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COMMUTATIVE CANCELLATIVE SEMIGROUPS WITHOUT IDEMPOTENTS

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A commutative cancellative idempotent-free semigroup (CCIF-) S can be described in terms of a commutative cancellative semigroup C with identity, an ideal of C, and a function of $C \times C$ into integers. If C is an abelian group, S has an archimedean component as an ideal; S is called an $\bar{\mathfrak{N}}$-semigroup. A CCIF-semigroup of finite rank has nontrivial homomorphism into nonnegative real numbers.

1. **Introduction.** In this paper, a commutative cancellative semigroup without idempotent is called a CCIF-semigroup (in which, by "IF" we mean "idempotent-free") and a commutative cancellative semigroup with identity is called a CCI-semigroup. In particular, an $\mathfrak{N}$-semigroup is an archimedean CCIF-semigroup. The structure of $\mathfrak{N}$-semigroups has been much studied [1, 2, 3, 6, 7, 8] and also it is well known that every CCIF-semigroup is a semilattic of $\mathfrak{N}$-semigroups. In this paper CCIF-semigroups will be studied by means of the representation by the generalized $\mathscr{I}$- and φ-functions and also through homomorphisms into the nonnegative real numbers.

Throughout this paper, $\boldsymbol{R}$ denotes the set of real numbers; R the set of rational numbers; $\boldsymbol{R}_+$ the set of positive real numbers; $\boldsymbol{R}_+^0$ the set of nonnegative real numbers; Z_+ the set of positive integers and Z_+^0 the set of nonnegative integers. Each of these is a semigroup under the usual addition. If S is a semigroup and if X is a subsemigroup of the group $\boldsymbol{R}$, then the notation Hom(S, X) denotes the semigroup of homomorphisms of S into X under the usual operation.

At the end of §1 we show that if S is a CCIF-semigroup, Hom$(S, \boldsymbol{R}) \neq \{0\}$, and the homomorphism group is transitive in some sense. In Section 2 we shall try to generalize the representation of $\mathfrak{N}$-semigroups to CCIF-semigroups. It will be understood as the so-called Schreier's extension to build up complicated CCIF-semigroups from simpler CCIF-semigroups. Most of the results in [7] will be extended to CCIF-semigroups. In §3 we shall treat the important case, i.e., the case where the structure semigroup is a group. Such a CCIF-semigroup will be called an $\bar{\mathfrak{N}}$-semigroup. In §4 we shall show that every CCIF-semigroup of finite rank has a nontrivial homomorphism into $\boldsymbol{R}_+^0$. In particular we will characterize CCIF-semigroups S having the property Hom$(S, \boldsymbol{R}_+) \neq \varnothing$.

(1.1) *Let S be a* CCIF-*semigroup. Then $x \neq xy$ for all $x, y \in S$.*

Proof. Suppose, for some $x, y \in S$, we have $x = xy$. Then $xy = xy^2$ which implies $y = y^2$ by cancellation. This is a contradiction.

PROPOSITION 1.2. *Let S be a* CCIF-*semigroup.*

(1.2.1) Hom $(S, \boldsymbol{R})$ *is a nontrivial vector space over the field* $\boldsymbol{R}$.

(1.2.2) *For each $a \in S$ and each $r \in \boldsymbol{R}$, $r \neq 0$, there is an $h \in$* Hom $(S, \boldsymbol{R})$ *such that $h(a) = r$.*

Proof of (1.2.1). Let S be a CCIF-semigroup. Let $Q(S)$ be the quotient group of S (i.e., the group of quotients of S), and $D(S)$ be the divisible hull of $Q(S)$

$$D(S) = \bigoplus_{\alpha \in \Gamma} R_\alpha \oplus \bigoplus_{p \in \Delta} C(p^\infty) . \qquad (1.2.3)$$

$D(S)$ is a direct sum of copies R_α of the group of rational numbers under addition and quasi-cyclic groups $C(p^\infty)$ with respect to prime number p. We view S as a subsemigroup of $D(S)$. Let π_α be the projection of $D(S)$ upon R_α for each $\alpha \in \Gamma$. Let x be an element of S. Suppose $\pi_\alpha(x) = 0$ for each $\alpha \in \Gamma$. It follows that $x \in \bigoplus_{p \in \Delta} C(p^\infty)$, a torsion group. This is a contradiction as x has infinite order. Thus, for some $\alpha_0 \in \Gamma$, $\pi_{\alpha_0}(x) \neq 0$. Note that $\pi_{\alpha_0} \in$ Hom $(S, \boldsymbol{R})$ and is not the trivial homomorphism. It is obvious that Hom $(S, \boldsymbol{R})$ is a vector space over $\boldsymbol{R}$ in the usual way.

Proof of (1.2.2). Let $a \in S$ and $r \in \boldsymbol{R}$ be given. In establishing (1.2.1), we have shown that there exists $h_1 \in$ Hom $(S, \boldsymbol{R})$ with $h_1(a) \neq 0$. Let $s = h_1(a)$. Now define h by $h = (r/s)h_1$. Then $h(a) = r$, and $h \in$ Hom $(S, \boldsymbol{R})$.

2. Schreier Extension.

We consider the following problem. Let C be a CCI-semigroup and ε be its identity. Given C, find all CCIF-semigroups S such that there is a homomorphism $\mathscr{P}$ of S onto C satisfying the condition.

$$\{x \in S \mid \mathscr{P}(x) = \varepsilon\} \cong Z_+ .$$

In this section we shall show that S always exists for every C and shall describe S in terms of elements of C, integers and a certain function of $C \times C$ into the integers. The extension S is called a Schreier extension (of Z_+) by C. (The terminology is due to [5].) Schreier extension by C is significant because we shall see that every CCIF-semigroup is isomorphic to a Schreier extension by some CCI-semigroup C.

THEOREM 2.1. *Let C be a* CCI-*semigroup and C_1 a proper ideal* *of* C. (C_1 can be empty.) *Let* $I\colon C \times C \to Z$ *be a function which satisfies*

(2.1.1) $I(\alpha, \beta) \in Z_+^0$ if $\alpha\beta \notin C_1$

(2.1.2) $I(\alpha, \beta) = I(\beta, \alpha)$ *for all* $\alpha, \beta \in C$

(2.1.3) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$ *for all* $\alpha, \beta, \gamma \in C$

(2.1.4) $I(\varepsilon, \alpha) = 1$ (ε the identity element of C) *for all* $\alpha \in C$.

Given C, C_1, I, *the set* $(C, C_1; I)$ *with its operation is defined by*

$$(C, C_1; I) = \{(x, \alpha) \in Z \times C; x \in Z_+^0 \text{ if } \alpha \notin C_1\}$$

(2.1.5) $(x, \alpha)(y, \beta) = (x + y + I(\alpha, \beta), \alpha\beta)$.

Then $(C, C_1; I)$ *is a* CCIF-*semigroup*.

Conversely if S *is a* CCIF-*semigroup, then* $(S \cong C, C_1; I)$ *for some* C, C_1, I.

Proof. It is routine to prove that $(C, C_1; I)$ is a commutative cancellative simigroup. To show idempotent-freeness, assume $(x, \alpha)^2 = (x, \alpha)$, that is, $\alpha^2 = \alpha$ and $2x + I(\alpha, \alpha) = x$. It follows that $\alpha = \varepsilon$ and $x + 1 = 0$. Since C_1 is a proper ideal of C, $\varepsilon \notin C_1$, hence $x \geqq 0$ and we arrive at a contradiction.

Conversely assume that S is a CCIF-semigroup. Let $a \in S$, and define a relation ρ_a on S by

(2.1.6) $x\rho_a y$ iff $a^m x = a^n y$ for some $m, n \in Z_+$.

It is easy to see that ρ_a is a congruence relation. To show that S/ρ_a is cancellative, assume $xz\rho_a yz$. Then $a^m xz = a^n yz$ for some $m, n \in Z_+$. Since S is cancellative, we get $a^m x = a^n y$, i.e., $x\rho_a y$. Obviously $ax\rho_a x$ for all $x \in S$, that is, the ρ_a-class containing a is the identity of S/ρ_a. Let $C = S/\rho_a$. C is a CCI-semigroup. In each ρ_a-class define $x \leqq_a y$ by $x = a^m y$ for some $m \in Z_+^0$ where $a^0 y = y$. Because of cancellation, each ρ_a-class forms a chain with respect to $\leqq_a$. Let $T = \bigcap_{n=1}^{\infty} a^n S$ and let C_1 be the image of T under the natural homomorphism $S \to C$. If $T \neq \varnothing$, it is a proper ideal of S (since $a \notin T$) and thus C_1 is a proper ideal of C. Under the homomorphism $S \to C$ we have a partition of S: $S = \bigcup_{\xi \in C} S_\xi$. If $\xi \in C \backslash C_1$, S_ξ contains a maximal element with respect to $\leqq_a$; but if $\xi \in C_1$, S_ξ contains no maximal element. For each $\xi \in C$, define p_ξ to be $a \leqq_a$-maximal element in S_ξ if $\xi \in C \backslash C_1$, and p_ξ to be arbitrarily chosen from S_ξ if $\xi \in C_1$. Since C_1 is a proper ideal, $\varepsilon \notin C_1$, hence $p_\varepsilon = a$ because of (1.1). Then every element of S has a unique expression

$$x = a^m p_\xi \text{ where } m \in Z \text{ if } \xi \in C_1;\ m \in Z_+^0 \text{ if } \xi \in C \backslash C_1 .$$

Define $I\colon C \times C \to Z$ as follows:

$$p_\alpha p_\beta = a^{I(\alpha, \beta)} p_{\alpha B} .$$

It is easy to see that I satisfies (2.1.1), (2.1.2), (2.1.3) and (2.1.4). S is isomorphic to $(C, C_1; I)$ under the map $a^m p_\xi \mapsto (m, \xi)$.

The representation $(C, C_1; I)$ of S depends on the choice of a. The element a is called the standard element of the representation $(C, C_1; I)$ of S. S/ρ_a is called the structure CCI-semigroup of S with respect to a; C is the structure CCI-semigroup of $(C, C_1; I)$, and $(0, \varepsilon)$ is the standard element. A function $I: C \times C \to Z$ satisfying (2.1.1), (2.1.2), (2.1.3), (2.1.4) is called an $\mathscr{I}$-function on (C, C_1).

THEOREM 2.2. *Let C be a* CCI-*semigroup, and C_1 be a proper ideal of C.* (C_1 can be empty.) *Assume that $\varphi: C \to \boldsymbol{R}$ satisfies*

(2.2.1) $\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta) \in \begin{cases} Z & \text{if } \alpha\beta \in C_1 \\ Z_+^0 & \text{if } \alpha\beta \notin C_1. \end{cases}$

(2.2.2) $\varphi(\varepsilon) = 1.$

Given C, φ, and C_1, define $((C, C_1; \varphi))$ by

(2.2.3) $((C, C_1; \varphi)) = \{((x + \varphi(\alpha), \alpha)): \alpha \in C, x \in Z, x \in Z_+^0 \text{ if } \alpha \notin C_1\}$

and

(2.2.4) $((x + \varphi(\alpha), \alpha))((y + \varphi(\beta), \beta)) = ((x + y + \varphi(\alpha) + \varphi(\beta), \alpha\beta)).$

Then $((C, C_1; \varphi))$ is a CCIF-*semigroup.*

Conversely every CCIF-*semigroup is isomorphic to $((C, C_1; \varphi))$ for some C, φ and C_1, that is, $(C, C_1; I) \cong ((C, C_1; \varphi))$ under $(x, \alpha) \to ((x + \varphi(\alpha), \alpha))$, $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)$.*

Proof. Assume S is a CCIF-semigroup. By Theorem 2.1, we let $S = (C, C_1; I)$ for some C, I, C_1. By (1.2.2), there is an $h \in \mathrm{Hom}(S, \boldsymbol{R})$ such that $h(0, \varepsilon) \neq 0$. Define $\varphi: C \to \boldsymbol{R}$ by

(2.2.5) $\varphi(\alpha) = \dfrac{h(0, \alpha)}{h(0, \varepsilon)}.$

If $I(\alpha, \beta) \geqq 0$, then $(0, \alpha)(0, \beta) = (0, \varepsilon)^{I(\alpha,\beta)}(0, \alpha\beta)$ implies

$$h(0, \alpha) + h(0, \beta) = I(\alpha, \beta) \cdot h(0, \varepsilon) + h(0, \alpha\beta) .$$

If $I(\alpha, \beta) < 0$, then $(0, \alpha)(0, \beta)(0, \varepsilon)^{-I(\alpha,\beta)} = (0, \alpha\beta)$ implies

$$h(0, \alpha) + h(0, \beta) - I(\alpha, \beta) \cdot h(0, \varepsilon) = h(0, \alpha\beta) .$$

In both cases, using (2.2.5), we have

(2.2.6) $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)$ for all $\alpha, \beta \in C$. It is easy to see that φ satisfies (2.2.1) and (2.2.2); and $S = (C, C_1; I) \cong ((C, C_1; \varphi))$ under $(x, \alpha) \mapsto ((x + \varphi(\alpha), \alpha))$.

Conversely assume φ satisfies (2.2.1) and (2.2.2), define $((C, C_1; \varphi))$ by (2.2.3) and (2.2.4), and define I by (2.2.6). Then we can see that I satisfies (2.1.1), (2.1.2), (2.1.3) and (2.1.4), and $((x, \alpha)) \mapsto (x - \varphi(\alpha), \alpha)$ gives an isomorphism of $((C, C_1; \varphi))$ to $(C, C_1; I)$.

A function $\varphi: C \rightarrow \boldsymbol{R}$ is called a defining function on (C, C_1) if it satisfies (2.2.1) and (2.2.2); let $\mathrm{Dfn}(C, C_1, \boldsymbol{R})$ denote the set of all defining functions on (C, C_1). If φ satisfies (2.2.6) for a fixed I, φ is called a defining function belonging to I, and the set of all φ belonging to I is denoted by $\mathrm{Dfn}_I(C, C_1, \boldsymbol{R})$.

COROLLARY 2.3. *S is a CCIF-semigroup if and only if S is isomorphic to the subdirect product of a CCI-semigroup C and a subsemigroup of $\boldsymbol{R}$ by means of φ on C* (i.e., by means of φ with (2.2.1) and (2.2.2) in the sense of (2.2.4)).

COROLLARY 2.4. *Let S be a CCIF-semigroup. S is a subdirect product of a subsemigroup P of $\boldsymbol{R}_+^0$ and a CCI-semigroup C if and only if there exists $h \in \mathrm{Hom}((S, \boldsymbol{R}_+^0)$ with $h \neq 0$.*

The problem posed at the beginning of the section is solved, that is,

$$\mathscr{P}: ((x + \varphi(\alpha), \alpha)) \longrightarrow \alpha$$

has kernel $K = \{((x + 1, \varepsilon)): x \in Z_+^0\}$ and $K \cong Z_+$ under $((x + 1, \varepsilon)) \rightarrow x + 1$.

Let $S = (C, C_1; I)$.

PROPOSITION 2.5. *Let $\varphi_0 \in \mathrm{Dfn}_I(C, C_1, \boldsymbol{R})$ be fixed. If $f \in \mathrm{Hom}(C, \boldsymbol{R})$ then $\varphi = \varphi_0 + f \in \mathrm{Dfn}_I(C, C_1, \boldsymbol{R})$. Every element φ of $\mathrm{Dfn}_I(C, C_1, \boldsymbol{R})$ can be obtained in this manner.*

PROPOSITION 2.6 (2.6.1). *Let $\varphi_0 \in \mathrm{Dfn}_I(C, C_1, \boldsymbol{R})$ be fixed and $f \in \mathrm{Hom}(C, \boldsymbol{R})$. Define $h: S \rightarrow \boldsymbol{R}$ by*

$$h(x, \alpha) = s(x + \varphi_0(\alpha) + f(\alpha)) \,, \quad s \in \boldsymbol{R} \,.$$

Then $h \in \mathrm{Hom}(S, \boldsymbol{R})$ Every element h of $\mathrm{Hom}(S, \boldsymbol{R})$ satisfying $h(0, \varepsilon) \neq 0$ can be obtained in this manner.

(2.6.2) *Let $p: S \rightarrow C$ be the natural homomorphism. Then every h of $\mathrm{Hom}(S, \boldsymbol{R})$ satisfying $h(0, \varepsilon) = 0$ is obtained by $h = fp$ where $f \in \mathrm{Hom}(C, \boldsymbol{R})$.*

Proof (2.6.1). As the former half is easily proved, we prove the latter half. By (1.2.1) $\mathrm{Hom}(S, \boldsymbol{R}) \neq \{0\}$, so there is h such that $h(0, \varepsilon) \neq 0$. If $x \geqq 0$,

$$\begin{aligned} h(x, \alpha) &= h((0, \varepsilon)^x(0, \alpha)) = x \cdot h(0, \varepsilon) + h(0, \alpha) \\ &= h(0, \varepsilon)(x + \varphi(\alpha)) = s(x + \varphi(\alpha)) \end{aligned}$$

where $s = h(0, \varepsilon)$; $\varphi(\alpha) = h(0, \alpha)/h(0, \varepsilon)$, $\varphi \in \mathrm{Dfn}_I(C, C_1, \boldsymbol{R})$. If $x = 0$, $(0, \varepsilon)^x$ is regarded as void. If $x < 0$, $-x - 1 \geqq 0$, then

$$\begin{aligned} h(0, \alpha) &= h((-x-1, \varepsilon)(x, \alpha)) = h((0, \varepsilon)^{-x}(x, \alpha)) \\ &= (-x) \cdot h(0, \varepsilon) + h(x, \alpha) \end{aligned}$$

hence $h(x, \alpha) = h(0, \varepsilon)(x + \varphi(\alpha))$. By Proposition 2.5, φ is expressed as $\varphi_0 + f$. Thus we have the conclusion.

Proof. (2.6.2) Let $h \in \mathrm{Hom}(S, \boldsymbol{R})$ with $h(0, \varepsilon) = 0$. If $x \geqq 0$, $h(x, \alpha) = x \cdot h(0, \varepsilon) + h(0, \alpha) = h(0, \alpha)$. If $x < 0$, $h(0, \alpha) = (-x) \cdot h(0, \varepsilon) + h(x, \alpha) = h(x, \alpha)$. Hence $h(x, \alpha) = h(0, \alpha)$ for all $(x, \alpha) \in S$. Define $f: C \to \boldsymbol{R}$ by $f(\alpha) = h(x, \alpha)$ where $(x, \alpha) \in S$. By the above result, f is well defined. Now

$$fp(x, \alpha) = f(\alpha) = h(x, \alpha)\ , \quad \text{hence } h = fp\ .$$

It is easy to see that $fp \in \mathrm{Hom}(S, \boldsymbol{R})$ with $fp(0, \varepsilon) = 0$.

By the notation $S = (C, C_1; I) = ((C, C_1; \varphi))$ we mean that S has representation $(C, C_1; I)$ and $((C, C_1; \varphi))$ identifying (x, α) of $(C, C_1; I)$ with $((x + \varphi(\alpha), \alpha))$ of $((C, C_1; \varphi))$.

PROPOSITION 2.7. *Let S be a CCIF-semigroup. If $a \in S$ and if there is an $h \in \mathrm{Hom}(S, \boldsymbol{R}^0_+)$ such that $h(a) \neq 0$, then $C_1 = \varnothing$ using a as the standard element.*

Proof. Let $S = (C, C_1; I) = ((C, C_1; \varphi))$ and let a denote $(0, \varepsilon)$ in $(C, C_1; I)$ and at the same time $((1, \varepsilon))$ in $((C, C_1; \varphi))$. Let $\alpha \in C_1$. Then $(x, \alpha) \in (C, C_1; I)$ for all $x \in Z$. By Proposition 2.6

$$h(x, \alpha) = h(0, \varepsilon)(x + \varphi(\alpha))\ .$$

Since $h(0, \varepsilon) > 0$ and x is arbitrary, $h(x, \alpha) < 0$ if, $x < -\varphi(\alpha)$; a contradiction to the assumption. Hence $C_1 = \varnothing$.

A subsemigroup T of a commutative semigroup S is called confinal if, for every $x \in S$, there is a $y \in S$ such that $xy \in T$. Let $S = (C_1, C; I)$. The following are easily obtained.

LEMMA 2.8.

(2.8.1) *If $C \backslash C_1$ contains a cofinal subsemigroup of C, then $C_1 = \varnothing$.*

(2.8.2) *If C is an abelian group, then $C_1 = \varnothing$.*

We will now make a further investigation into defining functions and C_1.

Let U denote the group of units of C. Let φ be a function

$C \to \boldsymbol{R}$. Define a set $D_C(\varphi)$ by

$$D_C(\varphi) = \{\alpha \in C : \varphi(\xi) + \varphi(\eta) - \varphi(\alpha) < 0 \text{ for some } \xi, \eta \in C \text{ with } \alpha = \xi\eta\}.$$

We define defining functions from the point of C.

DEFINITION 2.9.

(2.9.1) A function $\varphi : C \to \boldsymbol{R}$ is called a *defining function on* C if it satisfies

$$\begin{cases} \varphi(\varepsilon) = \mathbf{1}\,. \\ \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta) \in Z \text{ for all } \alpha, \beta \in C\,. \\ D_C(\varphi) \subseteqq C \backslash U\,. \end{cases}$$

The set of defining functions on C is denoted by $\mathrm{Dfn}\,(C, \boldsymbol{R})$.

(2.9.2) A defining function on C is called a normal defining function on C if $D_C(\varphi) = \varnothing$, and a nonnormal defining function on C if $D_C(\varphi) \neq \varnothing$. $D_C(\varphi)$ is called the nonnormal domain of φ. The set of normal defining functions on C is denoted by $\mathrm{NDfn}\,(C, \boldsymbol{R})$.

PROPOSITION 2.10. *Let* $\varphi : C \to \boldsymbol{R}$ *be a defining function on* C. *Let* C_1 *be a proper ideal of* C *such that* $D_C(\varphi) \subseteqq C_1$. *Then* $\varphi \in \mathrm{Dfn}\,(C, C_1, \boldsymbol{R})$. *Conversely every defining function on* (C, C_1) *is a defining function on* C.

The following three cases are possible:
(i) φ is normal and $C_1 = \varnothing$
(ii) φ is normal and $C_1 \neq \varnothing$
(iii) φ is not normal and $C_1 \neq \varnothing$.

DEFINITION. In each case we consider the CCIF-semigroup $((C, C_1; \varphi))$. $((C, C_1; \varphi))$ is called a normal representation in case (i); seminormal representation in case (ii); nonnormal representation in case (iii). In case (i), $((C, C_1; \varphi))$ is denoted by $((C; \varphi))$. When φ is normal (nonnormal), the $\mathscr{I}$-function I defined by $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)$ is called normal (nonnormal); the corresponding semigroup is denoted by $(C, C_1; I)$, in particular $(C; I)$ in case (i).

PROPOSITION 2.11. *Let* $S = ((C, C_1; \varphi))$ *with standard element* a. *Then* $((C, C_1; \varphi))$ *is a normal representation if and only if* $\bigcap_{n=1}^{\infty} a^n S = \varnothing$.

PROPOSITION 2.12. *For every* CCI*-semigroup* C *there exist normal defining functions on* C. *If* C *is a* CCI*-semigroup and* C_1 *is a non-*

empty proper ideal of C, there exist nonnormal defining functions φ *such that the nonnormal domain of* φ *is contained in* C_1.

EXAMPLES 2.13. Let C be a CCI-semigroup.

(2.13.1) Define φ by

$$\varphi(\alpha) = 1 \quad \text{for all } \alpha \in C .$$

Then $\varphi \in \mathrm{NDfn}(C, \boldsymbol{R})$, and $((C; \varphi)) \cong Z_+ \times C$.

(2.13.2) Let U be the group of units of C. Let φ_0 be a nonnegative integer valued normal defining function on U. Define $\varphi: C \to Z_+^0$ by

$$\varphi(\alpha) = \begin{cases} \varphi_0(\alpha) & \text{if } \alpha \in U \\ c & \text{if } \alpha \notin U \end{cases}$$

where c is a constant nonnegative integer. Then φ is a normal defining function on C.

(2.13.3) Let C_1 be a nonempty proper ideal of C. Define φ by

$$\varphi(\alpha) = \begin{cases} 1 & \alpha \notin C_1 \\ -1 & \alpha \in C_1 . \end{cases}$$

The φ is a nonnormal defining function on C such that $D_C(\varphi) \subseteqq C_1$.

(2.13.4) Assume that ε is the only unit of C. Suppose $\varphi_0: C \backslash \{\varepsilon\} \to \boldsymbol{R}$ satisfies, for all $\alpha, \beta \in C \backslash \{\varepsilon\}$.

$$\varphi_0(\alpha) + \varphi_0(\beta) - \varphi_0(\alpha\beta) \in Z .$$

Define $\varphi: C \to \boldsymbol{R}$ by

$$\varphi(\alpha) = \begin{cases} 1 & \alpha = \varepsilon \\ \varphi_0(\alpha) & \alpha \neq \varepsilon . \end{cases}$$

Then φ is a defining function on C.

As another example, consider the case $C = Z_+^0$.

(2.14) Let $C = Z_+^0$. Let $\delta: Z_+ \to Z$ be a function with $\delta(1) = 0$ and let r be a real number. Define $\varphi: Z_+^0 \to \boldsymbol{R}$ by

$$\varphi(m) = \begin{cases} 1 & m = 0 \\ mr - \delta(m) & m > 0 . \end{cases}$$

If $D_{Z_+^0}(\varphi) \neq \emptyset$, take a proper ideal C_1 with $C_1 \supseteqq D_{Z_+^0}(\varphi)$. Then $\varphi \in \mathrm{Dfn}(C, C_1; \boldsymbol{R})$. Every defining function on C is obtained in this manner. In particular if δ satisfies

$$\delta(m) + \delta(n) \leqq \delta(m + n) \quad \text{for all } m, n \in Z_+ ,$$

then φ is a normal defining function on C.

We are interested in the important case, i.e., case where C is a group. In the next section we discuss the structure of $((C, \varphi))$ where C is a group. Then we will see that Example (2.14) is isomorphic to a Schreier extension by a group.

3. $\bar{\mathfrak{N}}$-Semigroups.

DEFINITION 3.1. If S is a commutative semigroup and $v \in S$ such that for all $x \in S$ there exist $m \in Z_+$ and $y \in S$ with $v^m = xy$, then S is called a *subarchimedean* semigroup and the element v is called a *pivot element of* S.

DEFINITION 3.2. An $\bar{\mathfrak{N}}$-semigroup is a subarchimedean CCIF-semigroup.

LEMMA 3.3. *The pivot elements of a subarchimedean semigroup form an archimedean component and ideal of the semigroup.*

Proof. Let A be the set of pivot elements of a subarchimedean semigroup S. Let $v \in A$ and $x \in S$. There exist $m \in Z_+$ and $y \in S$ such that $v^m = xy$. Then $(vz)^m = x(yz^m)$ for every $z \in S$; hence $vz \in A$. Thus A is an ideal of S. To see that A is archimedean, let $u, v \in A$. Then there exist $m \in Z_+$ and $y \in S$ such that $v^m = uy$, therefore $v^{m+1} = u(yv)$ and $yv \in A$. Therefore A is archimedean. Let A_0 be the archimedean component containing $v \in A$. Obviously $A \subseteqq A_0$. Let $u \in A_0$, so $u^n = vy$ for some $n \in Z_+$, some $y \in S$. Let $z \in S$. As $v \in A$, $v^k = zt$ for some $k \in Z_+$, some $t \in S$. Then $u^{nk} = v^k y^k = z(ty^k)$, hence $u \in A$, $A_0 \subseteqq A$. Thus we have proved $A = A_0$.

LEMMA 3.4. *A homomorphic image of a subarchimedean semigroup is a subarchimedean semigroup.*

Proof. Let S be a subarchimedean semigroup, and f a surjective homomorphism of S onto a semigroup T. Let v be a privot element of S. Then for all $x \in S$ there exist $m \in Z_+$ and $y \in S$ such that $v^m = xy$. Hence $(f(v))^m = f(x)f(y)$, and we see that $f(v)$ is a pivot element of T.

LEMMA 3.5. *Let S be a* CCIF-*semigroup. S is subarchimedean if and only if S/ρ_a is subarchimedean for* (some) *all $a \in S$.*

Proof. If S is subarchimedean then S/ρ_a being a homomorphic image of S is subarchimedean for all $a \in S$ by Lemma 3.4. Conversely, if $a \in S$ and S/ρ_a is subarchimedean let $\bar{x}$ denote the ρ_a-class of $x \in S$. Let $\bar{v}$ be a pivot element of S/ρ_a. Then for all $\bar{x} \in S/\rho_a$ there exists $m \in Z_+$ and $\bar{y} \in S/\rho_a$ such that $\bar{v}^m = \bar{x}\bar{y}$. Hence, by the definition of ρ_a we have $v^m a^k = xya^l$ for some $k, l \in Z_+$. Therefore, $(va)^{m+k} = x(ya^{l+m}v^k)$ and we see that va is a pivot element of S.

LEMMA 3.6. *If S is an $\bar{\mathfrak{N}}$-semigroup then* $\operatorname{Hom}(S, \boldsymbol{R}_+^0) \neq \{0\}$.

Proof. By Lemma 3.3, S contains an $\mathfrak{N}$-semigroup A which is an ideal of S. By [2, 7, 8] $\operatorname{Hom}(A, \boldsymbol{R}_+) \neq \{\varnothing\}$. Let $h \in \operatorname{Hom}(A, \boldsymbol{R}_+)$. Then $h \neq 0$. Define $\bar{h}: S \rightarrow \boldsymbol{R}$ by $\bar{h}(x) = h(ax) - h(a)$ for $a \in A$ and $x \in S$. Let $a, b \in A$, and $x \in S$. Then $h(ax)+h(b) = h((ax)b) = h((bx)a) = h(bx) + h(a)$, so $h(ax) - h(a) = h(bx) - h(b)$. Thus $\bar{h}$ is well defined. Also, $\bar{h}(xy) = h(a^2xy) - h(a^2) = h(ax) - h(a) + h(ay) - h(a) = \bar{h}(x) + \bar{h}(y)$, hence $\bar{h}$ is a homomorphism. If $\bar{h}(x) < 0$ for some $x \in S$, choose $n \in Z_+$ such that $h(a) + n\bar{h}(x) < 0$. Since $ax^n \in A$, $h(ax^n) > 0$, but $h(ax^n) = h(a) + n\bar{h}(x) < 0$, a contradiction. Hence $\bar{h} \in \operatorname{Hom}(S, \boldsymbol{R}_+^0)$. As $\bar{h} \mid A = h \neq 0$, $\operatorname{Hom}(S, \boldsymbol{R}_+^0) \neq \{0\}$.

LEMMA 3.7. *Let S be an $\bar{\mathfrak{N}}$-semigroup. Then $a \in S$ is a pivot element if and only if S/ρ_a is an abelian group.*

Proof. Let A be the archimedian ideal of pivot elements of S, and let $a \in A$. Then $A/(\rho_a \mid A)$ is an abelian group, and for all $x \in S$ we have $(x, xa) \in \rho_a$ where $xa \in A$. Hence $S/\rho_a \cong A/(\rho_a \mid A)$ and S/ρ_a is an abelian group. Conversely if S/ρ_a is an abelian group then for all $x \in S$ there exists $y \in S$ such that $\bar{a} = \bar{x}\bar{y}$ in S/ρ_a. (See the notation in the proof of Lemma 3.5.) Thus $a^m = xya^l$ for some $m, l \in Z_+$. Hence $a \in A$.

THEOREM 3.8. *Let S be a* CCIF*-semigroup, and for $a \in S$ let ρ_a be defind by* (2.1.6). *The following are equivalent:*

(3.8.1) *S is an $\bar{\mathfrak{N}}$-semigroup.*

(3.8.2) *S/ρ_a is subarchimedean for all $a \in S$.*

(3.8.3) *S/ρ_a is subarchimedean for some $a \in S$.*

(3.8.4) *Some archimedean component of S is an ideal of S.*

(3.8.5) *S/ρ_a is an abelian group for some $a \in S$.*

(3.8.6) *$S \cong (G; I)$ where G is an abelian group and I is an $\mathscr{I}$-function on G.*

(3.8.7) *S is isomorphic to a subdirect product of an abelian group G and a subsemigroup of $\boldsymbol{R}_+^0$ by means of a defining function φ on G.*

Proof. By Lemma 3.5, the first three conditions are equivalent.

By Lemma 3.7, (3.8.1) implies (3.8.5); obviously (3.8.5) implies (3.8.3). By Lemma 3.3 and Lemma 3.7, (3.8.5) implies (3.8.4). Assume (3.8.4). Let I be the ideal and archimedean component, and let $a \in I$, $x \in S$. Since $ax \in I$, $a^m = axy$ for some $m \in Z_+$ and some $y \in I$, hence $a^m = x(ay)$, that is, a is a pivot element of S. By Lemma 3.7, (3.8.5) holds. By Theorem 2.1 and Lemma 2.8, (3.8.5) implies (3.8.6). Conversely if $S \cong (G; I)$, then $G \cong S/\rho_{(0,\varepsilon)}$. Thus the first six conditions are equivalent. To see that (3.8.1) and (3.8.6) imply (3.8.7), let S be an $\bar{\mathfrak{N}}$-semigroup. By Lemma 3.6, there exists a nontrivial homomorphism h of S into $\boldsymbol{R}_+^0$, and by (3.8.6), $S \cong (G; I)$ for some abelian group G and an $\mathscr{I}$-function I. Let $\varphi(\alpha) = h(0, \alpha)/h(0, \varepsilon)$ for all $\alpha \in G$. (Clearly we can assume $h(0, \varepsilon) \neq 0$.) Then by the proof of Theorem 2.2 we have (3.8.7). Finally if we assume (3.8.7), $S \cong ((G; \varphi))$ for some $\varphi\colon G \to \boldsymbol{R}_+^0$, then when we define $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha, \beta)$, we have $S \cong (G; I)$ as before. Hence (3.8.7) implies (3.8.6). The proof has been completed.

COROLLARY 3.9. *Let S be a* CCIF-*semigroup. S is an $\mathfrak{N}$-semigroup if and only if S/ρ_a is an abelian group for all $a \in S$.*

Proof. Let A be the set of pivot elements of S. If S is an $\mathfrak{N}$-semigroup then $S = A$ and so S/ρ_a is an abelian group for all $a \in S$. Conversely if S/ρ_a is an abelian group for all $a \in S$ then $S = A$ by Lemma 3.7. Hence S is archimedian, hence an $\mathfrak{N}$-semigroup.

4. Homomorphisms into $\boldsymbol{R}_+^0$. As seen in § 3 every $\bar{\mathfrak{N}}$-semigroup has a nontrivial homomorphism into $\boldsymbol{R}_+^0$. The following question is raised.

Is a CCIF-semigroup nontrivially homomorphic into $\boldsymbol{R}_+^0$? We cannot answer this question in general, but in some special case it is affirmative.

Let S be a CCIF-semigroup. As defined in § 1, $Q(S)$ denotes the quotient group and $D(S)$ the divisible hull of $Q(S)$.

$$D(S) \cong \bigoplus_{p \in \Delta} C(p^\infty) \oplus \bigoplus_{\alpha \in \Gamma} R_\alpha$$

where R_α is a copy of the additive group of rationals and $C(p^\infty)$ is a quasicyclic group. The cardinality $|\Gamma|$ of Γ is called the *rank* of S. In the present case the rank of S is not zero since $\bigoplus_{p \in \Delta} C(p^\infty)$ is torsion while S is torsion-free.

In particular, assume that S is of finite rank. Let T be the torsion subgroup of $D(S)$, then $D(S) = T \oplus R_1 \oplus \cdots \oplus R_n$ where n is the rank of S. We can assume $R_i \neq \{0\}$ for $i = 1, \cdots, n$. Let $P_i = R_1 \oplus \cdots \oplus R_i$ for each $i = 1, 2, \cdots, n$. Then $P_n = P_{n-1} \oplus R_n$ if $n > 1$; and $D(S) = T \oplus P_n$ if $n \geqq 1$. Let $\alpha, \bar{\sigma}, \sigma, \pi_n, \tau_n$ be the respective projection homomorphisms:

$$\alpha: D(S) \longrightarrow T \ , \quad \bar{\sigma}: D(S) \longrightarrow P_n \ , \quad \sigma = \bar{\sigma} \,|\, S \ ,$$
$$\pi_n: P_n \longrightarrow P_{n-1} \ , \quad \tau: P_n \longrightarrow R_n \quad (n \geqq 1)$$

THEOREM 4.1. *If S is a* CCIF-*semigroup of finite rank, then* $\mathrm{Hom}\,(S, R_+^0) \neq \{0\}$. ($R_+^0$ is the additive semigroup of nonnegative rationals.)

Proof. S is viewed as a subsemigroup of $D(S)$. We will prove the theorem by induction on n. Let $V_n = \pi_n\sigma(S)$, $W_n = \tau_n\sigma(S)$, $V = \sigma(S)$, $T' = \alpha(S)$. As $D(S) = T \oplus P_n$, we have

$$S = T' \oplus_s V \ , \text{ and if } n > 1 \ , \ V = V_n \oplus_s W_n \ ,$$

where $\oplus_s$ denotes a subdirect sum, $V \subseteqq P_n$, $V_n \subseteqq P_{n-1}$, $W_n \subseteqq R_n$, and $T' \subseteqq T$, hence T' is a torsion group. First we prove

(4.1.1) V does not contain 0.

Suppose V contains 0. There is $x \in T'$ such that $(x, 0) \in S$. Since T' is a torsion group, $mx = 0$ for some $m \in Z_+$. Then $(0, 0) = (x, 0)^m \in S$. This is a contradiction as S has no idempotent.

In case $n = 1$, $S = T' \oplus_s W_1$ where $W_1 = V \subset R_1$. By (4.1.1), W_1 must be isomorphic to a positive rational semigroup R_1', say, under f, i.e., $f(W_1) = R_1'$, hence $f\tau_1\sigma \in \mathrm{Hom}\,(S, R_+^0) \backslash \{0\}$.

Assume $n > 1$ and that the theorem holds for all semigroups of rank i such that $i \leqq n - 1$. As denoted above,

$$S = T' \oplus_s V \ , \quad V = V_n \oplus_s W_n$$

where $V_n \subseteqq P_{n-1}$, $W_n \subseteqq R_n$. We can assume $V_n \neq \{0\}$, otherwise it is reduced to the case $n = 1$.

If V_n is a CCIF-semigroup, V_n has a nontrivial homomorphism f from V_n into R_+^0 by the induction assumption, hence $f\pi_n\sigma \in \mathrm{Hom}\,(S, R_+^0) \backslash \{0\}$.

If V_n is a CCI-semigroup which is not a group, then $V_n = V_n' \cup H$ where $V_n' \neq \varnothing$, $H \neq \varnothing$, V_n' is an ideal of V_n and it is a CCIF-semigroup, and H is a group. Define S' by $S' = ((\pi_n\sigma)^{-1}(V_n')) \cap S$ and $W_n' = \tau_n\sigma(S')$. Then S' is an ideal of S and

$$S' = V_n' \oplus_s W_n' \ .$$

By the preceding paragraph, $\mathrm{Hom}\,(S', R_+^0)$ contains a nontrivial element f. However, since S' is an ideal of S, f can be extended to $\bar{f} \in \mathrm{Hom}(S, R_+^0)$. In fact $\bar{f}$ is obtained by defining $\bar{f}(x) = f(ax) - f(a)$ where $x \in S$, $a \in S'$. It is easy to show that $\bar{f}$ is well defined and a homomorphism. Suppose $\bar{f}(x_1) < 0$ for some $x_1 \in S$. There exists $m \in Z_+$ such that $m\bar{f}(x_1) + f(a) < 0$. However

$$m\bar{f}(x_1) + f(a) = f(ax_1^m) \geqq 0$$

since $ax_1^m \in S'$. This contradicts the assumption. Therefore $\bar{f}(x) \geqq 0$ for all $x \in S$. Hence $\mathrm{Hom}(S, R_+^0) \neq \{0\}$. Assume V_n is a group. Let $\bar{W}_n = \{(0, z): z \in W_n\} \cap V$. It is obvious that $\bar{W}_n$ is a subsemigroup if $\bar{W}_n \neq \varnothing$. If $x \in V$, x has the form $x = (x_1, x_2) \in V_n \oplus_s W_n$, $x_1 \in V_n$, $x_2 \in W_n$. Since V_n is a group, there exists $y_2 \in W_n$ such that $y = (-x_1, y_2) \in V$. Then $xy = (0, x_2 + y_2) \in \bar{W}_n$. This proves that $\bar{W}_n \neq \varnothing$ and it is cofinal in V. Suppose $x \in V$ and $a, xa \in \bar{W}_n$. We write $x = (x_1, x_2)$, $a = (0, a_2)$ viewing them as in $V_n \oplus_s W_n$. Then $xa = (x_1, x_2 + a_2) \in \bar{W}_n$ implies $x_1 = 0$, hence $x \in \bar{W}_n$. Thus $\bar{W}_n$ is unitary in V. Since $\bar{W}_n$ does not contain $(0, 0)$ by (4.1.1), $\bar{W}_n$ is isomorphic to a positive rational semigroup R'_n under f: $\bar{W}_n \to R'_n$. By (4.1.2) below, f extends to $\bar{f} \in \mathrm{Hom}(V, R_+^0)$. Therefore $\bar{f}\sigma \in \mathrm{Hom}(S, R_+^0)\backslash\{0\}$.

(4.1.2) *Let S be a* CCIF-*semigroup and let U be a unitary cofinal subsemigroup of S. Then every homomorphism of U into R_+^0 extends to a homomorphism of S into R_+^0.*

This is immediately obtained from [4].

The proof of Theorem 4.1 has been completed.

REMARK 4.2. Let $S = R_+ \oplus (\bigoplus_{\alpha \in \Gamma} R_\alpha)$ where $|\Gamma| = \infty$, R_α is the group of rationals. We note that $\mathrm{Hom}(S, R_+^0) \neq \{0\}$, yet S is not of finite rank. Thus the converse of Theorem 4.1 does not hold.

Next we consider the relation between nontriviality of $\mathrm{Hom}(S, R_+^0)$ and the property

$$\bigcap_{n=1}^{\infty} a^n S = \varnothing \quad \text{for some } a \in S . \tag{4.3}$$

PROPOSITION 4.4. *If* $\mathrm{Hom}(S, R_+^0) \neq \{0\}$, *then there is an element $a \in S$ satisfying* (4.3).

Proof. Let $h \in \mathrm{Hom}(S, R_+^0)$, $h \neq 0$. There is $a \in S$ such that $h(a) \neq 0$. Choose a as a standerd element. We have $C_1 = \varnothing$ by Proposition 2.7 and then have (4.3) by Proposition 2.11.

The converse of Proposition 4.4 is still open.

Problem 4.5. Let S be a CCIF-semigroup. If $\bigcap_{n=1}^{\infty} a^n S = \varnothing$ for some $a \in S$, then is the following true

$$\mathrm{Hom}(S, \boldsymbol{R}_+^0) \neq \{0\}?$$

However, we give a few examples with respect to the related problems.

EXAMPLE 4.6. Let $\bigcap_{n=1}^{\infty} a^n S = \varnothing$. There does not necessarily exist $h \in \mathrm{Hom}(S, \boldsymbol{R}_+^0)$ such that $h(a) \neq 0$.

Let $S = ((\boldsymbol{Z}_+^0; \varphi))$ where $\varphi: \boldsymbol{Z}_+^0 \to \boldsymbol{Z}$ is defined by

$$\varphi(m) = 1 - m^2 .$$

It can be easily shown that φ is a normal defining function on $\boldsymbol{Z}_+^0$, and that if $a = ((1, 0))$, $\bigcap_{n=1}^{\infty} a^n S = \varnothing$. Every element f_t of $\mathrm{Hom}(\boldsymbol{Z}_+^0, \boldsymbol{R})$ has the form

$$f_t(m) = tm \qquad t \in \boldsymbol{R} ,$$

but there is no t satisfying

$$\varphi(m) + f_t(m) = 1 - m^2 + tm \geqq 0 \quad \text{for all } m \in \boldsymbol{Z}_+^0 .$$

By Proposition 2.6, (2.6.1), there is no $h \in \mathrm{Hom}(S, \boldsymbol{R}_+^0)$ with $h(a) \neq 0$. However the projection $h_0: S \to \boldsymbol{Z}_+^0$ is a nontrivial element of $\mathrm{Hom}(S, \boldsymbol{R}_+^0)$ such that $h_0(a) = 0$. Thus $\mathrm{Hom}(S, \boldsymbol{R}_+^0) \neq \{0\}$ and so Example 4.6 is not a counterexample to the converse of Proposition 4.4. In fact the semigroup S is an $\bar{\mathfrak{N}}$-semigroup.

EXAMPLE 4.7. We exhibit an example of a CCIF-semigroup S which satisfies

$$\bigcap_{n=1}^{\infty} a^n S \neq \varnothing \quad \text{for all } a \in S ,$$

and hence $\mathrm{Hom}(S, \boldsymbol{R}_+^0) = \{0\}$.

$$\text{Let } S = \{(a_1, \cdots, a_m): m, a_m \in \boldsymbol{Z}_+, a_i \in \boldsymbol{Z}, 1 \leqq i < m\}$$

and define a binary operation on S as follows: if $m \leqq n$,

$$\begin{aligned}(a_1, \cdots, a_m)(b_1, \cdots, b_n) &= (b_1, \cdots, b_n)(a_1, \cdots, a_m)\\ &= (a_1 + b_1, \cdots, a_m + b_m, b_{m+1}, \cdots, b_n) .\end{aligned}$$

Then, with this product, S is a CCIF-semigroup. Let $S_1 = \boldsymbol{Z}_+$ and $S_i = \boldsymbol{Z}^{i-1} \times \boldsymbol{Z}_+$ for $i > 1$. Then S is the union of the infinite chain of S_i's, $S = \bigcup_{i=1}^{\infty} S_i$ and $S_i S_j \subseteqq S_j$ if $i \leqq j$. If $a \in S_m$ then

$$\bigcap_{n=1}^{\infty} a^n S = \bigcup_{i>m} S_i .$$

DEFINITION 4.8. A semigroup S is called an $\mathfrak{N}'$-semigroup if S is isomorphic to a subsemigroup of an $\mathfrak{N}$-semigroup.

THEOREM 4.9. *Let S be a* CCIF-*semigroup. S is an $\mathfrak{N}'$-semigroup if and only if*

$$\mathrm{Hom}\,(S, \boldsymbol{R}_+) \neq \varnothing \ .$$

Proof. Assume that S is a subsemigroup of an $\mathfrak{N}$-semigroup T. By [6, 7] there is an $h \in \mathrm{Hom}\,(T, \boldsymbol{R}_+)$. Let h_1 be the restriction of h to S. Then $h_1 \in \mathrm{Hom}\,(S, \boldsymbol{R}_+)$.

Conversely let $\mathrm{Hom}\,(S, \boldsymbol{R}_+) \neq \varnothing$. By Proposition 2.7, $C_1 = \varnothing$. By Theorem 2.2 and its Corollaries, $S \cong (C; \varphi)$ where C is a CCI-semigroup and $\varphi \in \mathrm{DNfn}\,(C, \boldsymbol{R})$; and S is isomorphic to a subdirect product of a subsemigroup P of $\boldsymbol{R}_+$ and $C, S \cong P \times_S C$. Let Q be the group of quotients of C. Then $P \times_S C$ is a subsemigroup of the direct product $\boldsymbol{R}_+ \times Q$, but the last direct product is an $\mathfrak{N}$-semigroup. Consequently S is an $\mathfrak{N}'$-semigroup.

The two concepts, $\bar{\mathfrak{N}}$-semigroup and $\mathfrak{N}'$-semigroup, are independent of each other.

EXAMPLE 4.10. Let $S = Z_+ \cup (Z \times Z_+)$. A binary operation is defined to be the same as Example 4.7, that is, S is a subsemigroup of the semigroup in Example 4.7. S is an $\bar{\mathfrak{N}}$-semigroup, but we prove $\mathrm{Hom}\,(S, \boldsymbol{R}_+) = \varnothing$ as follows:

Let $x \in Z_+$ and $(a_1, a_2) \in Z \times Z_+$. There exists $(b_1, b_2) \in Z \times Z_+$ such that

$$x \cdot (b_1, b_2) = (a_1, a_2) \ .$$

Suppose $h \in \mathrm{Hom}\,(S, \boldsymbol{R}_+) \neq \varnothing$. Then

$$h(x) < h(a_1, a_2) \quad \text{for all } x \in Z_+ \text{ and all } (a_1, a_2) \in Z \times Z_+ \ .$$

In particular $h(1) < h(a_1, a_2)$, but there is $x \in Z_+$ such that $x \cdot h(1) > h(a_1, a_2)$. Accordingly $h(x) = x \cdot h(1) > h(a_1, a_2)$. This contradiction proves $\mathrm{Hom}\,(S, \boldsymbol{R}_+) = \varnothing$, hence S is not an $\mathfrak{N}'$-semigroup.

EXAMPLE 4.11. Let S be the free commutative semigroup generated by infinitely countable letters $a_1, a_2, \cdots, a_n, \cdots$. (The empty word is not considered.) S is obviously a CCIF-semigroup and $\mathrm{Hom}(S, \boldsymbol{R}_+) \neq \varnothing$ since

$$a_{i_1}^{m_1} \cdots a_{i_k}^{m_k} \longmapsto m_1 + \cdots + m_k$$

gives a homomorphism of S into Z_+. However S is not an $\bar{\mathfrak{N}}$-semigroup, as the greatest semilattice homomorphic image of S does not have a zero.

REMARK. According to his recent personal letter to one of the authors, Professor Yuji Kobayashi, Tokushima University, has negatively answered Problem 4.5 by showing a counter example.

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