

Pacific Journal of Mathematics

**AN INVERSION OF THE S_2 TRANSFORM FOR GENERALIZED
FUNCTIONS**

L. S. DUBE

AN INVERSION OF THE S_2 TRANSFORM FOR GENERALIZED FUNCTIONS

L. S. DUBE

Define S_2 transform of a member f of a certain space of generalized functions as

$$F(x) = \langle f(t), K(t, x) \rangle$$

where

$$K(t, x) = \begin{cases} \frac{\log x/t}{x-t}, & x \neq t \\ \frac{1}{x}, & x = t \end{cases}$$

$$(0 < t < \infty, 0 < x < \infty).$$

It is shown that

$$\lim_{n \rightarrow \infty} H_{n,x}[F(x)] = f(x)$$

in the weak distributional sense. Here $H_{n,x}$ is a certain linear generalized differential operator.

1. **Introduction.** Schwartz [6, p. 248] first introduced the Fourier transform of distributions in 1947. Since then, extensions of the classical integral transformations to generalized functions have become of continuing interest. Some references to this effect are [2], [3], [4], [5], [8], [9] and [10]. The Stieltjes and iterated Stieltjes transforms of a function $f(t)$ have been defined respectively as

$$\tilde{f}(u) = \int_0^\infty \frac{f(t)}{u+t} dt, \quad u > 0$$

and

$$\tilde{\tilde{f}}(x) = \int_0^\infty \frac{du}{x+u} \int_0^\infty \frac{f(t)}{u+t} dt, \quad x > 0.$$

If it is permissible to change the order of integration in the above integral, one gets

$$(1) \quad \tilde{\tilde{f}}(x) = \int_{0+}^\infty \frac{\log x/t}{x-t} f(t) dt,$$

where $\log x/t/(x-t)$ is defined by its limiting value $1/x$ at $t = x$. (1) is referred to as the S_2 transform of the function $f(t)$ (see [1, p. 4]). The inversion formula for (1) due to Boas and Widder [1, p. 30] is given by

$$(2) \quad \lim_{n \rightarrow \infty} H_{n,x}[\tilde{f}(x)] = f(x),$$

for almost all $x > 0$, where

$$H_{n,x}[\phi(x)] = \left(\frac{1}{n!(n-2)} \right)^2 [x^{2n-1} \{x^{2n-1} \phi^{(n-1)}(x)\}^{(2n-1)}]^{(n)},$$

$n = 1, 2, \dots$.

Pandey [4] has defined the Stieltjes transform of an arbitrary element f of a generalized function space $S'_\alpha(I)$ as

$$(3) \quad G(s) = \left\langle f(t), \frac{1}{s+t} \right\rangle,$$

for s lying in the complex plane with a cut along the negative real axis. He has also proved both complex and real (for $s > 0$) inversion formulae for the transform (3).

It is natural to ask whether one can extend the classical iterated Stieltjes transform to a space of generalized functions. If $G(u)$ is the Stieltjes transform of $f \in S'_\alpha(I)$ for $u > 0$, as defined by (3), it seems reasonable to define the iterated Stieltjes transform of f as

$$(4) \quad F(x) = \left\langle G(u), \frac{1}{x+u} \right\rangle, \quad x > 0.$$

In order that the above definition be meaningful, $G(u)$ must belong to the space $S'_\alpha(I)$ as a regular generalized function. This we have not been able to show. In fact, $\int_0^\infty G(u)/(x+u) du$ ceases to exist in a neighborhood of zero, as $G(u) = 0(1/u)$, when $u \rightarrow 0+$ ([4, Corollary to Lemma 2a]). In this paper, we provide a partial solution to the present problem by defining the S_2 transform of generalized functions as in §3. In our definition of S_2 transform, the difficulty that occurs in justifying (4) does not arise. The inversion formula (2) is extended to a space of generalized functions in the sense of weak distributional convergence.

The notation and terminology will follow that of [3] and [11]. “ I ” denotes the open interval $(0, \infty)$ and t, x and u are real variables in I . If f is a generalized function, then $f(t)$ is used to indicate that the testing functions on which f is defined have t as their variable. The space of C^∞ -functions on I having compact support is denoted by $D(I)$ and its dual $D'(I)$ is the Schwartz distribution space.

2. The testing function space $S_\alpha(I)$. Let α be a fixed real number satisfying $0 < \alpha < 1$. $S_\alpha(I)$ is defined as the collection of all

C^∞ -functions $\phi(t)$ on $I = (0, \infty)$ such that

$$\gamma_k(\phi) = \sup_{0 < t < \infty} \left| t^\alpha \left(t \frac{d}{dt} \right)^k \phi(t) \right| < \infty ,$$

for each $k = 0, 1, 2, \dots$.

The topology of $S_\alpha(I)$ is generated by the seminorms $\{\gamma_k\}$ [11, p. 8]. A sequence $\{\phi_n\}$ converges to a function ϕ in the topology of $S_\alpha(I)$ if and only if

$$t^\alpha \left(t \frac{d}{dt} \right)^k \phi_n(t) \longrightarrow t^\alpha \left(t \frac{d}{dt} \right)^k \phi(t)$$

as $n \rightarrow \infty$, uniformly in t , for each $k = 0, 1, 2, \dots$. It turns out that $S_\alpha(I)$ is a locally convex, sequentially complete Hausdorff topological vector space. The dual space $S'_\alpha(I)$ consists of all linear continuous functionals on $S_\alpha(I)$. The space $D(I)$ is contained in $S_\alpha(I)$ and the topology of $D(I)$ is stronger than that induced on it by $S_\alpha(I)$. Hence the restriction of any $f \in S'_\alpha(I)$ to $D(I)$ is in $D'(I)$.

Regular generalized functions in $S'_\alpha(I)$. The regular generalized functions in $S'_\alpha(I)$ are characterized as follows:

If $f(t)$ is a locally integrable function such that $\int_0^\infty (|f(t)|/t^\alpha) dt < \infty$, then $f(t)$ generates a regular generalized function in $S'_\alpha(I)$ through the definition:

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t) dt , \quad \phi \in S_\alpha(I) .$$

The proof of the above statement follows easily in the lines of [11, V, p. 53].

Now define a function $K(t, x)$ on $(0 < t < \infty; 0 < x < \infty)$ as

$$(5) \quad K(t, x) = \begin{cases} \frac{\log x/t}{x-t} , & t \neq x \\ 1/x , & t = x . \end{cases}$$

For each fixed $x > 0$, $K(t, x)$ as a function of t belongs to $S_\alpha(I)$. In fact, taking the substitution $t - x = u$, the function $K(t, x)$ can be written as a power series in u with the centre $t = x$ and the radius of convergence x , which will imply that $K(t, x)$ is infinitely differentiable at $t = x$. That $K(t, x)$ is infinitely differentiable at $t \neq x$ is obvious. It follows now by a simple computation that $\gamma_k(K(t, x)) < \infty$ for a fixed $x > 0$ and for each $k = 0, 1, 2, \dots$.

3. The generalized S_2 transform. For $f \in S'_\alpha(I)$, we define the S_2 transform of f as a function $F(x)$ obtained by applying f on the

kernel $K(t, x)$, i.e.

$$(6) \quad F(x) = \langle f(t), K(t, x) \rangle, \quad x > 0,$$

where $K(t, x)$ is defined by (5).

The right hand side of (6) has a sense as $K(t, x)$ belongs to the testing function space $S_\alpha(I)$.

Following the technique used in [3, Th. 1] and applying the mathematical induction, one can show that $F(x)$ is an infinitely differentiable function and that

$$F^{(n)}(x) = \left\langle f(t), \frac{\partial^n}{\partial x^n} K(t, x) \right\rangle \quad \text{for each } x > 0$$

and $n = 1, 2, \dots$.

4. Inversion and uniqueness. Now we prove an inversion theorem for our generalized S_2 transform as follows:

THEOREM 1. *Let $f \in S'_\alpha(I)$ for $0 < \alpha < 1$ and let $F(x)$ be the S_2 transform of f as defined by (5). Then for an arbitrary $\phi(x) \in D(I)$ one has*

$$\langle H_{n,x}F(x), \phi(x) \rangle \longrightarrow \langle f, \phi \rangle \quad \text{as } n \longrightarrow \infty$$

where the operator $H_{n,x}$ is defined by (3) and the differentiation therein is understood in the distributional sense.

Proof. By a simple computation, the operator $H_{n,x}$ can be expressed as a polynomial in $(x(d/dx))$ of degree $4n - 2$. Let us denote this polynomial by $P(x(d/dx))$. The theorem will be proved by justifying steps:

$$\langle H_{n,x}F(x), \phi(x) \rangle = \left\langle P\left(x \frac{d}{dx}\right)F(x), \phi(x) \right\rangle$$

$$(7) \quad = \int_0^\infty \left[P\left(x \frac{d}{dx}\right)F(x) \right] \phi(x) dx$$

$$(8) \quad = \int_0^\infty F(x) P\left(-x \frac{d}{dx} - 1\right) \phi(x) dx$$

$$(8)' \quad = \int_0^\infty \langle f(t), K(t, x) \rangle P\left(-x \frac{d}{dx} - 1\right) \phi(x) dx$$

$$(9) \quad = \left\langle f(t), \int_0^\infty K(t, x) P\left(-x \frac{d}{dx} - 1\right) \phi(x) dx \right\rangle$$

$$(10) \quad \longrightarrow \langle f(t), \phi(t) \rangle, \quad \text{as } n \longrightarrow \infty,$$

where $(4n - 2)$ is the degree of the polynomial P .

The step (7) is obvious due to the fact that the function

$$P\left(x\frac{d}{dx}\right)F(x)$$

generates a regular distribution in $D'(I)$. The step (8) is obtained by applying integration by parts in (7) successively and using the fact that the support of $\phi(x)$ is contained in some open interval (a, b) , $0 < a < b < \infty$, so that the limit terms in the integration vanish. The limits of integration in both (8)' and (9) are essentially from a to b as the support of $P\left(-x\frac{d}{dx} - 1\right)\phi(x)$ is contained in (a, b) . Hence following the Riemann sum technique as used in [5, Th. 2] one can easily show that (8)' equals (9). In order to show that (9)→(10), we need prove that

$$t^\alpha\left(t\frac{d}{dt}\right)^k\left[\int_0^\infty K(t, x)P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx - \phi(t)\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

uniformly for all $t \in (0, \infty)$, for each $k = 0, 1, 2, \dots$.

Now

$$(11) \quad \left(t\frac{d}{dt}\right)\int_0^\infty K(t, x)P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx$$

$$(12) \quad = \int_a^b \left(t\frac{d}{dt}\right)[K(t, x)]P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx .$$

It can easily be checked that

$$\left(t\frac{d}{dt}\right)K(t, x) = \begin{cases} \left(-x\frac{d}{dx} - 1\right)K(t, x), & \text{when } t \neq x \\ \left(x\frac{d}{dx}\right)K(t, x), & \text{when } t = x. \end{cases}$$

Therefore (12) can be written without any change in the value of the integral as

$$\begin{aligned} & \int_a^b \left[\left(-x\frac{d}{dx} - 1\right)K(t, x) \right] P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx \\ & = \int_a^b K(t, x)\left(x\frac{d}{dx}\right)\left[P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx \right], \\ & \hspace{15em} \text{(by integration by parts)} \\ & = \int_a^b K(t, x)P\left(-x\frac{d}{dx} - 1\right)\left(x\frac{d}{dx}\right)\phi(x)dx \\ & = \int_a^b \left[P\left(x\frac{d}{dx}\right)K(t, x) \right]\left(x\frac{d}{dx}\right)\phi(x)dx, \\ & \hspace{15em} \text{(by integration by parts).} \end{aligned}$$

Hence applying $(t(d/dt))$ successively on the integral in (11) we get for any non-negative integer k

$$\begin{aligned} \left(t \frac{d}{dt}\right)^k \int_0^\infty K(t, x) P\left(-x \frac{d}{dx} - 1\right) \phi(x) dx &= \int \left[P\left(x \frac{d}{dx}\right) K(t, x) \right] \left(x \frac{d}{dx}\right)^k \phi(x) dx \\ &= \int_0^\infty [H_{n,x}] K(t, x) \left(x \frac{d}{dx}\right)^k \phi(x) dx \\ &= \int_0^\infty F_n(t, x) \left(x \frac{d}{dx}\right)^k \phi(x) dx, \end{aligned}$$

where

$$F_n(t, x) = d_n^2 x^{n-1} t^n \int_0^\infty \frac{u^{2n-1}}{(x+u)^{2n}(t+u)^{2n}} du, \tag{[1, Cor. 6.1.1, p. 20]}$$

$$d_n = (2n - 1)! c_n; c_1 = 1 \quad \text{and} \quad c_n = \frac{1}{n!(n-2)!}, \quad n \geq 2.$$

Also in view of [1, Lemma 7.2, p. 21], for $n \geq 2$

$$\int_0^\infty F_n(t, x) dx = \left(\frac{n-1}{n}\right)^2 \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence as $n \rightarrow \infty$,

$$\begin{aligned} \left(t \frac{d}{dt}\right)^k \left[\int_0^\infty K(t, x) P\left(-x \frac{d}{dx} - 1\right) \phi(x) dx - \phi(t) \right] \\ = \int_0^\infty F_n(t, x) \left[\left(x \frac{d}{dx}\right)^k \phi(x) - \left(t \frac{d}{dt}\right)^k \phi(t) \right] dx; \end{aligned}$$

here $(x(d/dx))^k \phi(x) \in D(I)$.

Now it suffices to show that

$$(13) \quad t^\alpha \int_0^\infty F_n(t, x) [\psi(x) - \psi(t)] dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

uniformly for all $t > 0$ for any $\psi(x) \in D(I)$.

Taking the substitution $x = ty$ and using the fact that $F_n(t, x)$ is homogeneous of degree-1 we get

$$\begin{aligned} \int_0^\infty F_n(t, x) [\psi(x) - \psi(t)] dx &= \int_0^\infty F_n(1, x) [\psi(xt) - \psi(t)] dx \\ &= \left(\int_0^{1-\gamma} + \int_{1-\gamma}^{1+\gamma} + \int_{1+\gamma}^\infty \right) F_n(1, x) [\psi(xt) - \psi(t)] dx \end{aligned}$$

where η is taken to be a positive number less than $1/2$.

In view of [1, Lemmas 7.2 and 8.2] it follows that

$$(14) \quad \int_0^{1-\eta} F_n(1, x) dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$(15) \quad \int_{1-\eta}^{1+\eta} F_n(1, x) dx \longrightarrow 1 \quad \text{as } n \longrightarrow \infty$$

and

$$(16) \quad \int_{1+\eta}^{\infty} F_n(1, x) dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty .$$

Let $\sup_{0 < t < \infty} t^\alpha \psi(t) = M$, which clearly exists. Then

$$(17) \quad \left| t^\alpha \int_0^{1-\eta} F_n(1, x) [\psi(xt) - \psi(t)] dx \right| \leq 2M \int_0^{1-\eta} F_n(1, x) dx \longrightarrow 0$$

as $n \rightarrow \infty$ uniformly for all $t > 0$ in view of (14).

Similarly, using (16) we get

$$(18) \quad \left| t^\alpha \int_{1+\eta}^{\infty} F_n(1, x) [\psi(xt) - \psi(t)] dx \right| \longrightarrow 0$$

as $n \rightarrow \infty$ uniformly for all $t > 0$.

Finally, in view of [7, Lemma 5, p. 287], and the fact that ψ has a compact support on I , for a given $\varepsilon > 0$ there exists a positive $\eta < 1/2$ such that

$$|t^\alpha [\psi(xt) - \psi(t)]| < \varepsilon ,$$

uniformly for all $t > 0$ and for all $x \in (1 - \eta, 1 + \eta)$. Hence the application of (15) leads to

$$(19) \quad \left| \int_{1-\eta}^{1+\eta} t^\alpha F_n(1, x) [\psi(xt) - \psi(t)] dx \right| < \varepsilon \int_{1-\eta}^{1+\eta} F_n(1, x) dx \longrightarrow \varepsilon$$

as $n \rightarrow \infty$ uniformly for all $t > 0$.

Combining (17), (18) and (19) in which ε is arbitrary, (13) is established. This completes the proof of the theorem.

THEOREM 2 (Uniqueness). *Let f and g be two members of $S'_\alpha(I)$ and let $F(x)$ and $G(x)$ be their S_2 transforms respectively as defined by (5). If $F(x) = G(x)$ for all $x > 0$ then $f = g$ in the sense of equality in $D'(I)$.*

Proof. For an arbitrary $\phi \in D(I)$,

$$\begin{aligned} \langle f - g, \phi \rangle &= \lim_{n \rightarrow \infty} \langle H_{n,x}(F(x) - G(x)), \phi(x) \rangle, \\ & \text{(by Theorem 1).} \\ &= 0, \quad \text{since } F(x) = G(x) \text{ for all } x > 0. \end{aligned}$$

Hence $f = g$ in $D'(I)$.

An open problem. We state the following open problem related to the present work:

Can one justify the definition of the iterated Stieltjes transform of generalized functions as given by (4)? In order to do this, some modifications in the asymptotic order of $G(u)$, and in the characterization of regular generalized functions of $S'_\alpha(I)$ as given in §2, might be needed.

Granted that (4) is well defined, can one prove the equivalence of the S_2 and iterated Stieltjes transforms of generalized functions?

Acknowledgement. The author is thankful to the referee for his suggestions.

REFERENCES

1. R. P. Boas, Jr. and D. V. Widder, *The iterated Stieltjes transform*, Trans. Amer. Math. Soc., **45** (1939), 1-72.
2. J. L. B. Cooper, *Laplace transformations of distributions*, Canad. J. Math., **18** (1966), 1325-1332.
3. L. S. Dube and J. N. Pandey, *On the Hankel transform of distributions*, Tohoku Math. J., **27** (1975), 337-354.
4. J. N. Pandey, *On the Stieltjes transform of generalized functions*, Proc. Camb. Phil. Soc., **71** (1972), 85-96.
5. J. N. Pandey and A. H. Zemanian, *Complex inversion for the generalized convolution transform*, Pacific J. Math., **25** (1968), 147-157.
6. L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
7. D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, 1946.
8. A. H. Zemanian, *Inversion formulas for the distributional Laplace transformations*, SIAM J. Appl. Math., **14** (1966), 159-166.
9. ———, *A distributional K-transformation*, SIAM J. Appl. Math., **14** (1966), 1350-1365.
10. ———, *A generalized Weierstrass transformation*, SIAM J. Appl. Math., **15** (1967), 1088-1105.
11. ———, *Generalized Integral Transformations*, Interscience Publishers, 1968.

Received December 10, 1974 and in revised form July 24, 1975. This work was supported by the National Research Council Grant No. 240-123.

CONCORDIA UNIVERSITY, MONTREAL, P. Q.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Graham Donald Allen, Francis Joseph Narcowich and James Patrick Williams, <i>An operator version of a theorem of Kolmogorov</i>	305
Joel Hilary Anderson and Ciprian Foias, <i>Properties which normal operators share with normal derivations and related operators</i>	313
Constantin Gelu Apostol and Norberto Salinas, <i>Nilpotent approximations and quasinilpotent operators</i>	327
James M. Briggs, Jr., <i>Finitely generated ideals in regular F-algebras</i>	339
Frank Benjamin Cannonito and Ronald Wallace Gatterdam, <i>The word problem and power problem in 1-relator groups are primitive recursive</i>	351
Clifton Earle Corzatt, <i>Permutation polynomials over the rational numbers</i>	361
L. S. Dube, <i>An inversion of the S_2 transform for generalized functions</i>	383
William Richard Emerson, <i>Averaging strongly subadditive set functions in unimodular amenable groups. I</i>	391
Barry J. Gardner, <i>Semi-simple radical classes of algebras and attainability of identities</i>	401
Irving Leonard Glicksberg, <i>Removable discontinuities of A-holomorphic functions</i> ...	417
Fred Halpern, <i>Transfer theorems for topological structures</i>	427
H. B. Hamilton, T. E. Nordahl and Takayuki Tamura, <i>Commutative cancellative semigroups without idempotents</i>	441
Melvin Hochster, <i>An obstruction to lifting cyclic modules</i>	457
Alistair H. Lachlan, <i>Theories with a finite number of models in an uncountable power are categorical</i>	465
Kjeld Laursen, <i>Continuity of linear maps from C^*-algebras</i>	483
Tsai Sheng Liu, <i>Oscillation of even order differential equations with deviating arguments</i>	493
Jorge Martinez, <i>Doubling chains, singular elements and hyper-Z l-groups</i>	503
Mehdi Radjabalipour and Heydar Radjavi, <i>On the geometry of numerical ranges</i>	507
Thomas I. Seidman, <i>The solution of singular equations, I. Linear equations in Hilbert space</i>	513
R. James Tomkins, <i>Properties of martingale-like sequences</i>	521
Alfons Van Daele, <i>A Radon Nikodým theorem for weights on von Neumann algebras</i>	527
Kenneth S. Williams, <i>On Euler's criterion for quintic nonresidues</i>	543
Manfred Wischnewsky, <i>On linear representations of affine groups. I</i>	551
Scott Andrew Wolpert, <i>Noncompleteness of the Weil-Petersson metric for Teichmüller space</i>	573
Volker Wrobel, <i>Some generalizations of Schauder's theorem in locally convex spaces</i>	579
Birge Huisgen-Zimmermann, <i>Endomorphism rings of self-generators</i>	587
Kelly Denis McKennon, <i>Corrections to: "Multipliers of type (p, p)"; "Multipliers of type (p, p) and multipliers of the group L_p-algebras"; "Multipliers and the group L_p-algebras"</i>	603
Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>Correction to: "a^*-closures to completely distributive lattice-ordered groups"</i>	606
Zvi Arad and George Isaac Glauberman, <i>Correction to: "A characteristic subgroup of a group of odd order"</i>	607
Roger W. Barnard and John Lawson Lewis, <i>Correction to: "Subordination theorems for some classes of starlike functions"</i>	607
David Westreich, <i>Corrections to: "Bifurcation of operator equations with unbounded linearized part"</i>	608