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**AN OPERATOR VERSION OF A THEOREM OF
KOLMOGOROV**

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AN OPERATOR VERSION OF A THEOREM OF KOLMOGOROV¹

G. D. ALLEN, F. J. NARCOWICH AND J. P. WILLIAMS

Let \mathcal{G} be a (separable) Hausdorff space and let K be a continuous nonnegative-definite kernel (covariance) from $\mathcal{G} \times \mathcal{G}$ to C . The well known theorem of Kolmogorov states that in the case \mathcal{G} is the set of integers there is a continuous mapping (stochastic process) $x(\cdot)$ from \mathcal{G} into a (separable) Hilbert space \mathcal{H} such that $K(s, t) = \langle x(s), x(t) \rangle$. The theorem is also known for any separable Hausdorff space. The purpose of this paper is to replace the complex numbers C by the algebra $B(\mathcal{H}, \mathcal{H})$ of bounded linear operators from a Hilbert space into itself. The factorization is then $K(t, s) = X(t)^*X(s)$ with X a continuous map from \mathcal{G} to $B(\mathcal{H}, \mathcal{H})$ for a suitable Hilbert space \mathcal{H} . If \mathcal{G} is separable we may take $\mathcal{H} = \mathcal{H}$.

Two proofs of this theorem are given. The first, for \mathcal{G} separable and \mathcal{H} of arbitrary dimension, uses an extension of the technique of [1] to obtain a triangular factorization for nonnegative-definite matrices with operator entries to construct the desired stochastic process $X(\cdot)$. The second, for \mathcal{G} arbitrary and \mathcal{H} of infinite dimension uses the techniques of reproducing kernel Hilbert spaces, and is a bit simpler.

Main results. Let \mathcal{H} be a complex Hilbert space and let $B(\mathcal{H}, \mathcal{H})$ be the bounded linear operators on. Let \mathcal{G} be a Hausdorff space and let $K: \mathcal{G} \times \mathcal{G} \rightarrow B(\mathcal{H}, \mathcal{H})$ be a (jointly) continuous function. We say that K is *nonnegative-definite* if for every $t_1, \dots, t_n \in \mathcal{G}$ and $x_1, \dots, x_n \in \mathcal{H}$ the sum

$$(1) \quad \sum_{i,j=1}^n (K(t_i, t_j)x_j, x_i) \geq 0.$$

The generalization of the Kolmogorov theorem we wish to prove is contained in

THEOREM 1. *Let \mathcal{G} be a separable Hausdorff space. If $K(\cdot, \cdot)$ is a continuous nonnegative-definite function from $\mathcal{G} \times \mathcal{G}$ into $B(\mathcal{H}, \mathcal{H})$ then there exists a separable Hilbert space \mathcal{H} and a continuous function $X(t)$ from \mathcal{G} into $B(\mathcal{H}, \mathcal{H})$ such that*

$$X^*(t)X(s) = K(t, s).$$

¹ This generalization was suggested to the authors by Professor P. Masani in January 1975.

In order to prove this theorem we require a number of facts about operator-valued matrices and about the solution of operator equations. The first result, is due to Douglas [2]. (See also Fillmore-Williams [4].) We will denote the *range* of the operator A by $\mathcal{R}(A)$, and the kernel of A by $\mathcal{N}(A)$.

LEMMA 1. *Let A and B be bounded operators on \mathcal{H} . Then the following conditions are equivalent:*

- (i) $\mathcal{R}(A) \subset \mathcal{R}(B)$,
- (ii) $A = BC$, for some bounded operator C on \mathcal{H} ,
- (iii) $AA^* \leq \lambda^2 BB^*$, for some $\lambda > 0$.

Moreover, the operator C can be chosen so that $\mathcal{N}(C^*) \supset \mathcal{N}(B)$ and $\mathcal{R}(C) \subset \overline{\mathcal{R}(B)}$.

COROLLARY. *If B is bounded and nonnegative then $\mathcal{R}(\sqrt{B}) \supset \mathcal{R}(B)$.*

If we restrict K , of Theorem 1, to a finite subset of \mathcal{S} the kernel K becomes a $n \times n$ matrix whose (i, j) entry is $K_{ij} = K(t_i, t_j)$, $1 \leq i, j \leq n$. This matrix is nonnegative-definite in the sense that for every $x_1, x_2, \dots, x_n \in \mathcal{H}$,

$$(2) \quad \sum_{i,j=1}^n (K_{ij}x_j, x_i) \geq 0.$$

Denote by \mathcal{H}_n the space which is a direct sum of n copies of \mathcal{H} , $\mathcal{H}_n = \mathcal{H} \oplus \dots \oplus \mathcal{H}$, with the natural inner product. Suppose that K is an operator on \mathcal{H}_n ; that is, K is an $n \times n$ operator-valued matrix. Then (2) means that $(Kx, x) \geq 0$ for every $x = (x_1, \dots, x_n) \in \mathcal{H}_n$, that is K is a nonnegative operator on \mathcal{H}_n . Note that if K is nonnegative-definite, $K_{ij} = K_{ji}^*$, for all $1 \leq i, j \leq n$. If K is an $n \times n$ operator-valued matrix and $m \leq n$, we write K_m for the upper left $m \times m$ submatrix of K .

LEMMA 2. *Let K be an $n \times n$ nonnegative definite, bounded operator-valued matrix. Then there is a positive constant λ so that*

$$(3) \quad K_{ii} \geq \lambda K_{ij} K_{ij}^*, \quad 1 \leq i < j \leq n.$$

Proof. Let $V_i: \mathcal{H} \rightarrow \mathcal{H}_n$ where $V_i h = (0, \dots, h, 0 \dots 0)$, h being in the i th position. If $h \in \mathcal{H}$, then

$$(4) \quad \begin{aligned} |K_{ij}^* h|^2 &= |K_{ji} h|^2 = |V_j^* K V_i h|^2 \leq |K V_i h|^2 \\ &= (V_i^* K^2 V_i h, h) \leq |K| (V_i^* K V_i h, h) = |K| (K_{ii} h, h). \end{aligned}$$

Thus $K_{ij} K_{ij}^* \leq |K| K_{ii}$.

We must show that

$$(10) \quad T_{n-1, n-1} T_{n-1, n} = K_{n-1, n} - \sum_{i=1}^{n-2} T_{i, n-1}^* T_{i, n}$$

has a bounded solution for $T_{n-1, n}$. By the Remark we have for any $z_{n-1} \in \mathcal{H}$ a vector $z_{n-2} \in \mathcal{H}$ such that

$$T_{n-2, n-2} z_{n-2} + T_{n-2, n-1} z_{n-1} = 0.$$

Thus, proceeding sequentially we can solve the equations

$$(11) \quad \sum_{j=i}^{n-1} T_{ij} z_j = 0, \quad i = n-2, n-3, \dots, 1$$

for $z_{n-2}, z_{n-3}, z_{n-4}, \dots, z_1$, given z_{n-1} . Now, if $z = (z_1, \dots, z_n)$, an application of (11) gives

$$(12) \quad \begin{aligned} (Kz, z) &= \sum_{j=1}^{n-2} (K_{nj} z_j, z_n) + \sum_{j=1}^{n-2} (K_{jn} z_n, z_j) + (K_{n-1, n} z_n, z_{n-1}) \\ &+ (K_{n, n-1} z_{n-1}, z_n) + (K_{nn} z_n, z_n) + (T_{n-1, n-1}^2 z_{n-1}, z_{n-1}). \end{aligned}$$

By (9),

$$(13) \quad \begin{aligned} \sum_{j=1}^{n-2} (K_{jn} z_n, z_j) &= \sum_{j=1}^{n-2} ((T_{jj} T_{jn} + \sum_{i=1}^{j-1} T_{ij}^* T_{in}) z_n, z_j) \\ &= \sum_{j=1}^{n-2} (T_{jn} z_n, T_{jj} z_j) + \sum_{j=1}^{n-2} \sum_{i=1}^{j-1} (T_{ij}^* T_{in} z_n, z_j). \end{aligned}$$

We interpret all sums over not well defined limits to be zero (e.g. $\sum_{i=1}^0 (\cdot) = 0$). From (11) we have

$$T_{jj} z_j = - \sum_{i=j+1}^{n-1} T_{ji} z_i.$$

Substitution into (13) gives

$$(14) \quad \begin{aligned} \sum_{j=1}^{n-2} (K_{jn} z_n, z_j) &= - \sum_{j=1}^{n-2} \sum_{i=j+1}^{n-1} (T_{jn} z_n, T_{ji} z_i) + \sum_{j=1}^{n-2} \sum_{i=1}^{j-1} (T_{in} z_n, T_{ij} z_j) \\ &= - \sum_{j=1}^{n-3} \sum_{i=j+1}^{n-2} (T_{jn} z_n, T_{ji} z_i) + \sum_{j=1}^{n-2} \sum_{i=1}^{j-1} (T_{in} z_n, T_{ij} z_j) \\ &\quad - \sum_{j=1}^{n-2} (T_{jn} z_n, T_{j, n-1} z_{n-1}). \end{aligned}$$

The last term of (14) is

$$- \sum_{j=1}^{n-2} (T_{j, n-1}^* T_{jn} z_n, z_{n-1}).$$

Interchanging limits in the second term on the right hand side of (14), the equation (14) becomes

we will employ Lemma 1 (iii). (T_{13} is obtained in the same way as T_{12} .) According to the Remark above we take $T_{11}z_1 = -T_{12}z_2$, for some $z_2 \in \mathcal{H}$. If $z = (z_1, z_2, z_3)$, then

$$\begin{aligned} 0 \leq (K_3z, z) &= (T_{11}^2z_1, z_1) + (T_{11}T_{12}z_2, z_1) + (T_{12}^*T_{11}z_1, z_2) \\ &+ ((T_{22}^2 + T_{12}^*T_{12})z_2, z_2) + (T_{11}T_{13}z_3, z_1) \\ &+ (K_{23}z_3, z_2) + (K_{32}z_2, z_3) + (K_{33}z_3, z_3), \end{aligned}$$

which, since $T_{11}z_1 = -T_{12}z_2$ equals

$$(T_{22}^2z_2, z_2) + ((K_{23} - T_{12}^*T_{13})z_3, z_2) + ((K_{32} - T_{13}^*T_{12})z_2, z_3) + (K_{33}z_3, z_3).$$

In matrix form this means, for every $z_2, z_3 \in \mathcal{H}$,

$$\left(\begin{pmatrix} T_{22}^2 & K_{23} - T_{12}^*T_{13} \\ (K_{23} - T_{12}^*T_{13})^* & K_{33} \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}, \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \right) \geq 0.$$

By Lemma 2, then, there is a positive λ such that

$$T_{22}^2 \geq \lambda(K_{23} - T_{12}^*T_{13})(K_{23} - T_{12}^*T_{13})^*,$$

and hence by the Corollary and Lemma 2 (ii) T_{23} exists and is a bounded operator. Moreover, by Lemma 1 $\mathcal{R}(T_{23}) \subset \overline{\mathcal{R}(T_{22})}$. (This last fact together with the Remark is interpreted to mean that for any $y \in \mathcal{H}$ there is an $x \in \mathcal{H}$ so that $T_{22}x + T_{23}y = 0$.)

To show that T_{33} exists is now routine. Let $z = (z_1, z_2, z_3)$. Then

$$\begin{aligned} ((K_{33} - T_{13}^*T_{13} - T_{23}^*T_{23})z_3, z_3) &= (Kz, z) - |T_{22}z_2 + T_{23}z_3|^2 \\ &- |T_{11}z_1 + T_{12}z_2 + T_{13}z_3|^2 \\ &\geq -|T_{22}z_2 + T_{23}z_3|^2 - |T_{11}z_1 + T_{12}z_2 + T_{13}z_3|^2. \end{aligned}$$

This inequality, combined with the Remark above gives the nonnegativity of $K_3 - T_{13}^*T_{13} - T_{23}^*T_{23}$ and hence the existence of and boundedness of T_{33} .

We pass to the induction. Assume that $T_k^*T_k = K_k, k = 1, 2, \dots, n - 1$. Solve for T_{1n} in the same way as for T_{12} . Proceeding, once again, by induction we assume that the T_{kn} exist and are bounded for $k = 2, 3, \dots, n - 2$, and also that $\mathcal{R}(T_{kn}) \subset \overline{\mathcal{R}(T_{kk})}$, which makes the Remark applicable. The formula for the T_{km} are given by

$$T_{kk}T_{km} = K_{km} - \sum_{i=1}^{k-1} T_{ik}^*T_{im}, \quad k \leq m,$$

or

$$(9) \quad K_{km} = \sum_{i=1}^k T_{ik}^*T_{im}.$$

We must show that

$$(10) \quad T_{n-1, n-1} T_{n-1, n} = K_{n-1, n} - \sum_{i=1}^{n-2} T_{i, n-1}^* T_{i, n}$$

has a bounded solution for $T_{n-1, n}$. By the Remark we have for any $z_{n-1} \in \mathcal{H}$ a vector $z_{n-2} \in \mathcal{H}$ such that

$$T_{n-2, n-2} z_{n-2} + T_{n-2, n-1} z_{n-1} = 0.$$

Thus, proceeding sequentially we can solve the equations

$$(11) \quad \sum_{j=i}^{n-1} T_{ij} z_j = 0, \quad i = n-2, n-3, \dots, 1$$

for $z_{n-2}, z_{n-3}, z_{n-4}, \dots, z_1$, given z_{n-1} . Now, if $z = (z_1, \dots, z_n)$, an application of (11) gives

$$(12) \quad \begin{aligned} (Kz, z) &= \sum_{j=1}^{n-2} (K_{nj} z_j, z_n) + \sum_{j=1}^{n-2} (K_{jn} z_n, z_j) + (K_{n-1, n} z_n, z_{n-1}) \\ &+ (K_{n, n-1} z_{n-1}, z_n) + (K_{nn} z_n, z_n) + (T_{n-1, n-1}^2 z_{n-1}, z_{n-1}). \end{aligned}$$

By (9),

$$(13) \quad \begin{aligned} \sum_{j=1}^{n-2} (K_{jn} z_n, z_j) &= \sum_{j=1}^{n-2} ((T_{jj} T_{jn} + \sum_{i=1}^{j-1} T_{ij}^* T_{in}) z_n, z_j) \\ &= \sum_{j=1}^{n-2} (T_{jn} z_n, T_{jj} z_j) + \sum_{j=1}^{n-2} \sum_{i=1}^{j-1} (T_{ij}^* T_{in} z_n, z_j). \end{aligned}$$

We interpret all sums over not well defined limits to be zero (e.g. $\sum_{i=1}^0 (\cdot) = 0$). From (11) we have

$$T_{jj} z_j = - \sum_{i=j+1}^{n-1} T_{ji} z_i.$$

Substitution into (13) gives

$$(14) \quad \begin{aligned} \sum_{j=1}^{n-2} (K_{jn} z_n, z_j) &= - \sum_{j=1}^{n-2} \sum_{i=j+1}^{n-1} (T_{jn} z_n, T_{ji} z_i) + \sum_{j=1}^{n-2} \sum_{i=1}^{j-1} (T_{in} z_n, T_{ij} z_j) \\ &= - \sum_{j=1}^{n-3} \sum_{i=j+1}^{n-2} (T_{jn} z_n, T_{ji} z_i) + \sum_{j=1}^{n-2} \sum_{i=1}^{j-1} (T_{in} z_n, T_{ij} z_j) \\ &\quad - \sum_{j=1}^{n-2} (T_{jn} z_n, T_{j, n-1} z_{n-1}). \end{aligned}$$

The last term of (14) is

$$- \sum_{j=1}^{n-2} (T_{j, n-1}^* T_{jn} z_n, z_{n-1}).$$

Interchanging limits in the second term on the right hand side of (14), the equation (14) becomes

$$\begin{aligned}
 & - \sum_{j=1}^{n-3} \sum_{i=j+1}^{n-2} (T_{jn}z_n, T_{ji}z_i) + \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} (T_{in}z_n, T_{ij}z_j) \\
 & - \sum_{j=1}^{n-2} (T_{j,n-1}^* T_{jn}z_n, z_{n-1}).
 \end{aligned}$$

Upon interchanging i and j we obtain

$$(15) \quad \sum_{j=1}^{n-2} (K_{jn}z_n, z_j) = - \sum_{j=1}^{n-2} (T_{j,n-1}^* T_{jn}z_n, z_{n-1}).$$

Similarly

$$(16) \quad \sum_{j=1}^{n-2} (K_{nj}z_j, z_n) = - \left(\sum_{j=1}^{n-2} T_{jn}^* T_{j,n-1} z_{n-1}, z_n \right).$$

Substituting (15) and (16) into (12) and writing the result in matrix form we have

$$\begin{aligned}
 0 & \leq (K_n x, x) \\
 & = \left(\left(\begin{array}{cc} T_{n-1,n-1}^2 & K_{n-1,n} - \sum_{j=1}^{n-2} T_{j,n-1}^* T_{jn} \\ \left(K_{n-1,n} - \sum_{j=1}^{n-2} T_{j,n-1}^* T_{jn} \right)^* & K_{nn} \end{array} \right) \begin{pmatrix} z_{n-1} \\ z_n \end{pmatrix}, \begin{pmatrix} z_{n-1} \\ z_n \end{pmatrix} \right)
 \end{aligned}$$

An application of Lemmas 1 and 2 and the Corollary (ii), (iii) gives that there is a bounded operator $T_{n-1,n}$ satisfying (10) and moreover that $\mathcal{R}(T_{n-1,n}) \subseteq \overline{\mathcal{R}(T_{n-1,n-1})}$.

To show that T_{nn} exists a similar argument is used. This completes the induction and the Lemma is proved.

Lemma 3 works in any Hilbert space \mathcal{H} , finite or infinite dimensional. The following result, a considerable improvement of Lemma 3, applies only to infinite dimensional Hilbert spaces.

LEMMA 3'. (a) Suppose $\dim \mathcal{H} = \infty$ of K is a nonnegative $n \times n$ matrix with entries $K_{ij} \in B(\mathcal{H}, \mathcal{H})$ then there exist X_1, X_2, \dots, X_n in $B(\mathcal{H}, \mathcal{H})$ such that $K_{ij} = X_i^* X_j (1 \leq i, j \leq n)$. Hence $K = X^* X$ where X is the $n \times n$ matrix whose first row is $(X_1 X_2 \dots X_n)$ and whose other entries are all 0.

(b) If A is an $n \times n$ matrix with entries $A_{ij} \in B(\mathcal{H}, \mathcal{H})$ then there exists a partial isometry $U = (U_{ij})$ in $B(\mathcal{H}_n, \mathcal{H}_n)$ and a matrix X as in (a) such that $A = UX, X = U^* A$.

(c) If $A \geq 0$ then U may be chosen to be an isometry in (b).

Proof. (a) Let V_i be the isometry from \mathcal{H} into \mathcal{H}_n given by $V_i h = (0, 0, \dots, 0, h, 0, \dots)$ where the vector h appears as the i th coordinate. If h, k belong to \mathcal{H} then $(K_{ij}h, k) = (KV_j h, V_i k) = (\sqrt{K} V_j h, \sqrt{K} V_i k)$. Hence $K_{ij} = (\sqrt{K} V_i)^* (\sqrt{K} V_j)$. Let Φ be an isometry

from \mathcal{H} onto \mathcal{H}_n . Then $X_i = \Phi^* \sqrt{K} V_i \in B(\mathcal{H}, \mathcal{H})$ and $X_i^* X_j = K_{ij}$.

(b), (c) Choose X as in (a) so that $A^* A = X^* X$. Then

$$V \sqrt{A^* A} f = X f$$

defines an isometry V from $\mathcal{R}(\sqrt{A^* A})^-$ onto $\mathcal{R}(X)^-$. Since $\mathcal{R}(X)^\perp = \Phi^*(\mathcal{N}(\sqrt{A^* A})) \oplus \mathcal{H} \oplus \mathcal{H} \oplus \dots \oplus \mathcal{H}$ it is clear V can be extended to an isometry on \mathcal{H}_n . This proves (c) and to complete the proof of (b) use the polar factorization $A = W \sqrt{A^* A}$ and put $U = W V^*$.

REMARK. Lemma 3' is also valid for infinite matrices K (or A) that define bounded operators on the direct sum of countably many copies of \mathcal{H} .

Proof of Theorem 1. Define the Hilbert space $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ where $\mathcal{H}_i = \mathcal{H}$, $i = 1, 2, \dots$, with the natural inner product. Let $V_i (i \geq 1)$ be the isometry from \mathcal{H} into \mathcal{K} given by $V_i h = (h_1, h_2, \dots)$ where $h_i = h$ and $h_j = 0$ for $j \neq i$. Let $\mathcal{B} = \{t_i : i = 1, 2, \dots\}$ be a dense set of points in \mathcal{G} . Define the non negative-definite, bounded operator-valued matrices

$$K^{(n)} = K(t_i, t_j), \quad i, j = 1, \dots, n.$$

By Lemma 3 there is an upper triangular operator-valued matrix $T^{(n)}$ for which $T^{(n)*} T^{(n)} = K^{(n)}$ and moreover from the construction, if $m \leq n$ then $K^{(m)} = K_m^{(n)} = (T_m^{(n)})^* (T_m^{(n)})$. Let T be the formal infinite upper triangular matrix whose n^{th} column is the n^{th} column of $T^{(n)}$, $n = 1, 2, \dots$. For each $t_i \in \mathcal{B}$ define

$$\tilde{X}(t_i) = \sum_{i=1}^l V_i T_{il}.$$

Then, if $m = \min(k, l)$,

$$\begin{aligned} \tilde{X}(t_k)^* \tilde{X}(t_i) &= \left(\sum_{j=1}^k V_j T_{jk} \right)^* \left(\sum_{i=1}^l V_i T_{il} \right) \\ &= \sum_{j=1}^k \sum_{i=1}^l T_{jk}^* V_j^* V_i T_{il} \\ &= \sum_{i=1}^m T_{ik}^* T_{il} = K(t_k, t_i). \end{aligned}$$

From this it follows that

$$|\tilde{X}(t) - \tilde{X}(s)| \leq |K(t, t) - K(s, t)| + |K(s, s) - K(t, s)|,$$

for any t, s in \mathcal{B} . Using the completeness of $B(\mathcal{H}, \mathcal{K})$ and the continuity of K we can therefore extend \tilde{X} to a function X from \mathcal{G} into $B(\mathcal{H}, \mathcal{K})$ that satisfies the same inequalities for all t, s in

\mathcal{G} . The function X is then continuous and $X(t)^*X(s) = K(t, s)$.

In the following theorem the condition of separability is removed from \mathcal{G} . However, \mathcal{H} will be a nonseparable Hilbert space. The construction below seems to have originated with Naimark [5].

THEOREM 2. *Let \mathcal{G} be a Hausdorff space, and let $K(\cdot, \cdot)$ be as in Theorem 1. Then there is a Hilbert space \mathcal{H} and a continuous function $X(t)$ from \mathcal{G} into $B(\mathcal{H}, \mathcal{H})$ such that $X^*(t)X(s) = K(t, s)$.*

Proof. Let \mathcal{L} be the vector space of functions $\xi: \mathcal{G} \rightarrow \mathcal{H}$ that vanish at all but a finite number of points of \mathcal{G} , and for ξ, η in \mathcal{L} put

$$(\xi, \eta) = \sum_{s,t} (K(s, t)\xi(t), \eta(s)).$$

Let $\mathcal{N} = \{\xi \in \mathcal{L} : (\xi, \xi) = 0\}$. Then \mathcal{N} is a subspace of \mathcal{L} and

$$(\xi + \mathcal{N}, \xi + \mathcal{N}) = (\xi, \eta)$$

defines an inner product on $\mathcal{H}_0 = \mathcal{L}/\mathcal{N}$. Let \mathcal{H} be the completion of \mathcal{H}_0 . For $s \in \mathcal{G}$ and $h \in \mathcal{H}$ define

$$\xi_s h(t) = \begin{cases} h & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

Then $X(s)h = \xi_s h + \mathcal{N}$ defines a bounded operator $X(s)$ from \mathcal{H} into \mathcal{H} . A simple computation shows that $X(t)^*X(s) = K(t, s)$. This implies $|X(t) - X(s)|^2 \leq |K(t, t) - K(t, s)| + |K(s, s) - K(s, t)|$, so the continuity of the map $s \rightarrow X(s)$ follows from that of K .

REFERENCES

1. G. D. Allen, *An extension of Kolmogorov's theorem for continuous covariances*, Proc. Amer. Math. Soc., **39** (1973), 214-216.
2. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., **17** (1966), 413-415.
3. D. K. Faddeev and V. N. Fadeeva, *Computational Methods in Linear Algebra*, Fizmatgiz, Moscow, 1960; English transl., Freeman, San Francisco, Calif., 1963.
4. P. A. Fillmore and J. P. Williams, *On operator ranges*, Advances in Mathematics, **7** (1971), 254-281.
5. M. A. Naimark, *On a representation of additive operator set functions*, Comptes Rendus (Doklady) Acad. Sci. USSR, **41** (1943), 359-361.

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