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CONSTANTIN GELU APOSTOL AND NORBERTO SALINAS

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Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all (bounded, linear) operators on $\mathscr{H}$. This paper is concerned with some aspects of the uniform approximation of an operator in $\mathscr{L}(\mathscr{H})$ by nilpotent operators.

If T is an operator in $\mathscr{L}(\mathscr{H})$ we shall denote by $\sigma(T)$ the spectrum of T and $r(T)$ the spectral radius of T. We recall that an operator T in $\mathscr{L}(\mathscr{H})$ is said to be quasinilpotent if $\sigma(T) = \{0\}$ or equivalently $r(T) = \lim_{n\to\infty} \| T^n \|^{1/n} = 0$. In [5] [7] it wasproved that a quasinilpotent operator T in $\mathscr{L}(\mathscr{H})$ is the uniform limit of a sequence $\{Q_k\}$ of nilpotent operators on H (cf. [6, Problem 7]). For each positive integer k let o_k be the order of the nilpotent operator Q_k, that is o_k is the smallest positive integer such that $Q_k^{o_k} = 0$. In view of the result mentioned above it is natural to ask whether there exists any relationship between the rate of decrease of the sequence $\| T^{o_k} \|^{1/o_k}$ and the rate of decrease of the sequence $\| T - Q_k \|$. Furthermore, it is reasonable to expect that there exists a characterization of the set of quasinilpotent operators in terms of its nilpotent approximations. These questions seem to be rather hard and in this paper we present some insights into these problems (cf. Corollary 3.4).

In Theorem 3.5 we prove that the distance from an arbitrary operator T in $\mathscr{L}(\mathscr{H})$ to the set $\mathscr{N}(\mathscr{H})$ of all nilpotent operators in $\mathscr{L}(\mathscr{H})$ is at most $r(T)$ and in Theorem 3.1 we give precise estimates for the nilpotent approximations of T in the case that T is a biquasitriangular operator and zero is in the essential spectrum of T (see § 3 for the corresponding definitions). Another by-product of our discussion (cf. Proposition 4.4) is that if T is an operator on $\mathscr{H}$ such that $\liminf_{n\to\infty} \sqrt{n} \| T^n \|^{1/n} = 0$, then the quasinilpotent operator T is actually pseudonilpotent (cf. [10, Problem 1]).

We are indebted to D. A. Herrero for his very useful comments, specifically, for his suggestions about the proof of Theorem 3.5 and for providing us with the example of Remark 4.5.

2. **The central lemma.** The following lemma is central to our purposes.

LEMMA 2.1. *Let $T \in \mathscr{L}(\mathscr{H})$. Then for every $\alpha > 0$, $\beta > r(T)$*

and every positive integer n there exists $Q \in \mathscr{N}(\mathscr{H} \oplus \mathscr{H})$ of order at most n (i.e. $Q^n = 0$) *such that*

$$\|(T \oplus 0) - Q\| < \alpha \|T\| + \beta + \frac{\|T^n\|}{\alpha\beta^{n-1}} .$$

Proof. Let U and V be two isometries in $\mathscr{L}(\mathscr{H})$ such that $VV^* = 1 - UU^*$. We define for $1 \leqq k \leqq n$ the subspace $\mathscr{M}_k$ of $\mathscr{H} \oplus \mathscr{H}$ given by $\mathscr{M}_k = \{T^{k-1}x \oplus \alpha\beta^{k-1}V^{k-1}Ux, x \in \mathscr{H}\}$. Note that $\mathscr{M}_k$ is closed because it is the image under the isometry $\begin{pmatrix} 1 & 0 \\ 0 & V^{k-1}U \end{pmatrix}$ of the graph of the transpose of $T^{k-1}/(\alpha\beta^{k-1})$. It is easy to see that $\mathscr{M}_j \cap \mathscr{M}_k = \{0\}$ if $1 \leqq j,\ k \leqq n,\ j \neq k$. This is due to the fact that the second components of the elements in $\mathscr{M}_j$ and $\mathscr{M}_k$ are orthogonal. Let $\mathscr{M} = \sum_{k=1}^n \mathscr{M}_k = \{y_1 + \cdots + y_n,\ y_k \in \mathscr{M}_k,\ 1 \leqq k \leqq n\}$. Now we prove that $\mathscr{M}$ is closed. Let $\{y_m\}$ be a sequence in $\mathscr{M}$ s.t. $\lim_{m\to\infty} y_m = 0$. Since $\mathscr{M}_j \cap \mathscr{M}_k = \{0\}$, $j \neq k$, we can write uniquely $y_m = \sum_{k=1}^n y_{m,k}$, where $y_{m,k} = T^{k-1}x_{m,k} \oplus \alpha\beta^{k-1}V^{k-1}Ux_{m,k}$, for some $x_{m,k} \in \mathscr{H}$, $1 \leqq k \leqq n$, $m = 1, 2, \cdots$. Since $\lim_{m\to\infty} y_m = 0$, then $\lim_{m\to\infty} \sum_{k=1}^n \alpha\beta^{k-1}V^{k-1}Ux_{m,k} = 0$. Then $\lim_{m\to\infty} V^{k-1}Ux_{m,k} = 0$, $1 \leqq k \leqq n$ and hence $\lim_{m\to\infty} x_{m,k} = 0$, $1 \leqq k \leqq n$. Therefore, $\lim_{m\to\infty} y_{m,k} = 0$, $1 \leqq k \leqq n$ and hence $\mathscr{M}$ is closed. Now we define $Q \in \mathscr{N}(\mathscr{H} \oplus \mathscr{H})$ by

$$Q \mid \mathscr{M}_n \oplus \mathscr{M}^\perp = 0,\ Q \mid \sum_{k=1}^{n-1} \mathscr{M}_k = (T \oplus \beta V) \mid \sum_{k=1}^{n-1} \mathscr{M}_k .$$

Thus, the representing matrix of $Q \mid \mathscr{M}$ on $\sum_{k=1}^n \mathscr{M}_k$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 \cdots 0 & 0 \\ * & 0 & 0 \cdots 0 & 0 \\ 0 & * & 0 \cdots 0 & 0 \\ \vdots & & & \\ 0 & 0 & 0 \cdots * & 0 \end{pmatrix} .$$

Therefore it is clear that $Q^n = 0$. Let P_n be the projection onto kernel V^{*n}. Then

$$\begin{aligned} &[(T \oplus \beta P_n V) - Q] \sum_{k=1}^n (T^{k-1}x_k \oplus \alpha\beta^{k-1}V^{k-1}Ux_k) \\ &= \left[T\sum_{k=1}^{n-1} T^{k-1}x_k - T\sum_{k=1}^{n-1} T^{k-1}x_k \right] \oplus \beta P_n V \sum_{k=1}^n \alpha\beta^{k-1}V^{k-1}Ux_k \\ &\quad - \beta V \sum_{k=1}^n \alpha\beta^{k-1}V^{k-1}Ux_k \\ &= T^n x_n \oplus \left[P_n \alpha\beta^n V^n Ux + \sum_{k=1}^{n-1} \alpha\beta^k (P_n - 1) V^k Ux_k \right] \\ &= T^n x_n \oplus 0 . \end{aligned}$$

Since

$$\| x_n \| = \frac{1}{\alpha\beta^{n-1}} \| \alpha\beta^{n-1} V^{n-1} U x_n \| \leqq \frac{1}{\alpha\beta^{n-1}} \left\| \sum_{k=1}^{n} \alpha\beta^{k-1} V^{k-1} U x_k \right\|$$
$$\leqq \frac{1}{\alpha\beta^{n-1}} \left\| \sum_{k=1}^{n} T^k x_k \oplus \alpha\beta^{k-1} V^{k-1} U x_k \right\| .$$

We conclude that $\| [(T \oplus \beta P_n V) - Q] \mid \mathscr{M} \| \leqq \| T^n \| / \alpha\beta^{n-1}$. From now on given a subspace $\mathscr{K}$ of $\mathscr{H}$, $P_{\mathscr{K}}$ will denote the projection from $\mathscr{H}$ onto $\mathscr{K}$. Then

$$\begin{aligned} \| (T \oplus 0) - Q \| &\leqq \| (T \oplus 0) P_{\mathscr{M}^\perp} \| + \| [(T \oplus 0) - Q] P_{\mathscr{M}} \| \\ &\leqq \| T \| \, \| P_{\mathscr{H} \oplus 0} P_{\mathscr{M}^\perp} \| + \| [(T \oplus \beta P_n V) - Q] P_{\mathscr{M}} \| \\ &\quad + \| 0 \oplus \beta P_n V \| \leqq \| T \| \, \| P_{\mathscr{H} \oplus 0} P_{\mathscr{M}_1^\perp} \| \\ &\quad + \beta + \| T^n \| / \alpha\beta^{n-1} . \end{aligned}$$

Thus in order to complete the proof it suffices to show that $\| P_{\mathscr{H} \oplus 0} P_{\mathscr{M}_1^\perp} \| \leqq \alpha$. Notice that $\mathscr{M}_1^\perp = \{ y \oplus z \colon \langle y, x \rangle + \alpha \langle z, Ux \rangle = 0$ for all $x \in \mathscr{H} \}$. Hence $\mathscr{M}_1^\perp = \{ (-\alpha U^* z) \oplus z, z \in \mathscr{H} \}$. Therefore $\| P_{\mathscr{H} \oplus 0} P_{\mathscr{M}_1^\perp} \| = \| P_{\mathscr{H} \oplus 0} |_{\mathscr{M}_1^\perp} \| \leqq \alpha$.

Following [9] we shall denote by $R_e(T)$ the reducing essential spectrum of the operator T, that is $R_e(T)$ is the set of complex numbers λ such that there exists a projection P in $\mathscr{L}(\mathscr{H})$ of infinite rank and nullity such that $(T - \lambda)P$ and $(T^* - \bar{\lambda})P$ are compact operators.

THEOREM 2.2. *Let T be in $\mathscr{L}(\mathscr{H})$ such that $R_e(T) \neq \varnothing$ and suppose that $0 \in R_e(T)$. Then for every $\alpha > 0$, $\beta > r(T)$, $\gamma > 1$ and every positive integer n there exists $Q \in \mathscr{N}(\mathscr{H})$ of order at most n such that*

$$(*) \qquad \| T - Q \| \leqq \gamma \left(\alpha \| T \| + \beta + \frac{\| T^n \|}{\alpha\beta^{n-1}} \right) .$$

Proof. From [9, Theorem 4.6] it follows that for every $\varepsilon > 0$ there exists a unitary transformation $U_\varepsilon \colon \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ such that $\| U_\varepsilon T U_\varepsilon^* - (T \oplus 0) \| < \varepsilon$. On the other hand, from Lemma 2.1 we deduce that there exists a nilpotent operator Q' on $\mathscr{H} \oplus \mathscr{H}$ of order at most n such that $\| T \oplus 0 - Q' \| \leqq \alpha \| T \| + \beta + \| T^n \| / (\alpha\beta^{n-1})$. Letting now $Q = U_\varepsilon^* Q' U_\varepsilon$ we observe that $\| T - Q \| = \| U_\varepsilon T U_\varepsilon^* - Q' \| \leqq \varepsilon + \| (T \oplus 0) - Q' \| \leqq \varepsilon + \alpha \| T \| + \beta + \| T^n \| / (\alpha\beta^{n-1})$. The proof of the theorem is completed by choosing ε small enough.

3. Nilpotent approximation of biquasitriangular operators. In what follows, given an operator T in $\mathscr{L}(\mathscr{H})$ we denote by $E(T)$

the essential spectrum of T, i.e. $E(T) = \{\lambda \in \sigma(T): T - \lambda$ is not Fredholm$\}$. Also $E_l(T)$ and $E_r(T)$ will denote the left essential spectrum and the right essential spectrum of the operator T, that is $E_l(T) = \{\lambda \in E(T): T - \lambda$ is not semi-Fredholm with $\dim[\text{null}(T-\lambda)] < \infty\}$ and $E_r(T) = [E_l(T^*)]^*$.

Following [6] we say that an operator T on $\mathscr{H}$ is quasitriangular if there exists an increasing sequence $\{P_n\}$ of finite rank projections in $\mathscr{L}(\mathscr{H})$ tending strongly to the identity operator such that $\lim_{n\to\infty} \|(1 - P_n)TP_n\| = 0$ and we say that T is biquasitriangular if both T and T^* are quasitriangular [2]. In [3] it was shown that T is biquasitriangular if and only if the index of $T - \lambda$ is zero for every complex number λ such that $T - \lambda$ is semi-Fredholm. In particular, if T is biquasitrianglar $E(T) = E_l(T) \cap E_r(T)$.

In the following theorem we give some precise estimates for the nilpotent approximation of a biquasitriangular operator.

THEOREM 3.1. *Let T in $\mathscr{L}(\mathscr{H})$ be a biquasitriangular operator and suppose that $0 \in E(T)$. Then for every $\alpha > 0$, $\beta > r(T)$, $\gamma > 1$ and every positive integer n there exists a nilpotent operator Q in $\mathscr{L}(\mathscr{H})$ of order at most $4n$ such that $(*)$ is valid.*

Proof. From [11, Proposition 3.2] it follows that for every $\varepsilon > 0$ there exists a unitary transformation $V_\varepsilon: \mathscr{H} \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ and a compact operator K_ε in $\mathscr{L}(\mathscr{H})$ with $\|K_\varepsilon\| < \varepsilon$ such that

$$V_\varepsilon(T - K_\varepsilon)V_\varepsilon^* = \begin{pmatrix} S_1 & 0 & 0 & 0 \\ L_{21} & S_2 & 0 & 0 \\ L_{31} & L_{32} & S_3 & 0 \\ L_{41} & L_{42} & L_{43} & S_4 \end{pmatrix}$$

where $L_{ij} \in \mathscr{L}(\mathscr{H})$, $1 \leqq j < i \leqq 4$, and for each $1 \leqq j \leqq 4$, S_j is an operator in $\mathscr{L}(\mathscr{H})$ such that S_j is unitarily equivalent to $M_j \oplus N_j$ where M_j is a block diagonal operator on $\mathscr{H}$ with $E(M_j) \subset E(T)$ and N_j is a normal operator in $\mathscr{L}(\mathscr{H})$ such that $E(N_j) = E(T)$. Since $E(N_j) = R_e(N_j)$ [9, Theorem 3.10], $0 \in E(T)$ and $R_e(M_j \oplus N_j) = R_e(M_j) \cup R_e(N_j)$ [9, Lemma 4.10], it follows that $0 \in R_e(S_j)$, $1 \leqq j \leqq 4$. Now we chose $\varepsilon > 0$ small enough so that $r(T - K_\varepsilon) < \beta$. Since $\|S_j^k\| \leqq \|(T - K_\varepsilon)^k\|$, $1 \leqq j \leqq 4$, $k = 1, 2, \cdots$, we also have $r(S_j) < \beta$, $1 \leqq j \leqq 4$. Given a fixed number δ with $1 < \delta < \gamma$, from Theorem 2.2 we can find nilpotent operators Q_j in $\mathscr{L}(\mathscr{H})$ of order at most n such that $\|S_j - Q_j\| \leqq \delta(\alpha\|S_j\| + \beta + \|S_j^n\|/\alpha\beta^{n-1})$, $1 \leqq j \leqq 4$. Now let Q be the operator in $\mathscr{L}(\mathscr{H})$ defined by

$$V_\varepsilon Q V_\varepsilon^* = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ L_{21} & Q_2 & 0 & 0 \\ L_{31} & L_{32} & Q_3 & 0 \\ L_{41} & L_{42} & L_{43} & Q_4 \end{pmatrix}$$

It is obvious that Q is a nilpotent operator of order at most $4n$ and we see that

$$\begin{aligned} \| T - K_\varepsilon - Q \| &= \| V_\varepsilon(T - K_\varepsilon - Q)V_\varepsilon^* \| = \max_{1 \leqq j \leqq 4} \| S_j - Q_j \| \\ &\leqq \delta\left(\alpha \| T - K_\varepsilon \| + \beta + \frac{\| (T - K_\varepsilon)^n \|}{\alpha\beta^{n-1}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \| T - Q \| &\leqq \| K_\varepsilon \| + \| T - K_\varepsilon - Q \| \\ &\leqq \varepsilon + \delta\left(\alpha \| T - K_\varepsilon \| + \beta + \frac{\| (T - K_\varepsilon)^n \|}{\alpha\beta^{n-1}}\right) \end{aligned}$$

and hence the theorem follows by choosing ε small enough.

COROLLARY 3.2. *Let T be a biquasitriangular operator in $\mathscr{L}(\mathscr{H})$ such that $0 \in E(T)$. Then the distance* $\operatorname{dist}(T, \mathscr{N}(\mathscr{H})) \leqq r(T)$.

Proof. From Theorem 3.1 we deduce that for an arbitrary $\varepsilon > 0$ (taking $\alpha = \varepsilon$, $\beta = r(T) + \varepsilon$ and $\gamma = 1 + \varepsilon$) there exists a nilpotent operator Q in $\mathscr{L}(\mathscr{H})$ such that

$$\begin{aligned} &\| T - Q \| \\ &\quad \leqq (1 + \varepsilon)\left\{\varepsilon \| T \| + r(T) + \varepsilon + \frac{1}{\varepsilon}(r(T) + \varepsilon)\left[\frac{\| T^n \|^{1/n}}{r(T) + \varepsilon}\right]^n\right\}. \end{aligned}$$

Since $\lim_{n\to\infty} \| T^n \|^{1/n} = r(T)$, it follows that for n sufficiently large $[\| T^n \|^{1/n}/(r(T) + \varepsilon)]^n < \varepsilon^2$ and hence

$$\| T - Q \| < (1 + \varepsilon)[\varepsilon \| T \| + (r(T) + \varepsilon)(1 + \varepsilon)] .$$

Since ε is arbitrary the proof of the corollary is complete.

The following corollary expresses the fact that if T is a quasinilpotent operator on $\mathscr{H}$ and $\| T^n \|^{1/n}$ doesn't decrease very fast, then there exists a sequence $\{Q_n\}$ in $\mathscr{N}(\mathscr{H})$ where each Q_n has order o_n, $n = 1, 2, \cdots$ such that the rate of decrease of the sequence $\| T^{o_n} \|^{1/o_n}$ is the same as the rate of decrease of the sequence $\| T - Q_n \|$.

COROLLARY 3.3. *Let T in $\mathscr{L}(\mathscr{H})$ be quasinilpotent and let*

$\delta > 1$. *Then there exists a sequence* $\{Q_n\}$ *in* $\mathscr{N}(\mathscr{H})$ *such that* $Q_n^{4n} = 0$ *and for some* $c > 0$, $\| T - Q_n \| < c(\| T \| \delta^{-n} + \| T^n \|^{1/n})$.

Proof. It is a direct consequence of Theorem 3.1 where $\alpha = \delta^{-n}$, $\beta = \delta \| T^n \|^{1/n}$.

COROLLARY 3.4. *Let* T *in* $\mathscr{L}(\mathscr{H})$ *be quasinilpotent. Then there exists a sequence* $\{Q_n\}$ *in* $\mathscr{N}(\mathscr{H})$ *such that* $Q_n^{4n} = 0$ *and* $\| T - Q_n \| \leqq 3(1 + \| T \|)(n \|T^n \|)^{1/n+1}$.

Proof. It is an immediate consequence of Theorem 3.1 taking $\alpha = \beta = ((n \| T^n \|)/(1 + \| T \|))^{1/n+1}$ and $\gamma = 3/2$.

In the next theorem we shall need the following terminology: If X is a compact subset of the plane we denote by $\hat{X}$ the complement of the unbounded component of the complement of X. If $\varepsilon \geqq 0$ we denote by X_ε the set $X_\varepsilon = \{\lambda \in C: \inf_{\mu \in X} |\lambda - \mu| \leqq \varepsilon\}$. We notice that if $\varepsilon \geqq \sup_{\lambda \in X} |\lambda|$, then $X_\varepsilon = (\hat{X})_\varepsilon = (X_\varepsilon)^\wedge$ and $0 \in X_\varepsilon$.

THEOREM 3.5. *Let* T *be in* $\mathscr{L}(\mathscr{H})$. *Then the distance* $\operatorname{dist}(T, \mathscr{N}(\mathscr{H})) \leqq r(T)$.

Proof. Let $\varepsilon > 0$ and let $\sum = \sigma(T)\backslash \widehat{E(T)}_\varepsilon$. If $\sum \neq \emptyset$, then $\sum = \{\lambda_1, \cdots, \lambda_m\}$ where each λ_j is an isolated eigenvalue of $\sigma(T)$ such that $T - \lambda_j$ is Fredholm, $1 \leqq j \leqq m$. Then the spectral idempotent E_Σ corresponding to $\sum$, associated with T, has finite rank. Let $\mathscr{F} = \operatorname{range} E_\Sigma$, $\mathscr{M} = \mathscr{F}^\perp$ and let P be the projection from $\mathscr{H}$ onto $\mathscr{M}$. It readily follows that $\sigma(T|\mathscr{F}) = \sum$ and $\sigma(PT|\mathscr{M}) = \sigma(T)\backslash\sum$. Furthermore, $E[(PT)|\mathscr{M}] = E(T)$ and $E_l[PT|\mathscr{M}] = E_l(T)$. Now we recall that a normal operator M in $\mathscr{L}(\mathscr{H})$ is called diagonalizable if $\mathscr{H}$ has an orthonormal basis consisting of eigenvectors of M and M is said to have uniform infinite multiplicity if every eigenvalue of M has infinite multiplicity. Notice that in this case M is unitarily equivalent to $M \oplus M$ and that $\sigma(M) = E(M)$. From [3, Theorem 2.2] there exists a unitary transformation $U_\varepsilon: \mathscr{M} \to \mathscr{H} \oplus \mathscr{H}$ and a compact operator K_ε on $\mathscr{M}$ whose norm is so small that $\| K_\varepsilon \| < \varepsilon$, $\sigma[PT|\mathscr{M} - K_\varepsilon] \subset \widehat{E(T)}_{2\varepsilon}$ and such that we also have

$$U_\varepsilon[PT|\mathscr{M} - K_\varepsilon]U_\varepsilon^* = \begin{bmatrix} M & R \\ 0 & S \end{bmatrix},$$

where M is a diagonalizable normal operator of uniform infinite multiplicity in $\mathscr{L}(\mathscr{H})$ such that $\sigma(M) = E_l[PT|\mathscr{M}]$. Applying now the same reasoning to the operator S we further deduce that there exist a unitary transformation $V_\varepsilon: \mathscr{M}_\varepsilon \to \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ and a compact operator L_ε in $\mathscr{L}(\mathscr{M})$ with norm so small that $\| L_\varepsilon \| < 2\varepsilon$, $\sigma[PT \mid \mathscr{M} - \mathscr{L}_\varepsilon] \subset \widehat{E(T)}_{3\varepsilon}$ and such that

$$T_\varepsilon = V_\varepsilon[PT \mid \mathscr{M} - \mathscr{L}_\varepsilon]V_\varepsilon^* = \begin{pmatrix} M & A & B \\ 0 & C & D \\ 0 & 0 & N \end{pmatrix},$$

where M is as before, N is a diagonalizable normal operator of infinite uniform multiplicity in $\mathscr{L}(\mathscr{H})$, A, B, C, D are in $\mathscr{L}(\mathscr{H})$ and $\sigma(N) = E_r(C) = E_r\begin{pmatrix} C & D \\ 0 & N \end{pmatrix} \subset E_r[PT \mid \mathscr{M}]$. Now let $\delta = r(M)$, $\eta = r(N)$ and notice that $\eta \leqq \delta$. From the remark preceding the present theorem we see that $\sigma(M)_\delta = \widehat{\sigma(M)}_\delta$, $\sigma(N)_\eta = \widehat{\sigma(N)}_\eta \subset \sigma(M)_\delta$ and $0 \in \sigma(N)_\eta$. Employing a minor variation of [3, Proposition 1.11] it follows that there exist two normal operators M' and N' in $\mathscr{L}(\mathscr{H})$ such that $\sigma(M') = E(M') = \sigma(M)_{\delta+3\varepsilon} = \widehat{E_l(T)}_{\delta+3\varepsilon} = \widehat{E(T)}_{\delta+3\varepsilon}$, $\sigma(N') = E(N') = \sigma(N)_\eta$ and $\| M - M' \| = \delta + 3\varepsilon$, $\| N - N' \| = \eta$. Let T'_ε be the 3×3 operator matrix defined by

$$T'_\varepsilon = \begin{pmatrix} M' & A & B \\ 0 & C & D \\ 0 & 0 & N' \end{pmatrix}.$$

We claim that $E_l(T'_\varepsilon) \cap E_r(T'_\varepsilon) = \sigma(T'_\varepsilon) = \sigma(M')$. In order to prove this assertion we first observe that, since M' acts on an invariant subspace of T'_ε, it is obvious that $\sigma(M') = E_l(M') \subset E_l(T'_\varepsilon)$. Also, if $\lambda \notin E_r(T'_\varepsilon)$, then $\lambda \notin E_r\begin{pmatrix} C & D \\ 0 & N' \end{pmatrix}$ and hence $\lambda \notin E_r(N') = \sigma(N')$. Since $E_l(C) \subset \widehat{E_r(C)}$ and $E_r(C) = \sigma(N) \subset \sigma(N') = \widehat{\sigma(N')}$, it follows that $\lambda \notin E(C)(=E_r(C) \cup E_l(C))$. Therefore $\lambda \notin E\begin{pmatrix} C & D \\ 0 & N' \end{pmatrix}$, and since $\lambda \notin E_r(T'_\varepsilon)$ we deduce that $\lambda \notin E_r(M') = \sigma(M')$. Now we shall prove the other inclusion. To this end we note that $\sigma\begin{pmatrix} C & D \\ 0 & N \end{pmatrix} \subset \widehat{\sigma(T_\varepsilon)} \subset \widehat{E(T)}_{3\varepsilon}$, and hence $\sigma\begin{pmatrix} C & D \\ 0 & N \end{pmatrix} \subset \widehat{E(T)}_{\delta+3\varepsilon} = \sigma(M')$. In particular, it follows that $\sigma(C) \subset \sigma(M')$. Thus, if $\lambda \notin \sigma(M')$, then $\lambda \notin \sigma(C) \cup \sigma(N')$ and hence

$$\lambda \notin \sigma\begin{pmatrix} M' & A & B \\ 0 & C & D \\ 0 & 0 & N' \end{pmatrix} = \sigma(T'_\varepsilon).$$

The last observation establishes our claim. Let $\gamma = \operatorname{dist}(T'_\varepsilon, \mathscr{N}(\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}))$. Our first conclusion is that $\operatorname{dist}(PT \mid \mathscr{M}, \mathscr{N}(\mathscr{M})) \leqq \| L_\varepsilon \| + \operatorname{dist}(T_\varepsilon, \mathscr{N}(\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H})) \leqq 2\varepsilon + \gamma + \max(\| M - M' \|, \| N - N')$. Since $\| M' - M \| = \delta + 3\varepsilon \leqq r(T) + 3\varepsilon$ and $\| N' - N \| =$

$\eta \leqq \delta$, we deduce that dist $(PT \mid \mathscr{M}, \mathscr{N}(\mathscr{M})) \leqq \gamma + r(T) + 5\varepsilon$. Using the fact that $\mathscr{F}$ is finite dimensional it is elementary to check that dist $(T \mid \mathscr{F}, \mathscr{N}(\mathscr{F})) \leqq r(T \mid \mathscr{F}) \leqq r(T)$ and hence we conclude that

$$\begin{aligned}\operatorname{dist}(T, \mathscr{N}(\mathscr{H})) &\leqq \max[\operatorname{dist}(T \mid \mathscr{F}, \mathscr{N}(\mathscr{F}))\,,\\ &\operatorname{dist}(PT \mid \mathscr{M}, \mathscr{N}(\mathscr{M}))] \leqq \gamma + r(T) + 5\varepsilon\,.\end{aligned}$$

Since $0 \in E(T'_\varepsilon)$ and we have shown that T'_ε is a biquasitriangular operator we may conclude from Corollary 3.2 that dist $(T, \mathscr{N}(\mathscr{H})) \leqq r(T'_\varepsilon) + r(T) + 5\varepsilon \leqq 2r(T) + 8\varepsilon$. However, we are interested in a sharper estimate. At this point we make use of the fact that T'_ε enjoys the further property that $\sigma(T'_\varepsilon)$ is simple connected and coincides with $E(T'_\varepsilon)$ and therefore from [4, Proposition 1.6], $\gamma = 0$. Thus, dist $(T, \mathscr{N}(\mathscr{H})) \leqq r(T) + 5\varepsilon$ and since ε is arbitrary the proof of the theorem is complete.

4. **Concluding remarks.** In this section we pose some problems concerning the nilpotent approximation of quasinilpotent operators. We motivate these problems by presenting some pertinent observations.

PROPOSITION 4.1. *Let T be in $\mathscr{L}(\mathscr{H})$ and suppose that there exists a sequence $\{Q_n\}$ in $\mathscr{N}(\mathscr{H})$ such that $Q_n^{o_n} = 0$, $n = 1, 2, \cdots$ and* $\liminf_{n\to\infty} \| T - Q_n \|^{1/o_n} = 0$. *Then T is quasinilpotent.*

Proof. Since $T^{o_n} = T^{o_n} - Q_n^{o_n} = \sum_{j=1}^{o_n} T^{o_n - j}(T - Q_n)Q_n^{j-1}$ then for n sufficiently large there exists $c > 0$ such that $\| T^{o_n} \|^{1/o_n} \leqq \| T - Q_n \|^{1/o_n}$. $[\sum_{j=1}^{o_n} \| T \|^{o_n - j} \| Q_n \|^{j-1}]^{1/o_n} \leqq c \| T - Q_n \|^{1/o_n}$. This completes the proof of the proposition.

PROBLEM 1. Is the converse of Proposition 4.1 valid? (We expect a negative answer.)

REMARK 4.2. In a different circle of ideas we note that minor modifications of the arguments given in this paper show that if we replace $\mathscr{L}(\mathscr{H})$ by the Calkin algebra over $\mathscr{H}$ (i.e., the quotient algebra of $\mathscr{L}(\mathscr{H})$ by the ideal of compact operators on $\mathscr{H}$), then the conclusion of Theorem 3.5 still holds. Thus it is natural to ask:

PROBLEM 2. If $\mathscr{A}$ is a C^*-algebra and $\mathscr{N}$ is the set of nilpotent elements of $\mathscr{A}$, when does it follow that the distance dist $(A, \mathscr{N}) \leqq r(A)$, for every $A \in \mathscr{A}$?

If $\mathscr{A}$ is a finite type I von Neumann algebra, it is easy to see that the above question has an affirmative answer. On the other hand, it is worth noting that, by a well known fact in the theory of C^*-algebras [8], any noncommutative C^*-algebra contains nonzero nilpotent elements. However, if $\mathscr{B}$ is a semi-simple Banach algebra, then there exists no nilpotent element different from zero. In particular, if $\mathscr{B}$ is a semi-simple commutative Banach algebra such that the Gelfand transformation is not an isometry, then the corresponding version of the above question has a negative answer. This is the reason that in the preceding problem we imposed the condition that $\mathscr{A}$ be a C^*-algebra.

The following terminology was introduced in [10].

DEFINITION. Let $T \in \mathscr{L}(\mathscr{H})$. We say that T is pseudonilpotent if for every $\varepsilon > 0$ there exists a finite orthogonal family of projections $P_1, \cdots, P_n$ in $\mathscr{L}(\mathscr{H})$ such that $\sum_{k=1}^{n} P_k = 1$ and $\| \sum_{i \geqq j} P_i T P_j \| < \varepsilon$. The set of all pseudonilpotent operators in $\mathscr{L}(\mathscr{H})$ will be denoted by $\Psi(\mathscr{H})$.

In [10], it was observed that $\Psi(\mathscr{H}) = \Psi(\mathscr{H})^*$ and it was shown that $\mathscr{N}(\mathscr{H}) \subset_{\neq} \Psi(\mathscr{H}) \subset_{\neq} \overline{\mathscr{N}(\mathscr{H})}$ and that there are pseudonilpotent operators which are not quasinilpotent.

LEMMA 4.3. *Let* $T \in \mathscr{L}(\mathscr{H})$ *and suppose that there exists a sequence* $\{Q_n\}$ *in* $\mathscr{N}(\mathscr{H})$ *such that* $Q_n^{o_n} = 0$, $n = 1, 2, \cdots$ *and* $\liminf_{n \to \infty} \sqrt{o_n} \, \| T - Q_n \| = 0$. *Then* T *is pseudonilpotent.*

Proof. Given $\varepsilon > 0$, let n be sufficiently large so that $\sqrt{o_n} \, \| T - Q_n \| < \varepsilon$. For each $1 \leqq j \leqq o_n$ let P_j be the projection onto kernel $Q_n^j \ominus$ kernel Q_n^{j-1}. Also let $x \in \mathscr{H}$. Then

$$\begin{aligned}
\left\| \sum_{j=1}^{o_n} \sum_{i=j}^{o_n} P_i T P_j x \right\| &= \left\| \sum_{j=1}^{o_n} \sum_{i=j}^{o_n} P_i (T - Q_n) P_j x \right\| \\
&\leqq \sum_{j=1}^{o_n} \left\| \left(\sum_{i=j}^{o_n} P_i \right) (T - Q_n) P_j \right\| \| P_j x \| \\
&\leqq \| T - Q_n \| \sum_{j=1}^{o_n} \| P_j x \| \leqq \| T - Q_n \| \sqrt{o_n} \, \| x \| .
\end{aligned}$$

Therefore $\| \sum_{j=1}^{o_n} \sum_{i=j}^{o_n} P_i T P_j \| \leqq \sqrt{o_n} \, \| T - Q_n \| < \varepsilon$, as desired.

PROPOSITION 4.4. *Let* T *be in* $\mathscr{L}(\mathscr{H})$ *such that*

$$\liminf_{n \to \infty} \sqrt{n} \; \| T^n \|^{1/n} = 0 .$$

Then T *is pseudonilpotent and quasinilpotent.*

Proof. It is an immediate consequence of Corollary 3.3 and Lemma 4.3.

REMARK 4.5. Let $\{e_n\}$ be an orthonormal basis for $\mathscr{H}$ and let $\{\alpha_n\}$ be a sequence of complex numbers. We recall that an operator T in $\mathscr{L}(\mathscr{H})$ is called a unilateral weighted shift with weights α_n, $n = 1, 2, \cdots$ if $Te_n = \alpha_n e_{n+1}$. Now we let T in $\mathscr{L}(\mathscr{H})$ be a unilateral weighted shift with weights $\alpha_0 = \alpha_1 = \cdots = \alpha_{n_1} = 1$, $\alpha_{n_1+1} = \cdots = \alpha_{n_2} = 1/2, \cdots$, $\alpha_{n_{k-1}+1} = \alpha_{n_k} = 1/k, \cdots$, where $\{n_k\}$ is a strictly increasing sequence of natural numbers. Given an arbitrary slowly increasing sequence $\{c_n\}$ of real numbers $c_n > 1$, tending to infinite, it is not difficult to define the sequence $\{n_k\}$ inductively so that $\lim_{k\to\infty} c_{n_k}/k = \infty$. It follows that for $k > 1$, if $n_{k-1} < n \leqq n_k$, then $c_n \| T^n \|^{1/n} > c_{n_{k-1}} \| T^{n_k} \|^{1/n_k} > c_{n_{k-1}} k^{-1}$. Since T is a quasinilpotent compact operator, from [10, Theorem 3.4], $T \in \Psi(\mathscr{H})$ but

$$\lim_{n\to\infty} c_n \|T^n\|^{1/n} = \infty .$$

This shows that the sufficient condition of Proposition 4.4 (or any reasonable relaxation of it) is not necessary. Moreover, $T \otimes 1_{\mathscr{K}}$ provides a similar example with a noncompact quasinilpotent and pseudonilpotent operator.

Finally, we pose two questions concerning pseudonilpotent operators which were already asked in [10].

PROBLEM 3. Is every quasinilpotent operator pseudonilpotent?

REMARK 4.6. From results of [3] and [4] it follows that every operator in $\mathscr{L}(\mathscr{H})$ has a nontrivial invariant subspace if and only if every operator in $\overline{\mathscr{N}(\mathscr{H})}$ has the same property.

PROBLEM 4. Does every pseudonilpotent operator have a nontrivial invariant subspace?

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