Other smaller cores for N are well-known. They generally consist of smooth polynomial cylinder functions. For the purposes of this note, however, \mathcal{E} possesses a property that polynomial cores fail to have. Namely, the elements of \mathcal{E} suffice to generate partitions of unity on B [1, 6]. That is, given any two concentric balls $b_1 \subsetneq b_2$ in B , we can find $\varphi \in \mathcal{E}$ such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1$ on b_1 and $\varphi(x) = 0$ on $B - b_2$. Moreover, $\varphi(x)$, $|D\varphi(x)|_H$ and $|D^2\varphi(x)|_{\mathcal{H}-\mathcal{H}}$ can be assumed continuous and bounded on B . We call such a φ a partition function for b_1, b_2 . We point out that if H -differentiability were replaced with the usual Fréchet differentiability on B , it would not always be possible to find a nontrivial C^1 function φ vanishing off b_2 .

Locality of N can be stated in several ways. If two functions f, g in the domain of N have disjoint supports, then Nf and Ng have disjoint supports. Or, a stronger statement, that if f and g coincide a.e. on an open set, then $Nf = Ng$ a.e. on that set. Or, equivalently,

PROPOSITION 1. *If f is in the domain of N and if f vanishes a.e. on an open subset \mathcal{O} of B , then Nf vanishes a.e. on \mathcal{O} .*

Proof. Since \mathcal{E} is a core for f , we can find $f_n \in \mathcal{E}$ with $f_n \rightarrow f(L^2)$ and $Nf_n \rightarrow Nf(L^2)$. Fix $y \in \mathcal{O}$, and choose two open balls

b_1, b_2 centered at y , with $b_1 \subset b_2$ and $\bar{b}_2 \subset \mathcal{O}$. Choose $\varphi_y \in \mathcal{E}$ with $0 \leq \varphi_y(x) \leq 1$, $\varphi_y(x) = 0$ on b_1 , $\varphi_y(x) = 1$ on $B - b_2$ and with $\varphi_y(x)$, $|D\varphi_y(x)|_H$ and $|D^2\varphi_y(x)|_{\mathcal{X}-\mathcal{G}}$ all continuous and bounded on B . Since ∂b_2 has p_1 measure zero, we may without loss of generality assume each f_n vanishes on b_2 . Now $\varphi_y f_n \rightarrow \varphi_y f = f$ in L^2 . Also $\varphi_y f_n \in \mathcal{E}$, and

$$\begin{aligned} N\varphi_y f_n &= \lim_i \varphi_y(x) \{ \langle x, P_i Df_n(x) \rangle - \text{trace} (P_i D^2 f_n(x)) \} \\ &\quad + \lim_i f_n(x) \{ \langle x, P_i D\varphi_y(x) \rangle - \text{trace} (P_i D^2 \varphi_y(x)) \} \\ &\quad - 2 \lim_i \text{trace} P_i (D\varphi_y(x) \otimes Df_n(x)) . \end{aligned}$$

Dominated convergence ensures that the first limit exists, and the choice of support for f_n ensures that the subsequent terms are zero a.e. Hence $N\varphi_y f_n = \varphi_y \cdot Nf_n$, and so $N\varphi_y f_n \rightarrow \varphi_y \cdot Nf$ in L^2 . Since N is closed, $\varphi_y \cdot Nf = Nf$ follows. Thus Nf vanishes a.e. on b_1 . Since B is separable, it follows that Nf vanishes a.e. on \mathcal{O} .

It is expected that N should serve as the model for the Laplace-Beltrami operator on manifolds modelled on B . We will now show that we can easily locally define an operator L which extends the restriction of N to \mathcal{E} . For any open subset \mathcal{O} of B , we define

$$\begin{aligned} \mathcal{E}_\mathcal{O} = \{ &\text{real valued } f \text{ defined on } \mathcal{O}, \text{ with } |Df(x)|_H \\ &\text{and } |D^2f(x)|_{\mathcal{X}-\mathcal{G}} \text{ existing a.e. on } \mathcal{O}, \text{ such that} \\ &f, |Df|_H \text{ and } |D^2f|_{\mathcal{X}-\mathcal{G}} \text{ are locally in } L^2(p_1) \text{ on} \\ &\mathcal{O} \} . \end{aligned}$$

Then we may define L on $\mathcal{E}_\mathcal{O}$ by

PROPOSITION 2. *Given f in $\mathcal{E}_\mathcal{O}$, let $\{\mathcal{O}_n\}$ be any countable cover of \mathcal{O} by open balls such that for each \mathcal{O}_n there is a concentric \mathcal{O}'_n with $\mathcal{O}_n \subseteq \mathcal{O}'_n \subset \mathcal{O}$ and such that $f, |Df|_H$ and $|D^2f|_{\mathcal{X}-\mathcal{G}}$ are in L^2 on each \mathcal{O}'_n . Let φ_n be a partition function for $\{\mathcal{O}_n, \mathcal{O}'_n\}$. Extend $\varphi_n f$ to be zero outside \mathcal{O} . Then $\varphi_n f \in \mathcal{E}$, and we define $Lf = N\varphi_n f$ on \mathcal{O}_n . Then Lf is well defined, is locally in $L^2(p_1)$ on \mathcal{O} , and is independent of the choice of \mathcal{O}_n and φ_n .*

Proof. If x belongs to two members of the covering, say to \mathcal{O}_n and \mathcal{O}_m , then $\varphi_n f$ and $\varphi_m f$ agree on $\mathcal{O}_n \cap \mathcal{O}_m$ and Lf is well-defined by Proposition 1. Hence since B is separable, Lf is independent of the choice of \mathcal{O}_n and φ_n .

In Reference [4] it is shown that for $f \in \mathcal{E}$,

$$(2) \quad \|Nf\|_{L^2(p_1)}^2 \leq \| |Df|_H \|_{L^2(p_1)}^2 + \| |D^2f|_{\mathcal{X}-\mathcal{G}} \|_{L^2(p_1)}^2 .$$

Thus for f in $\mathcal{E}_\mathcal{O}$, it follows that Lf is square integrable on \mathcal{O}_n .

REMARK. A popular choice of $(\mathcal{Q}, d\mu)$ is the underlying probability space of the realization on $\mathcal{S}'(\mathbf{R}^d)$ of a Gaussian process over Schwartz space $\mathcal{S}(\mathbf{R}^d)$. That is, $\mathcal{Q} = \mathcal{S}'$ and $d\mu$ is a Gaussian Borel measure on \mathcal{S}' . Such measures $d\mu$ have as supporting sets Hilbert spaces $B \subset \mathcal{S}'$, such that there is an $H \subset B$ with (H, B) an abstract Wiener pair. $d\mu|_B = p_1$, the Wiener measure for (H, B) [7, 5]. Our Proposition 1 then may be applied in $L^2(B, p_1)$.

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