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**OSCILLATION OF EVEN ORDER DIFFERENTIAL EQUATIONS  
WITH DEVIATING ARGUMENTS**

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## OSCILLATION OF EVEN ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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**The purpose of this paper is to give some oscillation  
 criteria for even order differential equations with deviating  
 arguments.**

A continuous real-valued function  $f(t)$  which is defined for all large  $t$  is called *oscillatory* if it has arbitrary large zero, otherwise it is called *nonoscillatory*.

Our work extends some results obtained by Ladas and Lakshmi-kantham [3] and Chiou [1] for second order equations.

1. In this section, we are concerned with the equation

$$(1.1) \quad y^{(n)}(t) - \sum_{j=1}^m p_j(t)y(g_j(t)) = 0, \quad n \geq 2 \text{ an even integer,}$$

where the following assumptions are assumed to hold:

(I<sub>1</sub>)  $g_j(t) \leq t$  on  $[a, \infty)$ ,  $j = 1, 2, \dots, m$  and  $g_k(t) < t$  for some  $1 \leq k \leq m$ ;  $g'_j(t) \geq 0$  on  $[a, \infty)$  and  $g_j(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ,  $j = 1, 2, \dots, m$ .

(I<sub>2</sub>)  $p_j(t) \geq 0$ ,  $p'_j(t) \leq 0$  on  $[a, \infty)$ ,  $j = 1, 2, \dots, m$  and  $p_k(t) > 0$  on  $[a, \infty]$  for the same  $k$  as in (I<sub>1</sub>).

We shall give a sufficient condition for all bounded solutions of (1.1) to be oscillatory. Our result extends Ladas and Lakshmikantham's Theorems 2.1-2.4 in [3] to arbitrary even order equation (1.1).

**LEMMA 1.1** (Lemma 2 in [2]). *If  $y$  is a function, which together with its derivatives of order up to  $(n - 1)$  inclusive, is absolutely continuous and of constant sign on the interval  $[a, \infty)$  and  $y^{(n)}(t)y(t) \geq 0$  on  $[a, \infty)$ , then either*

$$y^{(j)}(t)y(t) \geq 0, \quad j = 0, 1, \dots, n,$$

or there is an integer  $l$ ,  $0 \leq l \leq n - 2$ , which is even when  $n$  is even and odd when  $n$  is odd, such that

$$y^{(j)}(t)y(t) \geq 0, \quad j = 0, 1, \dots, l,$$

and

$$(-1)^{n+j}y^{(j)}(t)y(t) \geq 0, \quad j = l + 1, \dots, n$$

for  $t$  in  $[a, \infty)$ .

**THEOREM 1.1.** *If  $(t - g_k(t))^n p_k(t) \geq n!$  for all sufficiently large  $t$ , then every bounded solution of (1.1) is oscillatory.*

*Proof.* Let  $y$  be a nonoscillatory bounded solution of (1.1). Without loss of generality, we can assume that  $y(t) > 0$  for  $t \geq T_1$ . There is a  $T_2 \geq T_1$  such that  $g_j(t) > T_1$  ( $j = 1, 2, \dots, m$ ) for  $t \geq T_2$ . There is a  $T_3 \geq T_2$  such that  $y^{(j)}(t)$  ( $j = 1, 2, \dots, n-1$ ) is of constant sign for  $t \geq T_3$ . By Lemma 1.1 and since  $y$  is bounded, we have

$$(1.2) \quad (-1)^j y^{(j)}(t) > 0 \quad (j = 0, 1, \dots, n-1) \quad \text{for } t \geq T_3.$$

It follows from (1.1) that

$$y^{(n+1)}(t) = \sum_{j=1}^m \{p'_j(t)y(g_j(t)) + p_j(t)y'(g_j(t))g'_j(t)\} \leq 0$$

for  $t \geq T_3$ .

By Taylor's theorem, there is a  $\xi$  between  $\tau$  and  $t$  such that

$$(1.3) \quad \begin{aligned} y^{(n-1)}(\tau) &= y^{(n-1)}(t) + y^{(n)}(t)(\tau - t) + y^{(n+1)}(\xi) \frac{(\tau - t)^2}{2!} \\ &\leq y^{(n-1)}(t) + y^{(n)}(t)(\tau - t) \\ &= y^{(n-1)}(t) - (t - \tau) \sum_{j=1}^m p_j(t)y(g_j(t)) \end{aligned}$$

for  $\tau \geq T_3$  and  $t \geq T_3$ .

Integrating (1.3) with respect to  $\tau$  from  $s$  to  $t > s$ , we have

$$\begin{aligned} y^{(n-2)}(t) - y^{(n-2)}(s) &\leq y^{(n-1)}(t)(t - s) - \frac{(t - s)^2}{2!} \sum_{j=1}^m p_j(t)y(g_j(t)) \\ &\leq y^{(n-1)}(t)(t - s) - \frac{(t - s)^2}{2!} p_k(t)y(g_k(t)) \end{aligned}$$

or

$$\begin{aligned} y^{(n-2)}(s) &\geq y^{(n-2)}(t) - y^{(n-1)}(t)(t - s) \\ &\quad + \frac{(t - s)^2}{2!} p_k(t)y(g_k(t)) \quad \text{for } t > s \geq T_3. \end{aligned}$$

In a similar way repeatedly, we shall have

$$(1.4) \quad \begin{aligned} y'(s) &\leq y'(t) - y''(t)(t - s) + y'''(t) \frac{(t - s)^2}{2!} \\ &\quad - \dots + y^{(n-1)}(t) \frac{(t - s)^{n-2}}{(n-2)!} \\ &\quad - p_k(t)y(g_k(t)) \frac{(t - s)^{n-1}}{(n-1)!} \quad \text{for } t > s \geq T_3. \end{aligned}$$

Integrating (1.4) from  $g_k(t)$  to  $t$ , we obtain

$$\begin{aligned} y(t) - y(g_k(t)) &\leq y'(t)(t - g_k(t)) - y''(t) \frac{(t - g_k(t))^2}{2!} \\ &\quad + y'''(t) \frac{(t - g_k(t))^3}{3!} - \dots + y^{(n-1)}(t) \frac{(t - g_k(t))^{n-1}}{(n-1)!} \\ &\quad - p_k(t)y(g_k(t)) \frac{(t - g_k(t))^n}{n!} \end{aligned}$$

or

$$\begin{aligned} (t - g_k(t))y'(t) - \frac{(t - g_k(t))^2}{2!}y''(t) + \frac{(t - g_k(t))^3}{3!}y'''(t) - \dots \\ + \frac{(t - g_k(t))^{n-1}}{(n-1)!}y^{(n-1)}(t) \\ + \left[ 1 - \frac{p_k(t)(t - g_k(t))^n}{n!} \right] y(g_k(t)) - y(t) \geq 0 \end{aligned}$$

for all sufficiently large  $t$ .

It follows from (1.2) that

$$1 - \frac{p_k(t)(t - g_k(t))^n}{n!} > 0$$

or

$$(t - g_k(t))^n p_k(t) > n! \quad \text{for all sufficiently large } t.$$

This is a contradiction and the proof is then complete.

EXAMPLE 1.1. If we consider the equation

$$(1.5) \quad y^{(4)}(t) - \frac{(2k\pi)^4}{\tau^4} y(t - \tau) = 0, \quad \tau > 0, \quad k = 1, 2, \dots,$$

then  $p(t) = (2k\pi)^4 \tau^{-4}$  satisfies the assumption and every bounded solution of (1.5) is oscillatory. A bounded oscillatory solution of (1.5) is  $y(t) = \sin(2k\pi/\tau)t$ ,  $k = 1, 2, \dots$ .

COROLLARY 1.1. Consider the equation

$$(1.6) \quad y^{(n)}(t) - \sum_{j=1}^m y(t - \tau_j) = 0, \quad \tau_j \geq 0 \quad (j = 1, 2, \dots, m).$$

If  $\tau_k \geq \sqrt[n]{n!}$  for some  $1 \leq k \leq m$ , then every bounded solution of (1.6) is oscillatory.

2. We shall consider the equations

$$(2.1) \quad y^{(n)}(t) + p(t)f(y(t), y(g(t))) = 0$$

and

$$(2.2) \quad y^{(n)}(t) + F(t, y(t), y(g(t))) = 0, \quad n \geq 2 \text{ an even integer,}$$

with the following conditions:

(II<sub>1</sub>)  $g(t)$  is a continuous function on  $[a, \infty)$  such that  $g(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

(II<sub>2</sub>)  $p(t)$  is a nonnegative continuous function on  $[a, \infty)$ .

(II<sub>3</sub>)  $f(u, v)$  is a continuous function on  $R^2$  and has the same sign as that of  $u$  and  $v$  if  $uv > 0$ .

(II<sub>4</sub>)  $F(t, u, v)$  is a continuous function on  $[a, \infty) \times R^2$ , non-decreasing in  $u$  and in  $v$  for each fixed  $t$  and has the same sign as that of  $u$  and  $v$  if  $uv > 0$ .

In this section, we shall give conditions which will ensure that every extensible solution  $y$  of (2.1) or (2.2) is either oscillatory or  $y''(t)y(t) > 0$  for all sufficiently large  $t$ . This generalizes to higher order equations some results due to Chiou [1, Theorems 2.2, 2.6, 2.8, 2.9, 2.12, 2.14, 2.15, 2.18, 2.19, 2.20, 2.22 and 2.23].

LEMMA 2.1 (Lemma 1 in [2]). *If  $y$  is a function which together with its derivatives of order up to  $(n - 1)$  inclusive, is absolutely continuous and of constant sign on the interval  $[a, \infty)$  and  $y^{(m)}(t)y(t) \leq 0$  on  $[a, \infty)$ , then there is an integer  $l$ ,  $0 \leq l \leq n - 1$ , which is odd when  $n$  is even and even when  $n$  is odd, such that*

$$y^{(j)}(t)y(t) \geq 0, \quad j = 0, 1, \dots, l,$$

and

$$(-1)^{n+j-1}y^{(j)}(t)y(t) \geq 0, \quad j = l + 1, \dots, n$$

for  $t$  in  $[a, \infty)$ .

LEMMA 2.2 (Corollary 2.3 in [4]). *If*

$$(2.3) \quad \int_{\infty}^{\infty} t^{n-1}F(t, \gamma, \gamma) dt = \pm \infty \quad \text{for each } \gamma \neq 0,$$

then every bounded solution of (2.2) is oscillatory.

In a similar way, we have

LEMMA 2.3. *If*

$$(2.4) \quad \int_{\infty}^{\infty} t^{n-1}p(t)dt = \infty,$$

then every bounded solution of (2.1) is oscillatory.

**THEOREM 2.1.** Assume that

(i) there exists a positive function  $q(t)$  such that

$$(2.5) \quad q(t) \leq \min \{g(t), t\}, \quad q'(t) > 0 \text{ and } q''(t) \leq 0 \text{ for } t \geq a.$$

(ii) there exist positive functions  $h(t)$  and  $h_1(t)$  for  $t \geq a > 0$  and a constant  $M > 0$  such that

$$(2.6) \quad \int^\infty \frac{dv}{h(v)} < \infty \text{ and } \liminf_{v \rightarrow \infty} \left| \frac{h_1(cv)f(u, v)}{h(v)} \right| \geq \varepsilon > 0$$

for  $u > M$ , every  $c > 0$  and for some  $\varepsilon = \varepsilon(c)$ .

If

$$(2.7) \quad \int^\infty \frac{q^{n-1}(t)p(t)}{h_1(g(t))} dt = \infty,$$

then every extensible solution of (2.1) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

*Proof.* Let  $y$  be a nonoscillatory solution of (2.1). Without loss of generality, we may assume that  $y(t) > 0$  for  $t \geq T_1$ . There is a  $T_2 \geq T_1$  such that  $g(t) \geq T_1$  for  $t \geq T_2$ . It follows from (2.1) that  $y^{(n)}(t) \leq 0$  for  $t \geq T_2$ . There is a  $T_3 \geq T_2$  such that each  $y^{(j)}(t)$ ,  $j = 1, 2, \dots, n - 1$ , is of constant sign for  $t \geq T_3$ . By Lemma 2.1,  $y'(t) > 0$  for  $t \geq T_3$ .

If  $y''(t) > 0$  for  $t \geq T_3$ , then our proof is done. Assume that  $y''(t) < 0$  for  $t \geq T_3$ . Then, by Lemma 2.1, we have

$$(2.8) \quad (-1)^{j-1}y^{(j)}(t) > 0 \quad (j = 1, 2, \dots, n - 1) \text{ for } t \geq T_3.$$

Since (2.7) implies (2.4) and since  $y'(t) > 0$  for  $t \geq T_3$ , it follows from Lemma 2.3 that  $y(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

Integrating (2.1) repeatedly from  $t$  to  $t' > 2t \geq 2T_3$  and using (2.8) as well as integration by parts, we have

$$(2.9) \quad y'(t) \geq \frac{1}{(n - 2)!} \int_t^{t'} (u - t)^{n-2} p(u) f(y(u), y(g(u))) du.$$

Dividing (2.9) by  $h(y(g(t)))$ , we have

$$(2.10) \quad \frac{y'(t)}{h(y(g(t)))} \geq \frac{1}{(n - 2)!} \int_t^{t'} \frac{(u - t)^{n-2} p(u) f(y(u), y(g(u)))}{h(y(g(u)))} du.$$

Since  $y'(t)$  is decreasing for  $t \geq T_3$ , there exist  $T_4 \geq T_3$  and  $c > 0$  such that  $cy(t) \leq t$  for  $t \geq T_4$ . From (2.5) and (2.6) we have

$$\begin{aligned}
 (2.11) \quad \frac{p(u)f(y(u), y(g(u)))}{h(y(g(u)))} &\geq \frac{p(u)}{h_1(g(u))} \frac{h_1(cy(g(u)))f(y(u), y(g(u)))}{h(y(g(u)))} \\
 &\geq \frac{\varepsilon p(u)}{h_1(g(u))}.
 \end{aligned}$$

It follows from (2.10) and (2.11) that

$$\begin{aligned}
 (2.12) \quad \frac{y'(t)}{h(y(q(t)))} &\geq \frac{\varepsilon}{(n-2)!} \int_t^{t'} \frac{(u-t)^{n-2} p(u)}{h_1(g(u))} du \\
 &\geq \frac{\varepsilon t^{n-2}}{(n-2)!} \int_{2t}^{t'} \frac{p(u)}{h_1(g(u))} du.
 \end{aligned}$$

Since  $q(t) \leq t$ ,  $q'(t) > 0$  and  $q''(t) \leq 0$ , we have

$$(2.13) \quad \frac{y'(q(t))q'(t)}{h(y(q(t)))} \geq \frac{y'(t)q'(t)}{h(y(q(t)))} \geq \frac{\varepsilon q^{n-2}(2t)q'(2t)}{2^{n-2}(n-2)!} \int_{2t}^{t'} \frac{p(u)}{h_1(g(u))} du.$$

Integrating (2.13) from  $T_4$  to  $T > T_4$  and using integration by parts, we get

$$\begin{aligned}
 \int_{T_4}^T \frac{dy(q(s))}{h(y(q(s)))} &\geq \frac{\varepsilon q^{n-1}(2T)}{2^{n-1}(n-1)!} \int_{2T}^{t'} \frac{p(u)}{h_1(g(u))} du - \frac{\varepsilon q^{n-1}(2T_4)}{2^{n-1}(n-1)!} \\
 &\quad \times \int_{2T_4}^{t'} \frac{p(u)}{h_1(g(u))} du + \frac{\varepsilon}{2^{n-2}(n-2)!} \int_{T_4}^T \frac{q^{n-1}(2t)p(2t)}{h_1(g(2t))} dt \\
 &\geq -\frac{q^{n-1}(2T_4)}{2^{n-1}(n-1)!} \int_{2T_4}^{t'} \frac{p(u)}{h_1(g(u))} du + \frac{\varepsilon}{2^{n-2}(n-1)!} \\
 &\quad \times \int_{T_4}^T \frac{q^{n-1}(2t)p(2t)}{h_1(g(2t))} dt.
 \end{aligned}$$

Using (2.12) for  $t = T_4$ , we have

$$\begin{aligned}
 \int_{T_4}^T \frac{dy(q(s))}{h(y(q(s)))} &\geq -\frac{\varepsilon q^{n-1}(2T_4)}{2^{n-1}(n-1)!} \frac{y'(T_4)}{h(y(q(T_4)))} \frac{(n-2)!}{\varepsilon T_4^{n-2}} \\
 &\quad + \frac{\varepsilon}{2^{n-2}(n-1)!} \int_{T_4}^T \frac{q^{n-1}(2t)p(2t)}{h_1(g(2t))} dt.
 \end{aligned}$$

Let  $T \rightarrow \infty$  and obtain

$$\int_{2T_4}^{\infty} \frac{q^{n-1}(s)p(s)}{h_1(g(s))} ds < \infty.$$

This contradicts (2.7) and the proof is complete.

If  $y(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , then by the monotonicity of  $F(t, u, v)$ , there exist  $\alpha > 0$  and  $T > 0$  such that

$$F(t, y(t), y(g(t))) \geq F(t, \alpha, \alpha) > 0 \quad \text{for } t \geq T.$$

By using this fact and Lemma 2.2 instead of Lemma 2.3 in the proof

of Theorem 2.1, we can prove the following

**THEOREM 2.2.** *Assume that (2.5) is satisfied and that there exist positive nondecreasing functions  $h(t)$  and  $h_1(t)$  for  $t \geq a > 0$  and a constant  $M > 0$  such that*

$$(2.14) \quad \int^\infty \frac{dv}{h(v)} < \infty \text{ and } \liminf_{v \rightarrow \infty} \left| \frac{h_1(cv)F(t, u, v)}{h(v)} \right| \geq \varepsilon F(t, \alpha, \alpha) > 0$$

for  $u > M$ , every  $c > 0$  and for some  $\varepsilon = \varepsilon(c)$  and  $\alpha > 0$ . If

$$(2.15) \quad \int^\infty \frac{q^{n-1}(t)F(t, \alpha, \alpha)}{h_1(g(t))} dt = \infty ,$$

then every extensible solution  $y$  of (2.2) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

The following example presents the occurrence of the case  $y''(t)y(t) > 0$  for all sufficiently large  $t$ .

**EXAMPLE 2.1.** If we consider the equation

$$(2.16) \quad y^{(4)}(t) + \frac{15}{16}(t - \tau)^{-3/2}(t - 2\tau)^{-5/6}[y(t - \tau)]^{1/3} = 0 ,$$

then  $F(t, u, v) = (15/16)(t - \tau)^{-3/2}(t - 2\tau)^{-5/6}v^{1/3}$  and  $g(t) = t - \tau$  satisfy conditions (II<sub>4</sub>) and (II<sub>1</sub>). Let  $q(t) = t - \tau$ ,  $h(v) = v^{5/4}$  and  $h_1(v) = v$ . Then conditions (2.5), (2.14) and (2.15) are satisfied and  $y(t) = (t - \tau)^{5/2}$  is a nonoscillatory solution of (2.16) with  $y''(t)y(t) > 0$  eventually.

**EXAMPLE 2.2.** If we consider the equation

$$(2.17) \quad y^{(6)}(t) + y(t) + \frac{6}{6 - \frac{1}{2}\pi} y(g(t)) = 0 ,$$

then  $F(t, u, v) = u + (6/(6 - (1/2)\pi))v$  and  $g(t) = t - (1/2)\pi$  satisfy conditions (II<sub>4</sub>) and (II<sub>1</sub>). Let  $q(t) = t^{1/2}$ ,  $h(v) = v^{3/2}$  and  $h_1(v) = v$ . Then conditions (2.5), (2.14) and (2.15) are also satisfied. In fact,  $y(t) = t \sin t$  is an oscillatory solution of (2.17) which is not bounded. Lemma 2.2 does not cover this example.

By using the techniques given in [1] and the modification in the proof of Theorem 2.1, we can prove the following theorems. We shall omit their proofs here.

**THEOREM 2.3.** *Let  $0 < g(t) \leq t$ . Assume that there exist positive nondecreasing continuous functions  $h(t)$ ,  $h_1(t)$  and  $h_2(t)$  for  $t \geq a > 0$  and that  $u \geq v$  implies*



$$\int^{\infty} \frac{dv}{h(v)} < \infty \text{ and } \liminf_{v \rightarrow \infty} \left| \frac{h_1(cu)h_2(u)f(u, v)}{h(u)h_2\left(\frac{\alpha t}{g(t)}v\right)} \right| \geq \frac{\varepsilon}{h_2\left(\frac{t}{g(t)}\right)} > 0$$

for every  $c > 0$  and for some  $\alpha > 1$  and  $\varepsilon > 0$ . If

$$\int^{\infty} \frac{t^{n-1}p(t)}{h_1(t)h_2\left(\frac{t}{g(t)}\right)} dt = \infty,$$

then every extensible solution  $y$  of (2.1) is either oscillatory or  $y''(t)y(t) \geq 0$  eventually.

**THEOREM 2.4.** Let  $0 < g(t) \leq t$ . Assume that there exist positive nondecreasing continuous functions  $h(t)$ ,  $h_1(t)$  and  $h_2(t)$  for  $t \geq a > 0$  and that  $u > v$  implies

$$\int^{\infty} \frac{dv}{h(v)} < \infty \text{ and } \liminf_{v \rightarrow \infty} \left| \frac{h_1(cu)h_2(u)F(t, u, v)}{h(u)h_2\left(\frac{\alpha t}{g(t)}v\right)} \right| \geq \frac{\varepsilon F(t, \beta, \beta)}{h_2\left(\frac{t}{g(t)}\right)} > 0$$

for every  $c > 0$  and for some  $\alpha > 1$ ,  $\beta > 0$  and  $\varepsilon > 0$ . If

$$\int^{\infty} \frac{t^{n-1}F(t, \beta, \beta)}{h_1(t)h_2\left(\frac{t}{g(t)}\right)} dt = \infty,$$

then every extensible solution  $y$  of (2.2) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

**THEOREM 2.5.** Let  $g(t)$  satisfy (2.5) and  $q(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Assume that there exist a positive nondecreasing function  $h_1(t)$  for  $t \geq a > 0$  and a constant  $M > 0$  such that

$$\liminf_{v \rightarrow \infty} \left| \frac{h_1(cv)f(u, v)}{v} \right| \geq \varepsilon > 0$$

for every  $c > 0$  and for some  $\varepsilon > 0$ . If (2.4) hold and

$$\limsup_{t \rightarrow \infty} q^{n-1}(t) \int_t^{\infty} \frac{p(s)}{h_1(g(s))} ds > \frac{(n-1)!}{\varepsilon},$$

then every extensible solution  $y$  of (2.1) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

**THEOREM 2.6.** Let  $q(t)$  satisfy (2.5) and  $q(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Assume that there exist a positive nondecreasing function  $h_1(t)$  for  $t \geq a > 0$  and a constant  $M > 0$  such that

$$\liminf_{v \rightarrow \infty} \left| \frac{h_1(cv)F(t, u, v)}{v} \right| \geq \varepsilon F(t, \alpha, \alpha) > 0$$

for every  $c > 0$  and for some  $\alpha > 1$  and  $\varepsilon > 0$ . If (2.3) hold and

$$\limsup_{t \rightarrow \infty} q^{n-1}(t) \int_t^\infty \frac{F(s, \alpha, \alpha)}{h_1(g(s))} ds > \frac{(n-1)!}{\varepsilon} \text{ for every } \alpha > 0,$$

then every extensible solution  $y$  of (2.2) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

**THEOREM 2.7.** Let  $0 < g(t) \leq t$ . Assume that there exist positive nondecreasing continuous functions  $h_1(t)$  and  $h_2(t)$  for  $t \geq a > 0$  and that  $u \geq v$  implies

$$\liminf_{v \rightarrow \infty} \left| \frac{h_1(cu)h_2(u)f(u, v)}{uh_2\left(\frac{\alpha t}{g(t)}v\right)} \right| \geq \frac{\varepsilon}{h_2\left(\frac{t}{g(t)}\right)} > 0$$

for every  $c > 0$  and for some  $\alpha > 1$  and  $\varepsilon > 0$ . If (2.4) hold and

$$\limsup_{t \rightarrow \infty} t^{n-1} \int_t^\infty \frac{p(s)}{h_1(s)h_2\left(\frac{s}{g(s)}\right)} ds > \frac{2^{n-1}(n-1)!}{\varepsilon},$$

then every extensible solution  $y$  of (2.1) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

**THEOREM 2.8.** Let  $0 < g(t) \leq t$ . Assume that there exist positive nondecreasing continuous functions  $h_1(t)$  and  $h_2(t)$  for  $t \geq a > 0$  and that  $u \geq v$  implies

$$\liminf_{v \rightarrow \infty} \left| \frac{h_1(cu)h_2(u)f(t, u, v)}{uh_2\left(\frac{\alpha t}{g(t)}v\right)} \right| \geq \frac{\varepsilon f(t, \beta, \beta)}{h_2\left(\frac{t}{g(t)}\right)} > 0$$

for every  $c > 0$  and for some  $\alpha > 1, \beta > 0$  and  $\varepsilon > 0$ . If (2.3) hold and

$$\limsup_{t \rightarrow \infty} t^{n-1} \int_t^\infty \frac{f(s, \beta, \beta)}{h_1(s)h_2\left(\frac{s}{g(s)}\right)} ds > \frac{2^{n-1}(n-1)!}{\varepsilon},$$

then every extensible solution  $y$  of (2.2) is either oscillatory or  $y''(t)y(t) > 0$  eventually.

**REMARK 2.1.** In a similar way, corresponding to Theorems 2.6, 2.12, 2.18 and 2.22 in [1] we can establish the same results as those

of Theorems 2.1, 2.3, 2.5 and 2.7 for the equation

$$y^{(n)}(t) + \sum_{j=1}^m p_j(t) f_j(y(t), y(g_j(t))) = 0, \quad n \geq 2 \text{ an even integer,}$$

where  $p_j$ ,  $g_j$  and  $f_j$  are continuous functions,  $p_j(t) \geq 0$ ,  $g_j(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $f_j(u, v)$  has the same sign as that of  $u$  and  $v$  if  $uv > 0$ ,  $j = 1, 2, \dots, m$ .

REMARK 2.2. If  $n = 2$ , then the case of  $y''(t)y(t) > 0$  for all large  $t$  couldn't occur. Consequently, under the assumptions in each theorem all extensible solutions of (2.1) or (2.2) with  $n = 2$  are oscillatory [1, pp. 384-397].

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