COMMUTATIVE CANCELLOGATIVE SEMIGROUPS WITHOUT IDEMPOTENTS

H. B. Hamilton, T. E. Nordahl and Takayuki Tamura
COMMUTATIVE CANCELLATIVE SEMIGROUPS WITHOUT IDEMPOTENTS

H. B. HAMILTON, T. E. NORDAHL AND T. TAMURA

A commutative cancellative idempotent-free semigroup (CCIF-) \( S \) can be described in terms of a commutative cancellative semigroup \( C \) with identity, an ideal of \( C \), and a function of \( C \times C \) into integers. If \( C \) is an abelian group, \( S \) has an archimedean component as an ideal; \( S \) is called an \( \mathbb{N} \)-semigroup. A CCIF-semigroup of finite rank has nontrivial homomorphism into nonnegative real numbers.

1. Introduction. In this paper, a commutative cancellative semigroup without idempotent is called a CCIF-semigroup (in which, by "IF" we mean "idempotent-free") and a commutative cancellative semigroup with identity is called a CCI-semigroup. In particular, an \( \mathbb{N} \)-semigroup is an archimedean CCIF-semigroup. The structure of \( \mathbb{N} \)-semigroups has been much studied [1, 2, 3, 6, 7, 8] and also it is well known that every CCIF-semigroup is a semilattice of \( \mathbb{N} \)-semigroups. In this paper CCIF-semigroups will be studied by means of the representation by the generalized \( \mathfrak{J} \)- and \( \varphi \)-functions and also through homomorphisms into the nonnegative real numbers.

Throughout this paper, \( \mathbb{R} \) denotes the set of real numbers; \( \mathbb{Q} \) the set of rational numbers; \( \mathbb{R}^+ \) the set of positive real numbers; \( \mathbb{Z}^0_+ \) the set of nonnegative real numbers; \( \mathbb{Z}_+ \) the set of positive integers and \( \mathbb{Z}^0_+ \) the set of nonnegative integers. Each of these is a semigroup under the usual addition. If \( S \) is a semigroup and if \( X \) is a subsemigroup of the group \( \mathbb{R} \), then the notation \( \text{Hom}(S, X) \) denotes the semigroup of homomorphisms of \( S \) into \( X \) under the usual operation.

At the end of \( \S 1 \) we show that if \( S \) is a CCIF-semigroup, \( \text{Hom}(S, \mathbb{R}) \neq \{0\} \), and the homomorphism group is transitive in some sense. In Section 2 we shall try to generalize the representation of \( \mathbb{N} \)-semigroups to CCIF-semigroups. It will be understood as the so-called Schreier's extension to build up complicated CCIF-semigroups from simpler CCIF-semigroups. Most of the results in [7] will be extended to CCIF-semigroups. In \( \S 3 \) we shall treat the important case, i.e., the case where the structure semigroup is a group. Such a CCIF-semigroup will be called an \( \mathbb{N} \)-semigroup. In \( \S 4 \) we shall show that every CCIF-semigroup of finite rank has a nontrivial homomorphism into \( \mathbb{R}^0_+ \). In particular we will characterize CCIF-semigroups \( S \) having the property \( \text{Hom}(S, \mathbb{R}^+_+) \neq \emptyset \).

(1.1) Let \( S \) be a CCIF-semigroup. Then \( x \neq xy \) for all \( x, y \in S \).
Proof. Suppose, for some $x, y \in S$, we have $x = xy$. Then $xy = xy^2$ which implies $y = y^2$ by cancellation. This is a contradiction.

**Proposition 1.2.** Let $S$ be a CCIF-semigroup.

1.2.1 Hom $(S, R)$ is a nontrivial vector space over the field $R$.
1.2.2 For each $a \in S$ and each $r \in R$, $r \neq 0$, there is an $h \in \text{Hom} (S, R)$ such that $h(a) = r$.

**Proof of (1.2.1).** Let $S$ be a CCIF-semigroup. Let $Q(S)$ be the quotient group of $S$ (i.e., the group of quotients of $S$), and $D(S)$ be the divisible hull of $Q(S)$

$D(S) = \bigoplus_{a \in \Gamma} R_a \bigoplus_{p \in \mathfrak{p}} C(p^\infty)$. 

$D(S)$ is a direct sum of copies $R_a$ of the group of rational numbers under addition and quasi-cyclic groups $C(p^\infty)$ with respect to prime number $p$. We view $S$ as a subsemigroup of $D(S)$. Let $\pi_a$ be the projection of $D(S)$ upon $R_a$ for each $a \in \Gamma$. Let $x$ be an element of $S$. Suppose $\pi_a(x) = 0$ for each $a \in \Gamma$. It follows that $x \in \bigoplus_{p \in \mathfrak{p}} C(p^\infty)$, a torsion group. This is a contradiction as $x$ has infinite order. Thus, for some $a_0 \in \Gamma$, $\pi_{a_0}(x) \neq 0$. Note that $\pi_{a_0} \in \text{Hom} (S, R)$ and is not the trivial homomorphism. It is obvious that $\text{Hom} (S, R)$ is a vector space over $R$ in the usual way.

**Proof of (1.2.2).** Let $a \in S$ and $r \in R$ be given. In establishing (1.2.1), we have shown that there exists $h_a \in \text{Hom} (S, R)$ with $h_a(a) \neq 0$. Let $s = h_a(a)$. Now define $h$ by $h = (r/s)h_a$. Then $h(a) = r$, and $h \in \text{Hom} (S, R)$.

2. Schreier Extension. We consider the following problem. Let $C$ be a CCI-semigroup and $\varepsilon$ be its identity. Given $C$, find all CCIF-semigroups $S$ such that there is a homomorphism $\mathcal{P}$ of $S$ onto $C$ satisfying the condition.

$$\{x \in S \mid \mathcal{P}(x) = \varepsilon \} \cong Z_+.$$ 

In this section we shall show that $S$ always exists for every $C$ and shall describe $S$ in terms of elements of $C$, integers and a certain function of $C \times C$ into the integers. The extension $S$ is called a Schreier extension (of $Z_+$) by $C$. (The terminology is due to [5].) Schreier extension by $C$ is significant because we shall see that every CCIF-semigroup is isomorphic to a Schreier extension by some CCI-semigroup $C$.

**Theorem 2.1.** Let $C$ be a CCI-semigroup and $C_1$ a proper ideal
of $C$. (C can be empty.) Let $I: C \times C \to Z$ be a function which satisfies

(2.1.1) $I(\alpha, \beta) \in \mathbb{Z}^+ \text{ if } \alpha \beta \notin C$

(2.1.2) $I(\alpha, \beta) = I(\beta, \alpha)$ for all $\alpha, \beta \in C$

(2.1.3) $I(\alpha, \beta) + I(\alpha \beta, \gamma) = I(\alpha, \beta \gamma) + I(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in C$

(2.1.4) $I(\varepsilon, \alpha) = 1$ (\varepsilon the identity element of $C$) for all $\alpha \in C$.

Given $C, C_1, I$, the set $(C, C_1; I)$ with its operation is defined by

$(C, C_1; I) = \{(x, \alpha) \in \mathbb{Z} \times C; x \in \mathbb{Z}_+ \text{ if } \alpha \notin C_1\}$

(2.1.5) $(x, \alpha) + (y, \beta) = (x + y + I(\alpha, \beta), \alpha \beta)$.

Then $(C, C_1; I)$ is a CCIF-semigroup.

Conversely if $S$ is a CCIF-semigroup, then $(S \cong C, C_1; I)$ for some $C, C_1, I$.

Proof. It is routine to prove that $(C, C_1; I)$ is a commutative cancellative semigroup. To show idempotent-freeness, assume $(x, \alpha)^2 = (x, \alpha)$, that is, $\alpha^2 = \alpha$ and $2x + I(x, x) = x$. It follows that $\alpha = \varepsilon$ and $x + 1 = 0$. Since $C_1$ is a proper ideal of $C$, $\varepsilon \notin C_1$, hence $x \geq 0$ and we arrive at a contradiction.

Conversely assume that $S$ is a CCIF-semigroup. Let $a \in S$, and define a relation $\rho_a$ on $S$ by

(2.1.6) $x \rho_a y$ iff $a^m x = a^n y$ for some $m, n \in \mathbb{Z}_+$.

It is easy to see that $\rho_a$ is a congruence relation. To show that $S/\rho_a$ is cancellative, assume $xx \rho_a yz$. Then $a^m x = a^n yz$ for some $m, n \in \mathbb{Z}_+$. Since $S$ is cancellative, we get $a^m x = a^n y$, i.e., $x \rho_a y$. Obviously $a \rho_a x$ for all $x \in S$, that is, the $\rho_a$-class containing $a$ is the identity of $S/\rho_a$. Let $C = S/\rho_a$. $C$ is a CCI-semigroup. In each $\rho_a$-class define $x \leq_a y$ by $x = a^m y$ for some $m \in \mathbb{Z}_+^\times$ where $a^0 y = y$. Because of cancellation, each $\rho_a$-class forms a chain with respect to $\leq_a$. Let $T = \bigcap_{a \in S} a^\mathbb{Z} \ S$ and let $C_i$ be the image of $T$ under the natural homomorphism $S \to C$. If $T \neq \emptyset$, it is a proper ideal of $S$ (since $a \in T$) and thus $C_i$ is a proper ideal of $C$. Under the homomorphism $S \to C$ we have a partition of $S$: $S = \bigcup_{i \in C} S_i$. If $\xi \in C \setminus C_i$, $S_i$ contains a maximal element with respect to $\leq_a$; but if $\xi \in C_i$, $S_i$ contains no maximal element. For each $\xi \in C$, define $p_i$ to be $a \leq_a$-maximal element in $S_i$ if $\xi \in C \setminus C_i$, and $p_i$ to be arbitrarily chosen from $S_i$ if $\xi \in C_i$. Since $C_i$ is a proper ideal, $\varepsilon \notin C_i$, hence $p_i = a$ because of (1.1). Then every element of $S$ has a unique expression

$x = a^m p_\xi$ where $m \in \mathbb{Z}$ if $\xi \in C_1; m \in \mathbb{Z}_+^\times$ if $\xi \in C \setminus C_i$.

Define $I: C \times C \to Z$ as follows:

$p_\alpha p_\beta = a^{I(\alpha, \beta)} p_{\alpha \beta}$.
It is easy to see that $I$ satisfies (2.1.1), (2.1.2), (2.1.3) and (2.1.4). $S$ is isomorphic to $(C, C; I)$ under the map $a^n p_i \mapsto (m, \xi)$.

The representation $(C, C; I)$ of $S$ depends on the choice of $a$. The element $a$ is called the standard element of the representation $(C, C; I)$ of $S$. $S/\rho_a$ is called the structure CCI-semigroup of $S$ with respect to $a$; $C$ is the structure CCI-semigroup of $(C, C; I)$, and $(0, \varepsilon)$ is the standard element. A function $I: C \times C \to Z$ satisfying (2.1.1), (2.1.2), (2.1.3), (2.1.4) is called an $I$-function on $(C, C_i)$.

**Theorem 2.2.** Let $C$ be a CCI-semigroup, and $C_i$ be a proper ideal of $C$. (C_i can be empty.) Assume that $\varphi: C \to R$ satisfies

\begin{align*}
\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta) &\in \{ Z \quad \text{if } \alpha \beta \in C_i \\
Z_+ &\quad \text{if } \alpha \beta \notin C_i.
\end{align*}

(2.2.2) $\varphi(\varepsilon) = 1$.

Given $C$, $\varphi$, and $C_i$, define $((C, C_i; \varphi))$ by

\begin{align*}
((C, C_i; \varphi)) &\equiv \{(x + \varphi(\alpha), \alpha): \alpha \in C, x \in Z, x \in Z_+ \text{ if } \alpha \in C_i\} \\
\text{and } (2.2.3) &\quad ((x + \varphi(\alpha), \alpha))(y + \varphi(\beta), \beta) = (x + y + \varphi(\alpha) + \varphi(\beta), \alpha \beta).
\end{align*}

Then $((C, C_i; \varphi))$ is a CCIIF-semigroup.

Conversely every CCIIF-semigroup is isomorphic to $((C, C_i; \varphi))$ for some $C$, $\varphi$ and $C_i$, that is, $(C, C_i; I) \cong ((C, C_i; \varphi))$ under $(x, \alpha) \mapsto ((x + \varphi(\alpha), \alpha))$, $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta)$.

**Proof.** Assume $S$ is a CCIIF-semigroup. By Theorem 2.1, we let $S = (C, C_i; I)$ for some $C_i$, $C_i$. By (1.2.2), there is an $h \in \text{Hom}(S, R)$ such $h(0, \varepsilon) \neq 0$. Define $\varphi: C \to R$ by

\begin{align*}
\varphi(\alpha) = \frac{h(0, \alpha)}{h(0, \varepsilon)}.
\end{align*}

If $I(\alpha, \beta) \geq 0$, then $0, \alpha)(0, \beta) = (0, \varepsilon)^{I(\alpha, \beta)}(0, \alpha \beta)$ implies

\begin{align*}
h(0, \alpha) + h(0, \beta) = I(\alpha, \beta) \cdot h(0, \varepsilon) + h(0, \alpha \beta).
\end{align*}

If $I(\alpha, \beta) < 0$, then $0, \alpha)(0, \beta)(0, \varepsilon)^{-I(\alpha, \beta)} = (0, \alpha \beta)$ implies

\begin{align*}
h(0, \alpha) + h(0, \beta) = I(\alpha, \beta) \cdot h(0, \varepsilon) = h(0, \alpha \beta).
\end{align*}

In both cases, using (2.2.5), we have

\begin{align*}
(2.2.6) I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta) \quad \text{for all } \alpha, \beta \in C. \quad \text{It is easy to see that } \varphi \text{ satisfies (2.2.1) and (2.2.2); and } S = (C, C; I) \cong ((C, C_i; \varphi)) \quad \text{under } (x, \alpha) \mapsto ((x + \varphi(\alpha), \alpha)).
\end{align*}

Conversely assume $\varphi$ satisfies (2.2.1) and (2.2.2), define $((C, C_i; \varphi))$ by (2.2.3) and (2.2.4), and define $I$ by (2.2.6). Then we can see that $I$ satisfies (2.1.1), (2.1.2), (2.1.3) and (2.1.4), and $((x, \alpha)) \mapsto (x - \varphi(\alpha), \alpha)$ gives an isomorphism of $((C, C_i; \varphi))$ to $(C, C_i; I)$. 

A function $\varphi: C \to R$ is called a defining function on $(C, C_1)$ if it satisfies (2.2.1) and (2.2.2); let $Dfn(C, C_1, R)$ denote the set of all defining functions on $(C, C_1)$. If $\varphi$ satisfies (2.2.6) for a fixed $I$, $\varphi$ is called a defining function belonging to $I$, and the set of all $\varphi$ belonging to $I$ is denoted by $Dfn_I(C, C_1, R)$.

**Corollary 2.3.** $S$ is a CCIF-semigroup if and only if $S$ is isomorphic to the subdirect product of a CCI-semigroup $C$ and a subsemigroup of $R$ by means of $\varphi$ on $C$ (i.e., by means of $\varphi$ with (2.2.1) and (2.2.2) in the sense of (2.2.4)).

**Corollary 2.4.** Let $S$ be a CCIF-semigroup. $S$ is a subdirect product of a subsemigroup $P$ of $R_+$ and a CCI-semigroup $C$ if and only if there exists $h \in \text{Hom}((S, R_+)$ with $h \neq 0$.

The problem posed at the beginning of the section is solved, that is,

$$\mathcal{P}: ((x + \varphi(\alpha), \alpha)) \mapsto \alpha$$

has kernel $K = \{(x + 1, \varepsilon): x \in \mathbb{Z}_+\}$ and $K \cong \mathbb{Z}_+$ under $((x + 1, \varepsilon)) \mapsto x + 1$.

Let $S = (C, C_1; I)$.

**Proposition 2.5.** Let $\varphi_0 \in Dfn_I(C, C_1, R)$ be fixed. If $f \in \text{Hom}(C, R)$ then $\varphi = \varphi_0 + f \in Dfn_I(C, C_1, R)$. Every element $\varphi$ of $Dfn_I(C, C_1, R)$ can be obtained in this manner.

**Proposition 2.6 (2.6.1).** Let $\varphi_0 \in Dfn_I(C, C_1, R)$ be fixed and $f \in \text{Hom}(C, R)$. Define $h: S \to R$ by

$$h(x, \alpha) = s(x + \varphi_0(\alpha) + f(\alpha)),$$

$s \in R$.

Then $h \in \text{Hom}(S, R)$ Every element $h$ of $\text{Hom}(S, R)$ satisfying $h(0, \varepsilon) \neq 0$ can be obtained in this manner.

(2.6.2) Let $p: S \to C$ be the natural homomorphism. Then every $h$ of $\text{Hom}(S, R)$ satisfying $h(0, \varepsilon) = 0$ is obtained by $h = fp$ where $f \in \text{Hom}(C, R)$.

**Proof (2.6.1).** As the former half is easily proved, we prove the latter half. By (1.2.1) $\text{Hom}(S, R) \neq \{0\}$, so there is $h$ such that $h(0, \varepsilon) \neq 0$. If $x \geq 0$,

$$h(x, \alpha) = h((0, \varepsilon)^*(0, \alpha)) = x \cdot h(0, \varepsilon) + h(0, \alpha)$$

$$= h(0, \varepsilon)(x + \varphi(\alpha)) = s(x + \varphi(\alpha))$$
where \( s = h(0, \varepsilon) \); \( \varphi(\alpha) = h(0, \alpha)h(0, \varepsilon) \), \( \varphi \in \text{Dfn}(C, C, R) \). If \( x = 0 \), (0, \varepsilon)\(^r\) is regarded as void. If \( x < 0 \), \(-x - 1 \geq 0\), then
\[
\begin{align*}
    h(0, \alpha) &= h((-x - 1, \varepsilon)(x, \alpha)) = h((0, \varepsilon)^{-1}(x, \alpha)) \\
    &= (-x) \cdot h(0, \varepsilon) + h(x, \alpha)
\end{align*}
\]
hence \( h(x, \alpha) = h(0, \varepsilon)(x + \varphi(\alpha)) \). By Proposition 2.5, \( \varphi \) is expressed as \( \varphi_0 + f \). Thus we have the conclusion.

**Proof.** (2.6.2) Let \( h \in \text{Hom}(S, R) \) with \( h(0, \varepsilon) = 0 \). If \( x \geq 0 \),
\[
    h(x, \alpha) = x \cdot h(0, \varepsilon) + h(0, \alpha) = h(0, \alpha)
\]
If \( x < 0 \), \( h(0, \alpha) = (-x) \cdot h(0, \varepsilon) + h(x, \alpha) = h(x, \alpha) \). Hence \( h(x, \alpha) = h(0, \alpha) \) for all \((x, \alpha) \in S\). Define \( f: C \to R \) by \( f(\alpha) = h(x, \alpha) \) where \((x, \alpha) \in S\). By the above result, \( f \) is well defined. Now
\[
    fp(x, \alpha) = f(\alpha) = h(x, \alpha), \text{ hence } h = fp.
\]
It is easy to see that \( fp \in \text{Hom}(S, R) \) with \( fp(0, \varepsilon) = 0 \).

By the notation \( S = (C, C; I) = ((C, C; \varphi)) \) we mean that \( S \) has representation \((C, C; I)\) and \((C, C; \varphi)\) identifying \((x, \alpha)\) of \((C, C; I)\) with \((x + \varphi(\alpha), \alpha)\) of \((C, C; \varphi))\).

**Proposition 2.7.** Let \( S \) be a CCIF-semigroup. If \( a \in S \) and if there is an \( h \in \text{Hom}(S, R_c) \) such that \( h(a) \neq 0 \), then \( C_i = \emptyset \) using \( a \) as the standard element.

**Proof.** Let \( S = (C, C; I) = ((C, C; \varphi)) \) and let \( a \) denote \((0, \varepsilon)\) in \((C, C; I)\) and at the same time \((1, \varepsilon)\) in \((C, C; \varphi))\). Let \( \alpha \in C_i \). Then \((x, \alpha) \in (C, C; I)\) for all \( x \in Z \). By Proposition 2.6
\[
    h(x, \alpha) = h(0, \varepsilon)(x + \varphi(\alpha)).
\]
Since \( h(0, \varepsilon) > 0 \) and \( x \) is arbitrary, \( h(x, \alpha) < 0 \) if, \( x < -\varphi(\alpha) \); a contradiction to the assumption. Hence \( C_i = \emptyset \).

A subsemigroup \( T \) of a commutative semigroup \( S \) is called cofinal if, for every \( x \in S \), there is a \( y \in S \) such that \( xy \in T \). Let \( S = (C, C; I) \). The following are easily obtained.

**Lemma 2.8.**
\begin{align*}
    (2.8.1) & \text{ If } C \setminus C_i \text{ contains a cofinal subsemigroup of } C, \text{ then } C_i = \emptyset. \\
    (2.8.2) & \text{ If } C \text{ is an abelian group, then } C_i = \emptyset.
\end{align*}

We will now make a further investigation into defining functions and \( C_i \).

Let \( U \) denote the group of units of \( C \). Let \( \varphi \) be a function
Define a set $D_c(\varphi)$ by

$$D_c(\varphi) = \{ a \in C : \varphi(\xi) + \varphi(\eta) - \varphi(\alpha) < 0 \text{ for some } \xi, \eta \in C \text{ with } \alpha = \xi \eta \}.$$

We define defining functions from the point of $C$.

**Definition 2.9.**

(2.9.1) A function $\varphi : C \to \mathbb{R}$ is called a defining function on $C$ if it satisfies

$$\begin{align*}
\varphi(e) &= 1, \\
\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta) &\in \mathbb{Z} \text{ for all } \alpha, \beta \in C.
\end{align*}$$

The set of defining functions on $C$ is denoted by $\text{Dfn}(C, \mathbb{R})$.

(2.9.2) A defining function on $C$ is called a normal defining function on $C$ if $D_c(\varphi) = \emptyset$, and a nonnormal defining function on $C$ if $D_c(\varphi) \neq \emptyset$. $D_c(\varphi)$ is called the nonnormal domain of $\varphi$. The set of normal defining functions on $C$ is denoted by $\text{NDfn}(C, \mathbb{R})$.

**Proposition 2.10.** Let $\varphi : C \to \mathbb{R}$ be a defining function on $C$. Let $C_i$ be a proper ideal of $C$ such that $D_c(\varphi) \subseteq C_i$. Then $\varphi \in \text{Dfn}(C, C_i, \mathbb{R})$. Conversely every defining function on $(C, C_i)$ is a defining function on $C$.

The following three cases are possible:

(i) $\varphi$ is normal and $C_i = \emptyset$

(ii) $\varphi$ is normal and $C_i \neq \emptyset$

(iii) $\varphi$ is not normal and $C_i \neq \emptyset$.

**Definition.** In each case we consider the CCIF-semigroup $((C, C_i; \varphi))$. $((C, C_i; \varphi))$ is called a normal representation in case (i); seminormal representation in case (ii); nonnormal representation in case (iii). In case (i), $((C, C_i; \varphi))$ is denoted by $((C; \varphi))$. When $\varphi$ is normal (nonnormal), the $\mathcal{F}$-function $I$ defined by $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta)$ is called normal (nonnormal); the corresponding semigroup is denoted by $(C, C_i; I)$, in particular $(C; I)$ in case (i).

**Proposition 2.11.** Let $S = ((C, C_i; \varphi))$ with standard element $a$. Then $((C, C_i; \varphi))$ is a normal representation if and only if $\bigcap_{n=1}^{\infty} a^n S = \emptyset$.

**Proposition 2.12.** For every CCI-semigroup $C$ there exist normal defining functions on $C$. If $C$ is a CCI-semigroup and $C_i$ is a non-
empty proper ideal of \( C \), there exist nonnormal defining functions \( \varphi \) such that the nonnormal domain of \( \varphi \) is contained in \( C_i \).

**Examples 2.13.** Let \( C \) be a CCI-semigroup.

(2.13.1) Define \( \varphi \) by

\[
\varphi(\alpha) = 1 \quad \text{for all } \alpha \in C.
\]

Then \( \varphi \in \text{NDfn} (C, R) \), and \( ((C; \varphi)) \equiv \mathbb{Z}_+ \times C \).

(2.13.2) Let \( U \) be the group of units of \( C \). Let \( \varphi_0 \) be a non-negative integer valued normal defining function on \( U \). Define \( \varphi: C \to \mathbb{Z}_+^0 \) by

\[
\varphi(\alpha) = \begin{cases} 
\varphi_0(\alpha) & \text{if } \alpha \in U \\
c & \text{if } \alpha \notin U
\end{cases}
\]

where \( c \) is a constant nonnegative integer. Then \( \varphi \) is a normal defining function on \( C \).

(2.13.3) Let \( C_i \) be a nonempty proper ideal of \( C \). Define \( \varphi \) by

\[
\varphi(\alpha) = \begin{cases} 
1 & \alpha \in C_i \\
-1 & \alpha \in C \setminus C_i
\end{cases}
\]

The \( \varphi \) is a nonnormal defining function on \( C \) such that \( D_\varphi(\varphi) \subseteq C_i \).

(2.13.4) Assume that \( \varepsilon \) is the only unit of \( C \). Suppose \( \varphi_0: C \setminus \{\varepsilon\} \to R \) satisfies, for all \( \alpha, \beta \in C \setminus \{\varepsilon\} \).

\[
\varphi_0(\alpha) + \varphi_0(\beta) = \varphi_0(\alpha \beta) \in \mathbb{Z}.
\]

Define \( \varphi: C \to R \) by

\[
\varphi(\alpha) = \begin{cases} 
1 & \alpha = \varepsilon \\
\varphi_0(\alpha) & \alpha \neq \varepsilon
\end{cases}
\]

Then \( \varphi \) is a defining function on \( C \).

As another example, consider the case \( C = \mathbb{Z}_+^\ast \).

(2.14) Let \( C = \mathbb{Z}_+^\ast \). Let \( \delta: \mathbb{Z}_+ \to \mathbb{Z} \) be a function with \( \delta(1) = 0 \) and let \( r \) be a real number. Define \( \varphi: \mathbb{Z}_+^0 \to R \) by

\[
\varphi(m) = \begin{cases} 
1 & m = 0 \\
mr - \delta(m) & m > 0
\end{cases}
\]

If \( D_{\varphi_+^0}(\varphi) \neq \emptyset \), take a proper ideal \( C_i \) with \( C_i \supseteq D_{\varphi_+^0}(\varphi) \). Then \( \varphi \in \text{Dfn} (C, C_i; R) \). Every defining function on \( C \) is obtained in this manner. In particular if \( \delta \) satisfies

\[
\delta(m) + \delta(n) \leq \delta(m + n) \quad \text{for all } m, n \in \mathbb{Z}_+,
\]
then \( \varphi \) is a normal defining function on \( C \).

We are interested in the important case, i.e., case where \( C \) is a group. In the next section we discuss the structure of \((C, \varphi)\) where \( C \) is a group. Then we will see that Example (2.14) is isomorphic to a Schreier extension by a group.

3. \( \mathfrak{N} \)-Semigroups.

**Definition 3.1.** If \( S \) is a commutative semigroup and \( v \in S \) such that for all \( x \in S \) there exist \( m \in \mathbb{Z}_+ \) and \( y \in S \) with \( v^m = xy \), then \( S \) is called a subarchimedean semigroup and the element \( v \) is called a pivot element of \( S \).

**Definition 3.2.** An \( \mathfrak{N} \)-semigroup is a subarchimedean CCIF-semigroup.

**Lemma 3.3.** The pivot elements of a subarchimedean semigroup form an archimedean component and ideal of the semigroup.

*Proof.* Let \( A \) be the set of pivot elements of a subarchimedean semigroup \( S \). Let \( v \in A \) and \( x \in S \). There exist \( m \in \mathbb{Z}_+ \) and \( y \in S \) such that \( v^m = xy \). Then \((vx)^m = x(yz^m)\) for every \( z \in S \); hence \( vz \in A \). Thus \( A \) is an ideal of \( S \). To see that \( A \) is archimedean, let \( u, v \in A \). Then there exist \( m \in \mathbb{Z}_+ \) and \( y \in S \) such that \( v^m = uy \), therefore \( v^{m+1} = u(yv) \) and \( yv \in A \). Therefore \( A \) is archimedean. Let \( A_0 \) be the archimedean component containing \( v \in A \). Obviously \( A \subseteq A_0 \). Let \( u \in A_0 \), so \( u^n = vy \) for some \( n \in \mathbb{Z}_+ \), some \( y \in S \). Let \( z \in S \). As \( v \in A \), \( v^k = zt \) for some \( k \in \mathbb{Z}_+ \), some \( t \in S \). Then \( u^{nk} = v^ky^k = z(ty^k) \), hence \( u \in A, A_0 \subseteq A \). Thus we have proved \( A = A_0 \).

**Lemma 3.4.** A homomorphic image of a subarchimedean semigroup is a subarchimedean semigroup.

*Proof.* Let \( S \) be a subarchimedean semigroup, and \( f \) a surjective homomorphism of \( S \) onto a semigroup \( T \). Let \( v \) be a pivot element of \( S \). Then for all \( x \in S \) there exist \( m \in \mathbb{Z}_+ \) and \( y \in S \) such that \( v^m = xy \). Hence \((f(v))^m = f(x)f(y)\), and we see that \( f(v) \) is a pivot element of \( T \).

**Lemma 3.5.** Let \( S \) be a CCIF-semigroup. \( S \) is subarchimedean if and only if \( S/\rho_a \) is subarchimedean for (some) all \( a \in S \).

*Proof.* If \( S \) is subarchimedean then \( S/\rho_a \) being a homomorphic image of \( S \) is subarchimedean for all \( a \in S \) by Lemma 3.4. Conversely,
if $a \in S$ and $S/\rho_a$ is subarchimedean let $\bar{x}$ denote the $\rho_a$-class of $x \in S$.
Let $\bar{v}$ be a pivot element of $S/\rho_a$. Then for all $\bar{x} \in S/\rho_a$ there exists
$m \in Z_+$ and $\bar{y} \in S/\rho_a$ such that $\bar{v}^m = \bar{x}\bar{y}$. Hence, by the definition of
$\rho_a$ we have $v^m a^k = xya^l$ for some $k, l \in Z_+$. Therefore, $(va)^{m+k} = x(ya^{l+m}v^k)$ and we see that $va$ is a pivot element of $S$.

**Lemma 3.6.** If $S$ is an $\bar{\mathcal{R}}$-semigroup then $\text{Hom}(S, R_+^*) \neq \{0\}$.

**Proof.** By Lemma 3.3, $S$ contains an $\bar{\mathcal{R}}$-semigroup $A$ which is an ideal of $S$. By [2, 7, 8] $\text{Hom}(A, R_+) \neq \{\emptyset\}$. Let $h \in \text{Hom}(A, R_+)$. Then $h \neq 0$. Define $\bar{h}: S \rightarrow R$ by $\bar{h}(x) = h(ax) - h(a)$ for $a \in A$ and $x \in S$. Let $a, b \in A$, and $x \in S$. Then $h(ax + b) = h((ax)b) = h((bx)a) = h(bx) + h(a)$, so $\bar{h}(ax) - h(a) = h(bx) - h(b)$. Thus $\bar{h}$ is well defined. Also, $\bar{h}(xy) = h(a^2xy) - h(a^2) = h(ax) - h(a) + h(ay) - h(a) = h(x) + h(y)$, hence $\bar{h}$ is a homomorphism. If $\bar{h}(x) < 0$ for some $x \in S$, choose $n \in Z_+$ such that $n h(x) < 0$. Since $ax^n \in A$, $h(ax^n) > 0$, but $h(ax^n) = h(a) + n\bar{h}(x) < 0$, a contradiction. Hence $\bar{h} \in \text{Hom}(S, R_+)$. As $\bar{h} | A = h \neq 0$, Hom $(S, R_+) \neq \{0\}$.

**Lemma 3.7.** Let $S$ be an $\bar{\mathcal{R}}$-semigroup. Then $a \in S$ is a pivot
element if and only if $S/\rho_a$ is an abelian group.

**Proof.** Let $A$ be the archimedean ideal of pivot elements of $S$, and let $a \in A$. Then $A/(\rho_a | A)$ is an abelian group, and for all $x \in S$ we have $(x, xa) \in \rho_a$ where $xa \in A$. Hence $S/\rho_a \cong A/(\rho_a | A)$ and $S/\rho_a$ is an abelian group. Conversely if $S/\rho_a$ is an abelian group then for all $x \in S$ there exists $y \in S$ such that $\bar{a} = \bar{x}\bar{y}$ in $S/\rho_a$. (See the notation in the proof of Lemma 3.5.) Thus $a^n = xya^l$ for some $m, l \in Z_+$. Hence $a \in A$.

**Theorem 3.8.** Let $S$ be a CCIF-semigroup, and for $a \in S$ let $\rho_a$
be defined by (2.1.6). The following are equivalent:

(3.8.1) $S$ is an $\bar{\mathcal{R}}$-semigroup.

(3.8.2) $S/\rho_a$ is subarchimedean for all $a \in S$.

(3.8.3) $S/\rho_a$ is subarchimedean for some $a \in S$.

(3.8.4) Some archimedean component of $S$ is an ideal of $S$.

(3.8.5) $S/\rho_a$ is an abelian group for some $a \in S$.

(3.8.6) $S \equiv (G; I)$ where $G$ is an abelian group and $I$ is an $\mathcal{F}$-function on $G$.

(3.8.7) $S$ is isomorphic to a subdirect product of an abelian
group $G$ and a subsemigroup of $R_+$ by means of a defining function $\varphi$ on $G$.

**Proof.** By Lemma 3.5, the first three conditions are equivalent.
By Lemma 3.7, (3.8.1) implies (3.8.5); obviously (3.8.5) implies (3.8.3). By Lemma 3.3 and Lemma 3.7, (3.8.5) implies (3.8.4). Assume (3.8.4). Let \( I \) be the ideal and archimedean component, and let \( a \in I \), \( x \in S \). Since \( ax \in I \), \( a^m = axy \) for some \( m \in \mathbb{Z}_+ \) and some \( y \in I \), hence \( a^m = x(ay) \), that is, \( a \) is a pivot element of \( S \). By Lemma 3.7, (3.8.5) holds. By Theorem 2.1 and Lemma 2.8, (3.8.5) implies (3.8.6). Conversely if \( S \cong (G; I) \), then \( G \cong S/\rho_{(a, i)} \). Thus the first six conditions are equivalent. To see that (3.8.1) and (3.8.6) imply (3.8.7), let \( S \) be an \( \mathbb{R} \)-semigroup. By Lemma 3.6, there exists a nontrivial homomorphism \( h \) of \( S \) into \( \mathbb{R}^+ \), and by (3.8.6), \( S \cong (G; I) \) for some abelian group \( G \) and an \( \mathcal{I} \)-function \( I \). Let \( \varphi(\alpha) = h(0, \alpha)/h(0, \varepsilon) \) for all \( \alpha \in G \). (Clearly we can assume \( h(0, \varepsilon) \neq 0 \).) Then by the proof of Theorem 2.2 we have (3.8.7). Finally if we assume (3.8.7), \( S \cong ((G; \varphi)) \) for some \( \varphi: G \to \mathbb{R}^+ \), then when we define \( I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha, \beta) \), we have \( S \cong (G; I) \) as before. Hence (3.8.7) implies (3.8.6). The proof has been completed.

**Corollary 3.9.** Let \( S \) be a CCIF-semigroup. \( S \) is an \( \mathbb{R} \)-semigroup if and only if \( S/\rho_a \) is an abelian group for all \( a \in S \).

**Proof.** Let \( A \) be the set of pivot elements of \( S \). If \( S \) is an \( \mathbb{R} \)-semigroup then \( S = A \) and so \( S/\rho_a \) is an abelian group for all \( a \in S \). Conversely if \( S/\rho_a \) is an abelian group for all \( a \in S \) then \( S = A \) by Lemma 3.7. Hence \( S \) is archimedean, hence an \( \mathbb{R} \)-semigroup.

4. Homomorphisms into \( \mathbb{R}^+ \). As seen in § 3 every \( \mathbb{R} \)-semigroup has a nontrivial homomorphism into \( \mathbb{R}^+ \). The following question is raised.

Is a CCIF-semigroup nontrivially homomorphic into \( \mathbb{R}^+ \)? We cannot answer this question in general, but in some special case it is affirmative.

Let \( S \) be a CCIF-semigroup. As defined in § 1, \( Q(S) \) denotes the quotient group and \( D(S) \) the divisible hull of \( Q(S) \).

\[
D(S) \cong \bigoplus_{p \in \mathbb{Q}} C(p^\infty) \bigoplus_{a \in \Gamma} \mathbb{R}_a
\]

where \( \mathbb{R}_a \) is a copy of the additive group of rationals and \( C(p^\infty) \) is a quasicyclic group. The cardinality \( |\Gamma| \) of \( \Gamma \) is called the rank of \( S \). In the present case the rank of \( S \) is not zero since \( \bigoplus_{p \in \mathbb{Q}} C(p^\infty) \) is torsion while \( S \) is torsion-free.

In particular, assume that \( S \) is of finite rank. Let \( T \) be the torsion subgroup of \( D(S) \), then \( D(S) = T \oplus \mathbb{R}_1 \oplus \cdots \oplus \mathbb{R}_n \) where \( n \) is
the rank of $S$. We can assume $R_i \neq \{0\}$ for $i = 1, \ldots, n$. Let $P_i = R_i \bigoplus \cdots \bigoplus R_i$ for each $i = 1, 2, \ldots, n$. Then $P_n = P_{n-1} \bigoplus R_n$ if $n > 1$; and $D(S) = T \bigoplus P_n$ if $n \geq 1$. Let $\alpha, \bar{\sigma}, \sigma, \pi_n, \tau_n$ be the respective projection homomorphisms:

$$
\alpha: D(S) \longrightarrow T, \quad \bar{\sigma}: D(S) \longrightarrow P_n, \quad \sigma = \bar{\sigma}|S,
\pi_n: P_n \longrightarrow P_{n-1}, \quad \tau: P_n \longrightarrow R_n \quad (n \geq 1)
$$

**Theorem 4.1.** If $S$ is a CCIF-semigroup of finite rank, then $\text{Hom}(S, R^0_+) \neq \{0\}$. ($R^0_+$ is the additive semigroup of nonnegative rationals.)

**Proof.** $S$ is viewed as a subsemigroup of $D(S)$. We will prove the theorem by induction on $n$. Let $V_n = \pi_n \sigma(S)$, $W_n = \tau_n \sigma(S)$, $V = \sigma(S)$, $T' = \alpha(S)$. As $D(S) = T \bigoplus P_n$, we have

$$
S = T' \bigoplus V, \quad \text{and if } n > 1, \quad V = V_n \bigoplus W_n,
$$

where $\bigoplus$ denotes a subdirect sum, $V \subseteq P_n$, $V_n \subseteq P_{n-1}$, $W_n \subseteq R_n$, and $T' \subseteq T$, hence $T'$ is a torsion group. First we prove

(4.1.1) $V$ does not contain 0.

Suppose $V$ contains 0. There is $x \in T'$ such that $(x, 0) \in S$. Since $T'$ is a torsion group, $mx = 0$ for some $m \in \mathbb{Z}_+$. Then $(0, 0) = (x, 0)^m \in S$. This is a contradiction as $S$ has no idempotent.

In case $n = 1$, $S = T' \bigoplus W_1$ where $W_1 = V \subseteq R_1$. By (4.1.1), $W_1$ must be isomorphic to a positive rational semigroup $R'_1$, say, under $f$, i.e., $f(W_1) = R'_1$, hence $f \tau_1 \sigma \in \text{Hom}(S, R^0_+)[\{0\}]$.

Assume $n > 1$ and that the theorem holds for all semigroups of rank $i$ such that $i \leq n - 1$. As denoted above,

$$
S = T' \bigoplus V, \quad V = V_n \bigoplus W_n
$$

where $V_n \subseteq P_{n-1}$, $W_n \subseteq R_n$. We can assume $V_n \neq \{0\}$, otherwise it is reduced to the case $n = 1$.

If $V_n$ is a CCIF-semigroup, $V_n$ has a nontrivial homomorphism $f$ from $V_n$ into $R^0_+$ by the induction assumption, hence $f \pi_n \sigma \in \text{Hom}(S, R^0_+)[\{0\}]$.

If $V_n$ is a CCI-semigroup which is not a group, then $V_n = V'_n \cup H$ where $V'_n \neq \emptyset$, $H \neq \emptyset$, $V'_n$ is an ideal of $V_n$ and it is a CCIF-semigroup, and $H$ is a group. Define $S'$ by $S' = ((\pi_n \sigma)^{-1}(V'_n)) \cap S$ and $W'_n = \tau_n \sigma(S')$. Then $S'$ is an ideal of $S$ and

$$
S' = V'_n \bigoplus W'_n.
$$

By the preceding paragraph, $\text{Hom}(S', R^0_+)$ contains a nontrivial
element \( f \). However, since \( S' \) is an ideal of \( S \), \( f \) can be extended to \( \tilde{f} \in \text{Hom}(S, R^+_+) \). In fact \( \tilde{f} \) is obtained by defining \( \tilde{f}(x) = f(ax) - f(a) \) where \( x \in S, a \in S' \). It is easy to show that \( \tilde{f} \) is well defined and a homomorphism. Suppose \( \tilde{f}(x) < 0 \) for some \( x \in S \). There exists \( m \in \mathbb{Z}_+ \) such that \( m\tilde{f}(x) + f(a) < 0 \). However
\[
m\tilde{f}(x) + f(a) = f(ax^m) \geq 0
\]
since \( ax^m \in S' \). This contradicts the assumption. Therefore \( \tilde{f}(x) \geq 0 \) for all \( x \in S \). Hence \( \text{Hom}(S, R^+_+) \neq \{0\} \). Assume \( V_n \) is a group. Let \( W_n = \{(0, z) : z \in W_n \} \cap V \). It is obvious that \( W_n \) is a subsemigroup if \( W_n \neq \emptyset \). If \( x \in V, x \) has the form \( x = (x_1, x_2) \in V_n \oplus W_n, x_1 \in V_n, x_2 \in W_n \). Since \( V_n \) is a group, there exists \( y_2 \in W_n \) such that \( y = (-x_1, y_2) \in V \). Then \( xy = (0, x_2 + y_2) \in \overline{W_n} \). This proves that \( \overline{W_n} \neq \emptyset \) and it is cofinal in \( V \). Suppose \( x \in V \) and \( a, xa \in \overline{W_n} \). We write \( x = (x_1, x_2), a = (0, a_2) \) viewing them as in \( V_n \oplus W_n \). Then \( xa = (x_1, x_2 + a_2) \in \overline{W_n} \) implies \( x_1 = 0 \), hence \( x \in \overline{W_n} \). Thus \( \overline{W_n} \) is unitary in \( V \). Since \( \overline{W_n} \) does not contain \((0, 0)\) by (4.1.1), \( \overline{W_n} \) is isomorphic to a positive rational semigroup \( R'_+ \) under \( f: \overline{W_n} \to R'_+ \). By (4.1.2) below, \( f \) extends to \( \tilde{f} \in \text{Hom}(V, R'_+) \). Therefore \( \tilde{f} \sigma \in \text{Hom}(S, R'_+) \setminus \{0\} \).

(4.1.2) Let \( S \) be a CCIF-semigroup and let \( U \) be a unitary cofinal subsemigroup of \( S \). Then every homomorphism of \( U \) into \( R'_+ \) extends to a homomorphism of \( S \) into \( R'_+ \).

This is immediately obtained from [4].

The proof of Theorem 4.1 has been completed.

Remark 4.2. Let \( S = R_+ \oplus (\bigoplus_{\alpha \in \Gamma} R_\alpha) \) where \( |\Gamma| = \infty \), \( R_\alpha \) is the group of rationals. We note that \( \text{Hom}(S, R^+_+) \neq \{0\} \), yet \( S \) is not of finite rank. Thus the converse of Theorem 4.1 does not hold.

Next we consider the relation between nontriviality of \( \text{Hom}(S, R^+_+) \) and the property

\[
\bigcap_{n=1}^{\infty} a^nS = \emptyset \quad \text{for some } a \in S.
\]

Proposition 4.4. If \( \text{Hom}(S, R^+_+) \neq \{0\} \), then there is an element \( a \in S \) satisfying (4.3).

Proof. Let \( h \in \text{Hom}(S, R^+_+) \), \( h \neq 0 \). There is \( a \in S \) such that \( h(a) \neq 0 \). Choose \( a \) as a standard element. We have \( C_1 = \emptyset \) by Proposition 2.7 and then have (4.3) by Proposition 2.11.

The converse of Proposition 4.4 is still open.
Problem 4.5. Let $S$ be a CCIF-semigroup. If $\bigcap_{n=1}^{\infty} a^nS = \emptyset$ for some $a \in S$, then is the following true
\[ \text{Hom}(S, R^0_+) \neq \{0\} \]

However, we give a few examples with respect to the related problems.

Example 4.6. Let $\bigcap_{n=1}^{\infty} a^nS = \emptyset$. There does not necessarily exist $h \in \text{Hom}(S, R^0_+)$ such that $h(a) \neq 0$.

Let $S = ((Z^+_1; \varphi))$ where $\varphi: Z^+_1 \rightarrow Z$ is defined by
\[ \varphi(m) = 1 - m^2. \]

It can be easily shown that $\varphi$ is a normal defining function on $Z^+_1$, and that if $a = ((1, 0))$, $\bigcap_{n=1}^{\infty} a^nS = \emptyset$. Every element $f_i$ of $\text{Hom}(Z^+_1, R)$ has the form
\[ f_i(m) = tm \quad t \in R, \]

but there is no $t$ satisfying
\[ \varphi(m) + f_i(m) = 1 - m^2 + tm \geq 0 \quad \text{for all } m \in Z^+_1. \]

By Proposition 2.6, (2.6.1), there is no $h \in \text{Hom}(S, R^0_+)$ with $h(a) \neq 0$. However, the projection $h_0: S \rightarrow Z^0_+$ is a nontrivial element of $\text{Hom}(S, R^0_+)$ such that $h_0(a) = 0$. Thus $\text{Hom}(S, R^0_+) \neq \{0\}$ and so Example 4.6 is not a counterexample to the converse of Proposition 4.4. In fact the semigroup $S$ is an $\mathbb{N}$-semigroup.

Example 4.7. We exhibit an example of a CCIF-semigroup $S$ which satisfies
\[ \bigcap_{n=1}^{\infty} a^nS \neq \emptyset \quad \text{for all } a \in S, \]

and hence $\text{Hom}(S, R^0_+) = \{0\}$.

Let $S = \{(a_1, \cdots, a_m): m, a_m \in Z^+_1, a_i \in Z, 1 \leq i < m\}$

and define a binary operation on $S$ as follows: if $m \leq n$,
\[
(a_{i_1}, \cdots, a_{i_m})(b_{j_1}, \cdots, b_{j_n}) = (b_{j_{i_1}}, \cdots, b_{j_{i_m}})(a_{i_1}, \cdots, a_{i_m}) = (a_{i_1} + b_{j_{i_1}}, \cdots, a_{i_m} + b_{j_{i_m}}, b_{j_{i_{m+1}}}, \cdots, b_{j_n}).
\]

Then, with this product, $S$ is a CCIF-semigroup. Let $S_i = Z^+_1$ and $S_i = Z^{i-1}_+ \times Z_m$ for $i > 1$. Then $S$ is the union of the infinite chain of $S_i$'s, $S = \bigcup_{i=1}^{\infty} S_i$ and $S_iS_j \subseteq S_j$ if $i \leq j$. If $a \in S_m$ then
\[ \bigcap_{n=1}^{\infty} a^nS = \bigcup_i S_i. \]
DEFINITION 4.8. A semigroup \( S \) is called an \( \mathcal{R}' \)-semigroup if \( S \) is isomorphic to a subsemigroup of an \( \mathcal{R} \)-semigroup.

THEOREM 4.9. Let \( S \) be a CCIF-semigroup. \( S \) is an \( \mathcal{R}' \)-semigroup if and only if

\[
\text{Hom}(S, R_+) \neq \emptyset.
\]

Proof. Assume that \( S \) is a subsemigroup of an \( \mathcal{R} \)-semigroup \( T. \) By [6, 7] there is an \( h \in \text{Hom}(T, R_+). \) Let \( h_1 \) be the restriction of \( h \) to \( S. \) Then \( h_1 \in \text{Hom}(S, R_+). \)

Conversely let \( \text{Hom}(S, R_+) \neq \emptyset. \) By Proposition 2.7, \( C_1 = \emptyset. \) By Theorem 2.2 and its Corollaries, \( S \cong (C; \varphi) \) where \( C \) is a CCI-semigroup and \( \varphi \in \text{DNfn}(C, R); \) and \( S \) is isomorphic to a subdirect product of a subsemigroup \( P \) of \( R_+ \) and \( C, S \cong P \times_s C. \) Let \( Q \) be the group of quotients of \( C. \) Then \( P \times_s C \) is a subsemigroup of the direct product \( R_+ \times Q, \) but the last direct product is an \( \mathcal{R} \)-semigroup. Consequently \( S \) is an \( \mathcal{R}' \)-semigroup.

The two concepts, \( \mathcal{R} \)-semigroup and \( \mathcal{R}' \)-semigroup, are independent of each other.

EXAMPLE 4.10. Let \( S = Z_+ \cup (Z \times Z_+). \) A binary operation is defined to be the same as Example 4.7, that is, \( S \) is a subsemigroup of the semigroup in Example 4.7. \( S \) is an \( \mathcal{R} \)-semigroup, but we prove \( \text{Hom}(S, R_+) = \emptyset \) as follows:

Let \( x \in Z_+ \) and \((a_1, a_2) \in Z \times Z_+. \) There exists \((b_1, b_2) \in Z \times Z_+ \) such that

\[
x \cdot (b_1, b_2) = (a_1, a_2).
\]

Suppose \( h \in \text{Hom}(S, R_+) \neq \emptyset. \) Then

\[
h(x) < h(a_1, a_2) \quad \text{for all} \quad x \in Z_+ \quad \text{and all} \quad (a_1, a_2) \in Z \times Z_+.
\]

In particular \( h(1) < h(a_1, a_2), \) but there is \( x \in Z_+ \) such that \( x \cdot h(1) > h(a_1, a_2). \) Accordingly \( h(x) = x \cdot h(1) > h(a_1, a_2). \) This contradiction proves \( \text{Hom}(S, R_+) = \emptyset, \) hence \( S \) is not an \( \mathcal{R}' \)-semigroup.

EXAMPLE 4.11. Let \( S \) be the free commutative semigroup generated by infinitely countable letters \( a_1, a_2, \ldots, a_n, \ldots. \) (The empty word is not considered.) \( S \) is obviously a CCIF-semigroup and \( \text{Hom}(S, R_+) \neq \emptyset \) since

\[
a_1^{m_1} \cdots a_k^{m_k} \mapsto m_1 + \cdots + m_k
\]
gives a homomorphism of \( S \) into \( Z_+. \) However \( S \) is not an \( \mathcal{R} \)-semi-
group, as the greatest semilattice homomorphic image of $S$ does not have a zero.

**REMARK.** According to his recent personal letter to one of the authors, Professor Yuji Kobayashi, Tokushima University, has negatively answered Problem 4.5 by showing a counter example.

**ACKNOWLEDGMENT.** The authors express their heart felt thanks to the referee of his kind advice to this paper.

**REFERENCES**


Received March 19, 1975.

**CALIFORNIA STATE UNIVERSITY, SACRAMENTO**
**CALIFORNIA STATE COLLEGE, STANISLAUS**
**AND**
**UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA**
Graham Donald Allen, Francis Joseph Narcowich and James Patrick Williams, An operator version of a theorem of Kolmogorov .................................................. 305
Joel Hilary Anderson and Ciprian Foias, Properties which normal operators share with normal derivations and related operators ................................................. 313
Constantin Gelu Apostol and Norberto Salinas, Nilpotent approximations and quasinilpotent operators ............................................................. 327
James M. Briggs, Jr., Finitely generated ideals in regular F-algebras .................. 339
Frank Benjamin Cannonito and Ronald Wallace Gatterdam, The word problem and power problem in 1-relator groups are primitive recursive ......................... 351
Clifton Earle Corzatt, Permutation polynomials over the rational numbers .......... 361
L. S. Dube, An inversion of the S_2 transform for generalized functions .......... 383
William Richard Emerson, Averaging strongly subadditive set functions in unimodular amenable groups, I ......................................................... 391
Barry J. Gardner, Semi-simple radical classes of algebras and attainability of identities ....................................................................................... 401
Irving Leonard Glicksberg, Removable discontinuities of A-holomorphic functions ................................................................. 417
Fred Halpern, Transfer theorems for topological structures .............................. 427
H. B. Hamilton, T. E. Nordahl and Takayuki Tamura, Commutative cancellative semigroups without idempotents .................................................. 441
Melvin Hochster, An obstruction to lifting cyclic modules .................................. 457
Alistair H. Lachlan, Theories with a finite number of models in an uncountable power are categorical ............................................................... 465
Kjeld Laursen, Continuity of linear maps from C*-algebras .............................. 483
Tsai Sheng Liu, Oscillation of even order differential equations with deviating arguments ................................................................................... 493
Jorge Martinez, Doubling chains, singular elements and hyper-Z l-groups ........ 503
Mehdi Radjabalipour and Heydar Radjavi, On the geometry of numerical ranges ............................................................... 507
Thomas I. Seidman, The solution of singular equations, I. Linear equations in Hilbert space ................................................................. 513
R. James Tomkins, Properties of martingale-like sequences .............................. 521
Alfons Van Daele, A Radon Nikodym theorem for weights on von Neumann algebras ....................................................................................... 527
Kenneth S. Williams, On Euler's criterion for quintic nonresidues ................... 543
Manfred Wischnewsky, On linear representations of affine groups, I .............. 551
Scott Andrew Wolpert, Noncompleteness of the Weil-Petersson metric for Teichmüller space ................................................................. 573
Volker Wrobel, Some generalizations of Schauder's theorem in locally convex spaces ....................................................................................... 579
Birge Huisgen-Zimmermann, Endomorphism rings of self-generators .............. 587
Kelly Denis McKennon, Corrections to: “Multipliers of type (p, p)”; “Multipliers of type (p, p) and multipliers of the group L_p-algebras”; “Multipliers and the group L_p-algebras” ......................................................... 603
Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, Correction to: “a*-closures to completely distributive lattice-ordered groups” .......................................................... 606
Zvi Arad and George Isaac Glauberman, Correction to: “A characteristic subgroup of a group of odd order” ......................................................... 607
Roger W. Barnard and John Lawson Lewis, Correction to: “Subordination theorems for some classes of starlike functions” ................................. 607
David Westreich, Corrections to: “Bifurcation of operator equations with unbounded linearized part” ....................................................... 608