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**CARATHÉODORY AND HELLY-NUMBERS OF
CONVEX-PRODUCT-STRUCTURES**

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Let c_1 and c_2 be the Carathéodory-numbers of the convexity-structures \mathcal{C}_1 for X_1 , respectively \mathcal{C}_2 for X_2 . It is shown that the Carathéodory-number c of the convex-product-structure $\mathcal{C}_1 \oplus \mathcal{C}_2$ for $X_1 \times X_2$ satisfies the inequality $c_1 + c_2 - 2 \leq c \leq c_1 + c_2$; $c_1, c_2 \geq 2$.

The upper bound for c can be improved by one, resp. two, if a certain number, namely the so-called exchange-number, of one resp. each of the structures \mathcal{C}_1 and \mathcal{C}_2 is less than or equal to the Carathéodory-number of that structure.

A new definition of the Helly-number is given and Levi's theorem is proved with this new definition. Finally it is shown that the Helly-number of a convex-product-structure is the greater of the Helly-numbers of \mathcal{C}_1 and \mathcal{C}_2 .

1. Preliminary remarks and definitions. Existing notations and definitions have been taken from [3], [4] and, in particular, from [8]. Let \mathcal{C} be a collection of subsets of a set X ; by $\bigcap \mathcal{C}$ and $\bigcup \mathcal{C}$ we denote the intersection and the union respectively, of the elements of \mathcal{C} . \mathcal{C} is called a *convexity-structure* for X iff $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$ and $\bigcap \mathcal{F} \in \mathcal{C}$ for each subcollection $\mathcal{F} \subset \mathcal{C}$; the pair (X, \mathcal{C}) is called a *convexity-space*. The \mathcal{C} -hull of a set $S \subset X$, denoted by $\mathcal{C}(S)$, is defined by $\mathcal{C}(S) = \bigcap \{C \mid C \in \mathcal{C} \wedge S \subset C\}$. We shall write $\mathcal{C}(a_1, \dots, a_n)$ instead of $\mathcal{C}(\{a_1, \dots, a_n\})$, and $\mathcal{C}(p \cup (A \setminus a))$ instead of $\mathcal{C}(\{p\} \cup (A \setminus \{a\}))$.

Let X_i be a nonempty set and let \mathcal{C}_i be a convexity-structure for X_i ; $i = 1, 2$. Then $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1 \wedge B \in \mathcal{C}_2\}$ is a convexity-structure for the Cartesian-product $X_1 \times X_2$. The pair $(X_1 \times X_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ is called the *convex-product-space*, also called the *Eckhoff-space*. Note that the $\mathcal{C}_1 \oplus \mathcal{C}_2$ -hull of $E \subset X_1 \times X_2$ is given by $(\mathcal{C}_1 \oplus \mathcal{C}_2)(E) = \mathcal{C}_1(\pi_1 E) \times \mathcal{C}_2(\pi_2 E)$, where π_1 is the projection of $X_1 \times X_2$ on X_1 ; $i = 1, 2$. Also note that if $e_1, e_2, e_3 \in X_1 \times X_2$ with $e_1 \neq e_2$ and $\pi_i(e_1) = \pi_i(e_3)$ for $i = 1, 2$, then $e_3 \in (\mathcal{C}_1 \oplus \mathcal{C}_2)(e_1, e_2)$.

2. The Carathéodory-number and the exchange-number. A convexity-structure \mathcal{C} for X is said to possess the *Carathéodory-number* c if c is the smallest nonnegative integer such that $\mathcal{C}(S) = \bigcup \{\mathcal{C}(T) \mid T \subset S \wedge |T| \leq c\}$, for all $S \subset X$. The following lemma is an immediate consequence of this definition.

LEMMA 2.1. Let \mathcal{C} be a convexity-structure for X with Carathéodory-number c and let $f \in N$ ($N = 1, 2, 3, \dots$). Then the following holds:

- (i) $(\exists A)[A \subset X \wedge |A| = c \wedge \mathcal{C}(A) \not\subset \bigcup \{\mathcal{C}(A \setminus a) \mid a \in A\}]$;
- (ii) $(\exists A)[A \subset X \wedge |A| = f \wedge \mathcal{C}(A) \not\subset \bigcup \{\mathcal{C}(A \setminus a) \mid a \in A\}] \Rightarrow c \geq f$.

DEFINITION 2.1. The exchange-number of a convexity-structure \mathcal{C} for X is the smallest positive integer e , such that

$$(\forall A)(\forall p)[A \subset X \wedge p \in X \wedge e \leq |A| < \infty \implies \mathcal{C}(A) \subset \bigcup \{\mathcal{C}(p \cup (A \setminus a)) \mid a \in A\}.$$

Of course, if the exchange-number e of the convexity-structure \mathcal{C} for X exists then $e \geq 1$; if \mathcal{C} is a T_1 -convexity-structure (see [4]) then $e \geq 2$; if $A \subset X$, $|A| \geq e$ and $p \in \mathcal{C}(A)$ then $\mathcal{C}(A) = \bigcup \{\mathcal{C}(p \cup (A \setminus a)) \mid a \in A\}$, see [5], axiom C3; if the Carathéodory-number c of \mathcal{C} exists too, then $e \leq c + 1$, which follows directly from Lemma 2.1(ii).

EXAMPLE 2.1. Take $X = \mathbf{R}^n$ ($n \in N$) and $\mathcal{C} = \text{conv}$, the usual convexity-structure for the Euclidean-space \mathbf{R}^n . The classical theorem of Carathéodory implies that $c = n + 1$; see [2]. In [6] J. R. Reay proved that $e = n + 1$.

EXAMPLE 2.2. Let $\mathcal{C} = \{X\} \cup \{A \mid A \subset X \wedge |A| \leq f\}$, $f \in N$. Then $c = f + 1$ and $e = 2$.

EXAMPLE 2.3. Take $M \subset X$, $|M| = m$ ($m \in N$), and define $\mathcal{C} = \{X\} \cup \{A \mid A \subset X \wedge M \not\subset A\}$. Because $\mathcal{C}(A) = X$ if $M \subset A$, and $\mathcal{C}(A) = A$ if $M \not\subset A$, it follows that $c = m$ and $e = m + 1$.

EXAMPLE 2.4. The convexity-structure $\mathcal{C} = \{X\} \cup \{A \mid A \subset X \wedge |A| < \infty\}$, with $|X| = \infty$, has no Carathéodory-number, but the exchange-number is 2.

EXAMPLE 2.5. The convex-product-structure $\text{conv} \oplus \text{conv}$ for $\mathbf{R} \times \mathbf{R}$ has Carathéodory-number 2 and exchange-number 3.

THEOREM 2.1. Let \mathcal{C}_i be a convexity-structure for X_i , $X_i \neq \emptyset$; let c_i and e_i be the Carathéodory-number respectively the exchange-number of \mathcal{C}_i ; $i = 1, 2$. The Carathéodory-number c of $\mathcal{C}_1 \oplus \mathcal{C}_2$ exists and the following assertions hold:

- I. If $\min(c_1, c_2) = 1$ then
 - a. $c_1 + c_2 - 1 \leq c \leq c_1 + c_2$

- b. $(\exists i)[i \in \{1, 2\} \wedge e_i \leq c_i] \Rightarrow c = c_1 + c_2 - 1.$
- II. *If* $\min(c_1, c_2) \geq 2$ *then*
 - a. $c_1 + c_2 - 2 \leq c \leq c_1 + c_2$
 - b. $(\exists i)[i \in \{1, 2\} \wedge e_i \leq c_i] \Rightarrow c_1 + c_2 - 2 \leq c \leq c_1 + c_2 - 1$
 - c. $(\forall i)[i \in \{1, 2\} \wedge e_i \leq c_i] \Rightarrow c = c_2 + c_2 - 2.$

Proof. First we show that the Carathéodory-number c of $\mathcal{E}_1 \oplus \mathcal{E}_2$ exists and that $c \leq c_1 + c_2$. Let $\emptyset \neq E \subset X_1 \times X_2$ and $(a_1, a_2) \in (\mathcal{E}_1 \oplus \mathcal{E}_2)(E) = \mathcal{E}_1(\pi_1 E) \times \mathcal{E}_2(\pi_2 E)$. Hence there exists a set $S_i \subset \pi_i E$, $|S_i| \leq c_i$ such that $a_i \in \mathcal{E}_i(S_i)$; $i = 1, 2$. $S_i \subset \pi_i E$ implies, there exists a set $F_i \subset E$ such that $\pi_i F_i = S_i$ and $|F_i| = |S_i|$; $i = 1, 2$. So $(a_1, a_2) \in \mathcal{E}_1(S_1) \times \mathcal{E}_2(S_2) = \mathcal{E}_1(\pi_1 F_1) \times \mathcal{E}_2(\pi_2 F_2) \subset \mathcal{E}_1(\pi_1(F_1 \cup F_2)) \times \mathcal{E}_2(\pi_2(F_1 \cup F_2)) = (\mathcal{E}_1 \oplus \mathcal{E}_2)(F_1 \cup F_2)$. Obviously $|F_1 \cup F_2| \leq c_1 + c_2$. Because $(\mathcal{E}_1 \oplus \mathcal{E}_2)(F) \subset (\mathcal{E}_1 \oplus \mathcal{E}_2)(E)$ for each $F \subset E$, we have $(\mathcal{E}_1 \oplus \mathcal{E}_2)(E) = \bigcup \{(\mathcal{E}_1 \oplus \mathcal{E}_2)(F) \mid F \subset E \wedge |F| \leq c_1 + c_2\}$; hence c exists and $c \leq c_1 + c_2$.

To determine the lower bound for c we choose, according to Lemma 2.1.(i), a set $A_i \subset X_i$ such that $|A_i| = c_i$ and $\mathcal{E}_i(A_i) \not\subset \bigcup \{\mathcal{E}_i(A_i \setminus a) \mid a \in A_i\}$; $i = 1, 2$. Take $b_1 \in A_1$ and $b_2 \in A_2$, and consider the set $G = (A_1 \times \{b_2\}) \cup (\{b_1\} \times A_2) \subset X_1 \times X_2$. Obviously $|G| = c_1 + c_2 - 1$. There are two cases (take $c_1 \leq c_2$):

1. Let $c_1 = 1$. Then $A_1 = \{b_1\}$ and $G = \{b_1\} \times A_2$. So we have $(\mathcal{E}_1 \oplus \mathcal{E}_2)(G) = \mathcal{E}_1(b_1) \times \mathcal{E}_2(A_2) \not\subset \mathcal{E}_1(b_1) \times \bigcup \{\mathcal{E}_2(A_2 \setminus b) \mid b \in A_2\} = \bigcup \{(\mathcal{E}_1 \oplus \mathcal{E}_2)(G \setminus e) \mid e \in G\}$, and it follows from Lemma 2.1(ii) that $c \geq c_1 + c_2 - 1$.

2. Let $c_1 \geq 2$. Then also $c_2 \geq 2$. So there exists an element $d_i \in A_i$ with $d_i \neq b_i$; $i = 1, 2$. Note that $\pi_1((b_1, b_2)) = \pi_1((b_1, d_2))$ and $\pi_2((b_1, b_2)) = \pi_2((d_1, b_2))$. The last remark of §1 gives us that $(b_1, b_2) \in (\mathcal{E}_1 \oplus \mathcal{E}_2)((d_1, b_2), (b_1, d_2)) \subset (\mathcal{E}_1 \oplus \mathcal{E}_2)(G \setminus (b_1, b_2))$. Define $F = G \setminus \{(b_1, b_2)\}$. Clearly $|F| = c_1 + c_2 - 2$ and $\pi_i F = A_i$; $i = 1, 2$. Moreover $(\mathcal{E}_1 \oplus \mathcal{E}_2)(F) \not\subset \bigcup \{(\mathcal{E}_1 \oplus \mathcal{E}_2)(F \setminus e) \mid e \in F\}$. From Lemma 2.1(ii) it follows that $c \geq c_1 + c_2 - 2$.

We now prove I.b and II.b. Let us assume that $e_i \leq c_i$. Take $c_1, c_2 \geq 1$, $\emptyset \neq E \subset X_1 \times X_2$ and $(a_1, a_2) \in (\mathcal{E}_1 \oplus \mathcal{E}_2)(E)$. We show that there exists a set F such that

$$(a_1, a_2) \in (\mathcal{E}_1 \oplus \mathcal{E}_2)(F), F \subset E \text{ and } |F| \leq c_1 + c_2 - 1.$$

The first part of the proof of this theorem implies that there exists a set F_i such that $a_i \in \mathcal{E}_i(\pi_i F_i)$, $F_i \subset E$ and $|\pi_i F_i| = |F_i| \leq c_i$; $i = 1, 2$. We may assume that $|F_i| = c_i \forall i \in \{1, 2\}$, because if $|F_i| \leq c_i - 1$ for some $i \in \{1, 2\}$ then $|F_1 \cup F_2| \leq c_1 + c_2 - 1$, and we are done. If $F_1 \cap F_2 \neq \emptyset$, then define $F = F_1 \cup F_2$. So $|F| \leq c_1 + c_2 - 1$ and $(a_1, a_2) \in (\mathcal{E}_1 \oplus \mathcal{E}_2)(F)$. If $F_1 \cap F_2 = \emptyset$, we distinguish two cases:

1. $(\exists i)[i \in \{1, 2\} \wedge \pi_i F_1 \cap \pi_i F_2 \neq \emptyset]$. Take $i = 1$. Hence $\pi_1 F_1 \cap \pi_1 F_2 \neq \emptyset$. Now there exist elements $e_1 \in F_1$ and $e_2 \in F_2$ such that $\pi_1(e_1) = \pi_1(e_2)$. Note that $e_1 \neq e_2$. Define $F = (F_1 \setminus \{e_1\}) \cup F_2$. Clearly $|F| \leq c_1 + c_2 - 1$ and $\pi_i F_i \subset \pi_i F$; $i = 1, 2$. So $(a_1, a_2) \in \mathcal{C}_1(\pi_1 F_1) \times \mathcal{C}_2(\pi_2 F_2) \subset \mathcal{C}_1(\pi_1 F) \times \mathcal{C}_2(\pi_2 F) = (\mathcal{C}_1 \oplus \mathcal{C}_2)(F)$.

2. $(\forall i)[i \in \{1, 2\} \Rightarrow \pi_i F_1 \cap \pi_i F_2 = \emptyset]$. Take $e \in F_2$. Then $\pi_1(e) \notin \pi_1 F_1$. Because $e_1 \leq c_1$, there exists an element $e_1 \in F_1$ such that $a_1 \in \mathcal{C}_1(\pi_1(e) \cup \pi_1 F_1 \setminus \pi_1(e_1)) \subset \mathcal{C}_1(\pi_1(e \cup F_1 \setminus e_1))$. Define $F = (F_1 \setminus \{e_1\}) \cup F_2$. Obviously $|F| \leq c_1 + c_2 - 1$, $a_1 \in \mathcal{C}_1(\pi_1 F)$ and $a_2 \in \mathcal{C}_2(\pi_2 F_2) \subset \mathcal{C}_2(\pi_2 F)$. Hence $(a_1, a_2) \in (\mathcal{C}_1 \oplus \mathcal{C}_2)(F)$.

Finally we prove II.c. Take again $\emptyset \neq E \subset X_1 \times X_2$ and $(a_1, a_2) \in (\mathcal{C}_1 \oplus \mathcal{C}_2)(E)$. We shall show that there exists a set F such that

$$(a_1, a_2) \in (\mathcal{C}_1 \oplus \mathcal{C}_2)(F), \quad F \subset E, \quad \text{and} \quad |F| \leq c_1 + c_2 - 2.$$

In the proof of II.b we found a set $G_i \subset E$ such that $|\pi_i G_i| = |G_i| \leq c_i$, $a_i \in \mathcal{C}(\pi_i G_i)$ and $|G_1 \cup G_2| \leq c_1 + c_2 - 1$; $i = 1, 2$. As in the proof of II.b we may assume that $|G_i| = c_i$; $i = 1, 2$. If $|G_1 \cup G_2| > 1$ then define $F = G_1 \cup G_2$, hence $a_i \in \mathcal{C}_i(\pi_i F)$ and $|F| \leq c_1 + c_2 - 2$, so we are done. The case that $|G_1 \cap G_2| = 1$ still remains. Assume $G_1 \cap G_2 = \{e\}$, and $|G_i| \geq 2$ for $i = 1, 2$. Throughout the remainder of the proof we take $i, j \in \{1, 2\}$ with $i + j = 3$. Let $e_i \in G_j$, $e_i \neq e$. There are two cases:

1. If $\pi_i(e_i) \notin \pi_i G_i$ then, because $e_i \leq c_i$, there exists an element $u_i \in G_i$ such that $u_i \neq e_i$ and $a_i \in \mathcal{C}_i(\pi_i(e_i) \cup (\pi_i G_i \setminus \pi_i(u_i))) \subset \mathcal{C}_i(\pi_i(e_i \cup (G_i \setminus u_i)))$.

2. If $\pi_i(e_i) \in \pi_i G_i$ then there exists an element $v_i \in G_i$ such that $v_i \neq e_i$ and $\pi_i(e_i) = \pi_i(v_i)$; hence $\pi_i G_i = \pi_i(e_i \cup (G_i \setminus v_i))$, so $a_i \in \mathcal{C}_i(\pi_i G_i) = \mathcal{C}_i(\pi_i(e_i \cup (G_i \setminus v_i)))$.

We may conclude that in both cases there exists an element $d_i \in G_i$ such that $d_i \neq e_i$ and $a_i \in \mathcal{C}_i(\pi_i(e_i \cup (G_i \setminus d_i)))$.

If $d_1 = e = d_2$ then define $F = G_1 \cup G_2 \setminus \{e\}$. Hence $|F| \leq c_1 + c_2 - 2$ and, because $e_i \in G_j$, $e_i \neq e$ we have $(a_1, a_2) \in \mathcal{C}_1(\pi_1(e_1 \cup (G_1 \setminus e))) \times \mathcal{C}_2(\pi_2(e_2 \cup (G_2 \setminus e))) \subset (\mathcal{C}_1 \oplus \mathcal{C}_2)(G_1 \cup G_2 \setminus e) = (\mathcal{C}_1 \oplus \mathcal{C}_2)(F)$. If $d_i \neq e$ for some $i \in \{1, 2\}$ then, taking e.g. $i = 1$, we define $F = G_1 \cup G_2 \setminus \{d_1\}$. Clearly $|F| \leq c_1 + c_2 - 2$ and because $e_1 \in G_2$ we have $(a_1, a_2) \in \mathcal{C}_1(\pi_1(e_1 \cup (G_1 \setminus d_1))) \times \mathcal{C}_2(\pi_2 G_2) \subset (\mathcal{C}_1 \oplus \mathcal{C}_2)(G_1 \cup G_2 \setminus d_1) = (\mathcal{C}_1 \oplus \mathcal{C}_2)(F)$.

It follows that $c \leq c_1 + c_2 - 2$ and because $c \geq c_1 + c_2 - 2$, as we have seen already, we may conclude that $c = c_1 + c_2 - 2$. This completes the proof of Theorem 2.1.

EXAMPLE 2.6. Take $X_1 = \mathbf{R}^m$ and $X_2 = \mathbf{R}^n$ ($m, n \in \mathbf{N}$), and $\mathcal{C}_1 = \text{conv} = \mathcal{C}_2$ (see Example 2.1). Because $c_1 = e_1 = m + 1$ and $c_2 = e_2 = n + 1$ it follows from Theorem 2.1. (II.c) that the Carathéodory-number of the convex-product-structure $\text{conv} \oplus \text{conv}$ for \mathbf{R}^{m+n} is $c = c_1 + c_2 - 2 = m + n$.

EXAMPLE 2.7. Take $\mathcal{C}_i = \{X_i\} \cup \{A \mid A \subset X_i \wedge |A| \leq k_i\}$, $k_i \geq 1$, then because $e_i = 2 \leq k_i + 1 = c_i$, $\forall i \in \{1, 2\}$, it follows from Theorem 2.1(II.c) that $c = c_1 + c_2 - 2 = k_1 + k_2$.

EXAMPLE 2.8. Take $M_i \subset X_i$, $|M_i| = m_i$ and define $\mathcal{C}_i = \{X_i\} \cup \{A \mid A \subset X_i \wedge M_i \not\subset A\}$; $i = 1, 2$. Because $e_i = m_i + 1 > m_i = c_i$ it follows from Theorem 2.1(I.II.a) that $c \leq c_1 + c_2 = m_1 + m_2$. We shall show now that even $c = m_1 + m_2$. Consider a set $E \subset X_1 \times X_2$ such that $E = E_1 \cup E_2$, with $|E_i| = m_i$, $|\pi_i E_i| = 1$, $\pi_i E_j = M_i$ and $\pi_i E_i \cap \pi_i E_2 = \emptyset$; $i, j \in \{1, 2\}$ and $i + j = 3$. It is easy to see that $(\mathcal{C}_1 \oplus \mathcal{C}_2)(E) = X_1 \times X_2$. However, in general, $\bigcup \{(\mathcal{C}_1 \oplus \mathcal{C}_2)(E \setminus e) \mid e \in E\} = (\pi_1 E \times X_2) \cup (X_1 \times \pi_2 E) \neq X_1 \times X_2$. Hence $c = m_1 + m_2$.

EXAMPLE 2.9. Take $M \subset X_1$, $|M| = m$ and $\mathcal{C}_1 = \{X_1\} \cup \{A \mid A \subset X_1 \wedge M \not\subset A\}$. Take $X_2 = \mathbf{R}^n$ and $\mathcal{C}_2 = \text{conv}$. We know that $e_1 = m + 1 > m = c_1$, and $e_2 = n + 1 = c_2$. From Theorem 2.1(II.b) it follows that $c \leq c_1 + c_2 - 1 = m + n$. As in the previous example we can show that $c = m + n$. In order to prove this we have to look for a set $E \subset X_1 \times X_2$, $|E| = m + n$, such that the convex hull of E is not the union of the convex hulls of proper subsets of E . Take $E = E_1 \cup E_2$ with $|E_1| = m$, $|E_2| = n$, $|\pi_i E_j| = 1$, $\pi_i E_1 \cap \pi_i E_2 = \emptyset$, $\pi_1 E_1 = M$ and $\mathcal{C}_2(\pi_2 E) \neq \bigcup \{\mathcal{C}_2(\pi_2 E \setminus a) \mid a \in \pi_2 E\}$; $i, j \in \{1, 2\}$, $i + j = 3$. Note that $|\pi_2 E| = n + 1$. Now we have $(\mathcal{C}_1 \oplus \mathcal{C}_2)(E) = X_1 \times \mathcal{C}_2(\pi_2 E)$. But, in general, $\bigcup \{(\mathcal{C}_1 \oplus \mathcal{C}_2)(E \setminus e) \mid e \in E\} \neq X_1 \times \mathcal{C}_2(\pi_2 E)$. Hence $c = m + n$.

The main result of this section, Theorem 2.1, is a generalization of J. R. Reay's Theorem 1 (first part) together with Example (1) on pg. 229 of [7]. In fact, Reay proves that the Carathéodory-number c of the convex-product-structure, whose component spaces are Euclidian-spaces with dimensions m_1 and n_2 and with the usual convexity-structure, satisfies the condition $c = m_1 + n_2$. (See Ex. 2.6.)

3. The Helly-number. Let $\mathcal{F} \subset 2^X$, $|F| \geq k$, $k \in \mathbf{N}$, for some set X . Define $\bigcap_{(k)} F = \{\bigcap A \mid A \in \mathcal{F}^k\}$. I.e. $\emptyset \notin \bigcap_{(k)} \mathcal{F}$ implies that each intersection of k elements of \mathcal{F} is not empty.

DEFINITION 3.1. A convexity-structure \mathcal{C} for X has the *Helly-number* h if h is the smallest integer such that $[\mathcal{F} \subset \mathcal{C} \wedge$

$$|\mathcal{F}| < \infty \wedge \emptyset \notin \bigcap_{(t)} F \Rightarrow \bigcap \mathcal{F} \neq \emptyset.$$

Note that if $X = \emptyset$ then \mathfrak{h} does not exist, and if \mathfrak{h} exists then $\mathfrak{h} \geq 1$. The following characterization of the Helly-number gives rise to another definition of it; see also [1] and [4].

LEMMA 3.1. *Let \mathcal{C} be a convexity-structure for X with Helly-number \mathfrak{h} . Then the following assertions are equivalent:*

- (i) $\mathfrak{h} \leq \mathfrak{f}$;
- (ii) $[\mathcal{F} \subset \mathcal{C} \wedge |\mathcal{F}| = \mathfrak{f} + 1 \wedge \emptyset \notin \bigcap_{(t)} \mathcal{F}] \Rightarrow \bigcap \mathcal{F} \neq \emptyset$;
- (iii) $[\mathcal{A} \subset X \wedge |A| = \mathfrak{f} + 1] \Rightarrow \bigcap \{\mathcal{C}(A \setminus a) \mid a \in A\} \neq \emptyset$;
- (iv) $[A \subset X \wedge \mathfrak{f} + 1 \leq |A| < \infty] \Rightarrow \bigcap \{\mathcal{C}(A \setminus a) \mid a \in A\} \neq \emptyset$.

Proof. We shall go through the following implication-cycle:

(i) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).

(i) \Rightarrow (iv): Take $A \subset X$, $|A| = \mathfrak{f} + n$ ($n \in \mathbb{N}$) and define $\mathcal{F} = \{\mathcal{C}(A \setminus a) \mid a \in A\}$. Note that $|\mathcal{F}| \leq \mathfrak{f} + n$. If there exist elements $a, b \in A$, such that $a \neq b$ and $\mathcal{C}(A \setminus a) = \mathcal{C}(A \setminus b)$ then of course $\bigcap \{\mathcal{C}(A \setminus a) \mid a \in A\} \neq \emptyset$, and we are done. So we may assume that $|\mathcal{F}| = \mathfrak{f} + n$. If $\mathcal{F}' \subset \mathcal{F}$, with $|\mathcal{F}'| = \mathfrak{f}$, then $\mathcal{F}' \neq \mathcal{F}$ and there exists an element $a_1 \in A$ such that $\mathcal{C}(A \setminus a_1) \notin \mathcal{F}'$. From $a_1 \in A \setminus \{a\} \subset \mathcal{C}(A \setminus a)$ for each $a \in A$, $a \neq a_1$, it follows that $a_1 \in \bigcap \mathcal{F}'$, and hence $\emptyset \notin \bigcap_{(t)} \mathcal{F}$. Because $\mathfrak{h} \leq \mathfrak{f}$ we have $[\emptyset \notin \bigcap_{(t)} \mathcal{F} \Rightarrow \emptyset \notin \bigcap_{(t)} \mathcal{F}]$. From Definition 3.1 it follows that $\bigcap \{\mathcal{C}(A \setminus a) \mid a \in A\} = \bigcap \mathcal{F} \neq \emptyset$.

(iv) \Rightarrow (iii) is trivial and (ii) \Rightarrow (i) follows by induction. It remains to be shown that (iii) \Rightarrow (ii): Take $\mathcal{F} \subset \mathcal{C}$, $|F| = \mathfrak{f} + 1$ and $\emptyset \notin \bigcap_{(t)} \mathcal{F}$. Let $\mathcal{F} = \{F_i \mid i = 1, 2, \dots, \mathfrak{f} + 1\}$. Then $\emptyset \notin \bigcap_{(t)} \mathcal{F} \Rightarrow \bigcap \{\mathcal{F} \setminus \{F_i\}\} \neq \emptyset$, $\forall i = 1, 2, \dots, \mathfrak{f} + 1$. For each $i = 1, 2, \dots, \mathfrak{f} + 1$ we choose an element $a_i \in \bigcap \{\mathcal{F} \setminus \{F_i\}\}$. Define $A = \{a_i \mid i = 1, 2, \dots, \mathfrak{f} + 1\}$. Hence $|A| \leq \mathfrak{f} + 1$. If $|A| < \mathfrak{f} + 1$, then there exists an index i such that $a_i \in F_i$. But then $a_i \in \bigcap \mathcal{F}$. Hence $\bigcap \mathcal{F} \neq \emptyset$, and we are done. So we may assume that $|A| = \mathfrak{f} + 1$. From (iii) it follows that $\bigcap \{\mathcal{C}(A \setminus a_i) \mid i = 1, 2, \dots, \mathfrak{f} + 1\} \neq \emptyset$. For each $i = 1, 2, \dots, \mathfrak{f} + 1$ we have $A \setminus \{a_i\} \subset F_i$, so $\mathcal{C}(A \setminus a_i) \subset F_i$. Hence $\bigcap \{\mathcal{C}(A \setminus a_i) \mid i = 1, 2, \dots, \mathfrak{f} + 1\} \subset \bigcap \mathcal{F}$, and we conclude that $\bigcap \mathcal{F} \neq \emptyset$.

DEFINITION 3.1'. A convexity-structure \mathcal{C} for X has the *Helly-number* \mathfrak{h} if \mathfrak{h} is the smallest integer such that $[A \subset X \wedge |A| = \mathfrak{h} + 1] \Rightarrow \bigcap \{\mathcal{C}(A \setminus a) \mid a \in A\} \neq \emptyset$.

With the aid of Lemma 3.1. it is easy to verify that Definitions 3.1 and 3.1' are equivalent. We now prove the classical theorem of Levi with the aid of Definition 3.1'. See [5], Theorem H.

THEOREM 3.1 (Levi). *Let \mathcal{C} be a convexity-structure for X . Then the existence of a Radon-number r implies the existence of a Helly-number h , such that $h \leq r - 1$.*

Proof. Each $A \subset X$ with $|A| = r$ has a \mathcal{C} -Radon-partition; see [2]. So there exists a set $B \subset A$, with $\emptyset \neq B \neq A$ and $\mathcal{C}(B) \cap \mathcal{C}(A \setminus B) \neq \emptyset$. Because $\mathcal{C}(B) \subset \bigcap \{\mathcal{C}(A \setminus a) \mid a \in A \setminus B\}$ and $\mathcal{C}(A \setminus B) \subset \bigcap \{\mathcal{C}(A \setminus a) \mid a \in B\}$, we have $\bigcap \{\mathcal{C}(A \setminus a) \mid a \in A\} \supset \mathcal{C}(B) \cap \mathcal{C}(A \setminus B) \neq \emptyset$. Hence h exists and $h \leq r - 1$.

THEOREM 3.2. *Let \mathcal{C}_i be a convexity-structure for X_i , $X_i \neq \emptyset$, with Helly-number h_i ; $i = 1, 2$. Then the Helly-number h of the convex-product-structure $\mathcal{C}_1 \oplus \mathcal{C}_2$ for $X_1 \times X_2$ exists and $h = \max(h_1, h_2)$.*

Proof. We may assume that $h_1 \geq h_2$. Take $E \subset X_1 \times X_2$ with $|E| = h_1 + 1$. There are two possibilities for the projection $\pi_i E$ of E on X_i ; $i = 1, 2$.

—If $|\pi_i E| \leq h_i$ for some $i \in \{1, 2\}$, then there exist elements $e_1, e_2 \in E$ such that $e_1 \neq e_2$ and such that $\pi_i(e_1) = \pi_i(e_2)$. Clearly $\pi_i(e_1) \in \mathcal{C}_i(\pi_i(E \setminus e_1))$. Hence $\pi_i(e_1) \in \bigcap \{\mathcal{C}_i(\pi_i(E \setminus e)) \mid e \in E\} \neq \emptyset$.

—If $|\pi_i E| = h_i + 1$, for some $i \in \{1, 2\}$, then it follows from Definition 3.1' that $\bigcap \{\mathcal{C}_i(\pi_i E \setminus x) \mid x \in \pi_i E\} \neq \emptyset$. Hence $\bigcap \{\mathcal{C}_i(\pi_i(E \setminus e)) \mid e \in E\} \supset \bigcap \{\mathcal{C}_i(\pi_i E \setminus \pi_i(e)) \mid e \in E\} \neq \emptyset$ and we may conclude that $\bigcap \{\mathcal{C}_i(\pi_i(E \setminus e)) \mid e \in E\} \neq \emptyset$; $i = 1, 2$.

Hence $\bigcap \{(\mathcal{C}_1 \oplus \mathcal{C}_2)(E \setminus e)\} = \bigcap \{\mathcal{C}_1(\pi_1(E \setminus e)) \times \mathcal{C}_2(\pi_2(E \setminus e))\} \neq \emptyset$, and so $h \leq h_1 = \max(h_1, h_2)$.

Next we show that $h \geq h_1$. Assuming $h = h_1 - 1$, $A \subset X_1$ with $|A| = h_1$ and $b \in X_2$ we have, because $|A \times \{b\}| = h_1$, $\bigcap \{(\mathcal{C}_1 \oplus \mathcal{C}_2)((A \times \{b\}) \setminus (a, b)) \mid a \in A\} \neq \emptyset$. Hence $\bigcap \{\mathcal{C}_1(A \setminus a) \mid a \in A\} = \bigcap \{\mathcal{C}_1(\pi_1((A \times \{b\}) \setminus (a, b))) \mid a \in A\} \neq \emptyset$. This contradicts the fact that h_1 is the Helly-number of \mathcal{C}_1 , so that, indeed, $h \geq h_1 = \max(h_1, h_2)$ and the final conclusion is that $h = h_1 = \max(h_1, h_2)$.

It is well known that the Carathéodory-number c , the Helly-number h and the Radon-number r of the usual convexity-structure conv for \mathbf{R}^n satisfy the equalities $c = h = r - 1$ ($= n + 1$), that is, in Levi's theorem equality holds. There are however convexity-structures \mathcal{C} where the equality does not hold. This is even the case when \mathcal{C} is a convex-product-structure:

We know that (see also [8]):

$$\begin{aligned} c_1 + c_2 - 2 &\leq c \leq c_1 + c_2 \\ h &= \max(h_1, h_2) \\ \max(r_1, r_2) &\leq r \leq r_1 + r_2 - 2; \end{aligned}$$

c_i , h_i and r_i are resp. the Carathéodory-, Helly- and Radon-number of \mathcal{E}_i ; $i = 1, 2$.

If $c_i = h_i = r_i - 1$, $i = 1, 2$, then we have:

a. if $c_i \geq 3$ (e.g. when \mathcal{E}_i is T_1) then $c \neq h$.

b. if $c = c_1 + c_2$, then $c > r - 1$.

if the exchange-number, $e_i \leq c_i$ and if $r = r_1 + r_2 - 2$ (e.g. in the case $X_i = \mathbf{R}^{i-1}$, $\mathcal{E}_i = \text{conv}$) then $c \leq r - 1$.

c. if $r > \max(r_1, r_2)$ (e.g. in the same case as in b) then $h < r - 1$.

The results in this paper can be extended to convex-product-structures which are products of finitely many convexity-structures.

In a next paper we shall pay more attention to the properties of the exchange number. For example we shall show that under certain conditions, the exchange-number of a convex-product-structure exists and how it can be derived.

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