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OSCILLATION OF EVEN ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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The purpose of this paper is to give some oscillation criteria for even order differential equations with deviating arguments.

A continuous real-valued function $f(t)$ which is defined for all large t is called *oscillatory* if it has arbitrary large zero, otherwise it is called *nonoscillatory*.

Our work extends some results obtained by Ladas and Lakshmikantham [3] and Chiou [1] for second order equations.

1. In this section, we are concerned with the equation

$$y^{(n)}(t) - \sum_{j=1}^{m} p_j(t)y(g_j(t)) = 0 \,, \quad n \geqq 2 \text{ an even integer} \,, \tag{1.1}$$

where the following assumptions are assumed to hold:

(I_1) $g_j(t) \leqq t$ on $[a, \infty)$, $j = 1, 2, \cdots, m$ and $g_k(t) < t$ for some $1 \leqq k \leqq m$; $g'_j(t) \geqq 0$ on $[a, \infty)$ and $g_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2, \cdots, m$.

(I_2) $p_j(t) \geqq 0$, $p'_j(t) \leqq 0$ on $[a, \infty)$, $j = 1, 2, \cdots, m$ and $p_k(t) > 0$ on $[a, \infty]$ for the same k as in (I_1).

We shall give a sufficient condition for all bounded solutions of (1.1) to be oscillatory. Our result extends Ladas and Lakshmikatham's Theorems 2.1-2.4 in [3] to arbitrary even order equation (1.1).

LEMMA 1.1 (Lemma 2 in [2]). *If y is a function, which together with its derivatives of order up to $(n-1)$ inclusive, is absolutely continuous and of constant sign on the interval $[a, \infty)$ and $y^{(n)}(t)y(t) \geqq 0$ on $[a, \infty)$, then either*

$$y^{(j)}(t)y(t) \geqq 0 \,, \qquad j = 0, 1, \cdots, n \,,$$

or there is an integer l, $0 \leqq l \leqq n-2$, which is even when n is even and odd when n is odd, such that

$$y^{(j)}(t)y(t) \geqq 0 \,, \qquad j = 0, 1, \cdots, l \,,$$

and

$$(-1)^{n+j}y^{(j)}(t)y(t) \geqq 0 \,, \qquad j = l+1, \cdots, n$$

for t in $[a, \infty)$.

THEOREM 1.1. *If* $(t - g_k(t))^n p_k(t) \geqq n!$ *for all sufficiently large* t, *then every bounded solution of* (1.1) *is oscillatory.*

Proof. Let y be a nonoscillatory bounded solution of (1.1). Without loss of generality, we can assume that $y(t) > 0$ for $t \geqq T_1$. There is a $T_2 \geqq T_1$ such that $g_j(t) > T_1$ $(j = 1, 2, \cdots, m)$ for $t \geqq T_2$. There is a $T_3 \geqq T_2$ such that $y^{(j)}(t)$ $(j = 1, 2, \cdots, n-1)$ is of constant sign for $t \geqq T_3$. By Lemma 1.1 and since y is bounded, we have

$$(-1)^j y^{(j)}(t) > 0 \ (j = 0, 1, \cdots, n-1) \quad \text{for } t \geqq T_3\,. \tag{1.2}$$

It follows from (1.1) that

$$y^{(n+1)}(t) = \sum_{j=1}^{m} \{p_j'(t)y(g_j(t)) + p_j(t)y'(g_j(t))g_j'(t)\} \leqq 0$$

for $t \geqq T_3$.

By Taylor's theorem, there is a ξ between τ and t such that

$$\begin{aligned} y^{(n-1)}(\tau) &= y^{(n-1)}(t) + y^{(n)}(t)(\tau - t) + y^{(n+1)}(\xi)\frac{(\tau - t)^2}{2!} \\ &\leqq y^{(n-1)}(t) + y^{(n)}(t)(\tau - t) \\ &= y^{(n-1)}(t) - (t - \tau)\sum_{j=1}^{m} p_j(t)y(g_j(t)) \end{aligned} \tag{1.3}$$

for $\tau \geqq T_3$ and $t \geqq T_3$.

Integrating (1.3) with respect to τ from s to $t > s$, we have

$$\begin{aligned} y^{(n-2)}(t) - y^{(n-2)}(s) &\leqq y^{(n-1)}(t)(t-s) - \frac{(t-s)^2}{2!}\sum_{j=1}^{m} p_j(t)y(g_j(t)) \\ &\leqq y^{(n-1)}(t)(t-s) - \frac{(t-s)^2}{2!} p_k(t)y(g_k(t)) \end{aligned}$$

or

$$\begin{aligned} y^{(n-2)}(s) \geqq\ & y^{(n-2)}(t) - y^{(n-1)}(t)(t-s) \\ & + \frac{(t-s)^2}{2!} p_k(t)y(g_k(t)) \quad \text{for } t > s \geqq T_3\,. \end{aligned}$$

In a similar way repeatedly, we shall have

$$\begin{aligned} y'(s) \leqq\ & y'(t) - y''(t)(t-s) + y'''(t)\frac{(t-s)^2}{2!} \\ & - \cdots + y^{(n-1)}(t)\frac{(t-s)^{n-2}}{(n-2)!} \\ & - p_k(t)y(g_k(t))\frac{(t-s)^{n-1}}{(n-1)!} \quad \text{for } t > s \geqq T_3\,. \end{aligned} \tag{1.4}$$

Integrating (1.4) from $g_k(t)$ to t, we obtain

$$y(t) - y(g_k(t)) \leqq y'(t)(t - g_k(t)) - y''(t)\frac{(t - g_k(t))^2}{2!} + y'''(t)\frac{(t - g_k(t))^3}{3!} - \cdots + y^{(n-1)}(t)\frac{(t - g_k(t))^{n-1}}{(n-1)!} - p_k(t)y(g_k((t))\frac{(t - g_k(t))^n}{n!}$$

or

$$(t - g_k(t))y'(t) - \frac{(t - g_k(t))^2}{2!}y''(t) + \frac{(t - g_k(t))^3}{3!}y'''(t) - \cdots + \frac{(t - g_k(t))^{n-1}}{(n-1)!}y^{(n-1)}(t) + \left[1 - \frac{p_k(t)(t - g_k(t))^n}{n!}\right]y(g_k(t)) - y(t) \geqq 0$$

for all sufficiently large t.

It follows from (1.2) that

$$1 - \frac{p_k(t)(t - g_k(t))^n}{n!} > 0$$

or

$$(t - g_k(t))^n p_k(t) > n! \qquad \text{for all sufficiently large } t\,.$$

This is a contradiction and the proof is then complete.

EXAMPLE 1.1. If we consider the equation

$$y^{(4)}(t) - \frac{(2k\pi)^4}{\tau^4}y(t - \tau) = 0\,, \quad \tau > 0\,, \quad k = 1, 2, \cdots, \tag{1.5}$$

then $p(t) = (2k\pi)^4\tau^{-4}$ satisfies the assumption and every bounded solution of (1.5) is oscillatory. A bounded oscillatory solution of (1.5) is $y(t) = \sin(2k\pi/\tau)t$, $k = 1, 2, \cdots$.

COROLLARY 1.1. *Consider the equation*

$$y^{(n)}(t) - \sum_{j=1}^{m} y(t - \tau_j) = 0\,, \quad \tau_j \geqq 0 \ (j = 1, 2, \cdots, m)\,. \tag{1.6}$$

If $\tau_k \geqq \sqrt[n]{n!}$ *for some* $1 \leqq k \leqq m$, *then every bounded solution of* (1.6) *is oscillatory.*

2. We shall consider the equations

$$y^{(n)}(t) + p(t)f(y(t), y(g(t))) = 0 \tag{2.1}$$

and

$$y^{(n)}(t) + F(t, y(t), y(g(t))) = 0 \,, \qquad n \geqq 2 \text{ an even integer} \,, \tag{2.2}$$

with the following conditions:

(II_1) $g(t)$ is a continuous function on $[a, \infty)$ such that $g(t) \to \infty$ as $t \to \infty$.

(II_2) $p(t)$ is a nonnegative continuous function on $[a, \infty)$.

(II_3) $f(u, v)$ is a continuous function on R^2 and has the same sign as that of u and v if $uv > 0$.

(II_4) $F(t, u, v)$ is a continuous function on $[a, \infty) \times R^2$, nondecreasing in u and in v for each fixed t and has the same sign as that of u and v if $uv > 0$.

In this section, we shall give conditions which will ensure that every extensible solution y of (2.1) or (2.2) is either oscillatory or $y''(t)y(t) > 0$ for all sufficiently large t. This generalizes to higher order equations some results due to Chiou [1, Theorems 2.2, 2.6, 2.8, 2.9, 2.12, 2.14, 2.15, 2.18, 2.19, 2.20, 2.22 and 2.23].

LEMMA 2.1 (Lemma 1 in [2]). *If y is a function which together with its derivatives of order up to $(n-1)$ inclusive, is absolutely continuous and of constant sign on the interval $[a, \infty)$ and $y^{(n)}(t)y(t) \leqq 0$ on $[a, \infty)$, then there is an integer l, $0 \leqq l \leqq n-1$, which is odd when n is even and even when n is odd, such that*

$$y^{(j)}(t)y(t) \geqq 0 \,, \qquad j = 0, 1, \cdots, l \,,$$

and

$$(-1)^{n+j-1}y^{(j)}(t)y(t) \geqq 0 \,, \qquad j = l+1, \cdots, n$$

for t in $[a, \infty)$.

LEMMA 2.2 (Corollary 2.3 in [4]). *If*

$$\int^{\infty} t^{n-1}F(t, \gamma, \gamma)\, dt = \pm\infty \quad \textit{for each } \gamma \neq 0 \,, \tag{2.3}$$

then every bounded solution of (2.2) *is oscillatory.*

In a similar way, we have

LEMMA 2.3. *If*

$$\int^{\infty} t^{n-1}p(t)dt = \infty \,, \tag{2.4}$$

then every bounded solution of (2.1) *is oscillatory.*

THEOREM 2.1. *Assume that*

(i) *there exists a positive function* $q(t)$ *such that*

$$q(t) \leqq \min\{g(t), t\}, \; q'(t) > 0 \; \textit{and} \; q''(t) \leqq 0 \; \textit{for} \; t \geqq a . \tag{2.5}$$

(ii) *there exist positive functions* $h(t)$ *and* $h_1(t)$ *for* $t \geqq a > 0$ *and a constant* $M > 0$ *such that*

$$\int^{\infty} \frac{dv}{h(v)} < \infty \quad \textit{and} \quad \liminf_{v \to \infty} \left| \frac{h_1(cv) f(u, v)}{h(v)} \right| \geqq \varepsilon > 0 \tag{2.6}$$

for $u > M$, *every* $c > 0$ *and for some* $\varepsilon = \varepsilon(c)$.

If

$$\int^{\infty} \frac{q^{n-1}(t) p(t)}{h_1(g(t))} dt = \infty , \tag{2.7}$$

then every extensible solution of (2.1) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

Proof. Let y be a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $y(t) > 0$ for $t \geqq T_1$. There is a $T_2 \geqq T_1$ such that $g(t) \geqq T_1$ for $t \geqq T_2$. It follows from (2.1) that $y^{(n)}(t) \leqq 0$ for $t \geqq T_2$. There is a $T_3 \geqq T_2$ such that each $y^{(j)}(t)$, $j = 1, 2, \cdots, n-1$, is of constant sign for $t \geqq T_3$. By Lemma 2.1, $y'(t) > 0$ for $t \geqq T_3$.

If $y''(t) > 0$ for $t \geqq T_3$, then our proof is done. Assume that $y''(t) < 0$ for $t \geqq T_3$. Then, by Lemma 2.1, we have

$$(-1)^{j-1} y^{(j)}(t) > 0 \; (\mathrm{j} = 1, 2, \cdots, n-1) \; \text{for} \; t \geqq T_3 . \tag{2.8}$$

Since (2.7) implies (2.4) and since $y'(t) > 0$ for $t \geqq T_3$, it follows from Lemma 2.3 that $y(t) \to \infty$ as $t \to \infty$.

Integrating (2.1) repeatedly from t to $t' > 2t \geqq 2T_3$ and using (2.8) as well as integration by parts, we have

$$y'(t) \geqq \frac{1}{(n-2)!} \int_t^{t'} (u-t)^{n-2} p(u) f(y(u), y(g(u))) du . \tag{2.9}$$

Dividing (2.9) by $h(y(g(t)))$, we have

$$\frac{y'(t)}{h(y(q(t)))} \geqq \frac{1}{(n-2)!} \int_t^{t'} \frac{(u-t)^{n-2} p(u) f(y(u), y(g(u)))}{h(y(g(u)))} du . \tag{2.10}$$

Since $y'(t)$ is decreasing for $t \geqq T_3$, there exist $T_4 \geqq T_3$ and $c > 0$ such that $cy(t) \leqq t$ for $t \geqq T_4$. From (2.5) and (2.6) we have

$$\begin{aligned}\frac{p(u)f(y(u),\,y(g(u)))}{h(y(g(u)))} &\geqq \frac{p(u)}{h_1(g(u))}\frac{h_1(cy(g(u)))f(y(u),\,y(g(u)))}{h(y(g(u)))}\\ &\geqq \frac{\varepsilon p(u)}{h_1(g(u))}\,.\end{aligned}\tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\begin{aligned}\frac{y'(t)}{h(y(q(t)))} &\geqq \frac{\varepsilon}{(n-2)!}\int_t^{t'}\frac{(u-t)^{n-2}p(u)}{h_1(g(u))}du\\ &\geqq \frac{\varepsilon t^{n-2}}{(n-2)!}\int_{2t}^{t'}\frac{p(u)}{h_1(g(u))}du\,.\end{aligned}\tag{2.12}$$

Since $q(t) \leqq t$, $q'(t) > 0$ and $q''(t) \leqq 0$, we have

$$\frac{y'(q(t))q'(t)}{h(y(q(t)))} \geqq \frac{y'(t)q'(t)}{h(y(q(t)))} \geqq \frac{\varepsilon q^{n-2}(2t)q'(2t)}{2^{n-2}(n-2)!}\int_{2t}^{t'}\frac{p(u)}{h_1(g(u))}du\,.\tag{2.13}$$

Integrating (2.13) from T_4 to $T > T_4$ and using integration by parts, we get

$$\begin{aligned}\int_{T_4}^{T}\frac{dy(q(s))}{h(y(q(s)))} &\geqq \frac{\varepsilon q^{n-1}(2T)}{2^{n-1}(n-1)!}\int_{2T}^{t'}\frac{p(u)}{h_1(g(u))}du - \frac{\varepsilon q^{n-1}(2T_4)}{2^{n-1}(n-1)!}\\ &\quad\times\int_{2T_4}^{t'}\frac{p(u)}{h_1(g(u))}du + \frac{\varepsilon}{2^{n-2}(n-2)!}\int_{T_4}^{T}\frac{q^{n-1}(2t)p(2t)}{h_1(g(2t))}dt\\ &\geqq -\frac{q^{n-1}(2T_4)}{2^{n-1}(n-1)!}\int_{2T_4}^{t'}\frac{p(u)}{h_1(g(u))}du + \frac{\varepsilon}{2^{n-2}(n-1)!}\\ &\quad\times\int_{T_4}^{T}\frac{q^{n-1}(2t)p(2t)}{h_1(g(2t))}dt\,.\end{aligned}$$

Using (2.12) for $t = T_4$, we have

$$\begin{aligned}\int_{T_4}^{T}\frac{dy(q(s))}{h(y(q(s)))} &\geqq -\frac{\varepsilon q^{n-1}(2T_4)}{2^{n-1}(n-1)!}\frac{y'(T_4)}{h(y(q(T_4)))}\frac{(n-2)!}{\varepsilon T_4^{n-2}}\\ &\quad+\frac{\varepsilon}{2^{n-2}(n-1)!}\int_{T_4}^{t}\frac{q^{n-1}(2t)p(2t)}{h_1(g(2t))}dt\,.\end{aligned}$$

Let $T \longrightarrow \infty$ and obtain

$$\int_{2T_4}^{\infty}\frac{q^{n-1}(s)p(s)}{h_1(g(s))}ds < \infty\,.$$

This contradicts (2.7) and the proof is complete.

If $y(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, then by the monotonicity of $F(t, u, v)$, there exist $\alpha > 0$ and $T > 0$ such that

$$F(t,\,y(t),\,y(g(t))) \geqq F(t,\,\alpha,\,\alpha) > 0 \quad \text{for} \quad t \geqq T\,.$$

By using this fact and Lemma 2.2 instead of Lemma 2.3 in the proof of Theorem 2.1, we can prove the following

THEOREM 2.2. *Assume that* (2.5) *is satisfied and that there exist positive nondecreasing functions* $h(t)$ *and* $h_1(t)$ *for* $t \geqq a > 0$ *and a constant* $M > 0$ *such that*

$$(2.14) \quad \int^{\infty} \frac{dv}{h(v)} < \infty \quad and \quad \liminf_{v\to\infty} \left| \frac{h_1(cv)F(t, u, v)}{h(v)} \right| \geqq \varepsilon F(t, \alpha, \alpha) > 0$$

for $u > M$, *every* $c > 0$ *and for some* $\varepsilon = \varepsilon(c)$ *and* $\alpha > 0$. *If*

$$(2.15) \qquad \int^{\infty} \frac{q^{n-1}(t)F(t, \alpha, \alpha)}{h_1(g(t))} dt = \infty ,$$

then every extensible solution y *of* (2.2) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

The following example presents the occurrence of the case $y''(t)y(t) > 0$ for all sufficiently large t.

EXAMPLE 2.1. If we consider the equation

$$(2.16) \qquad y^{(4)}(t) + \frac{15}{16}(t - \tau)^{-3/2}(t - 2\tau)^{-5/6}[y(t - \tau)]^{1/3} = 0 ,$$

then $F(t, u, v) = (15/16)(t - \tau)^{-3/2}(t - 2\tau)^{-5/6}v^{1/3}$ and $g(t) = t - \tau$ satisfy conditions (II_4) and (II_1). Let $q(t) = t - \tau$, $h(v) = v^{5/4}$ and $h_1(v) = v$. Then conditions (2.5), (2.14) and (2.15) are satisfied and $y(t) = (t - \tau)^{5/2}$ is a nonoscillatory solution of (2.16) with $y''(t)y(t) > 0$ eventually.

EXAMPLE 2.2. If we consider the equation

$$(2.17) \qquad y^{(6)}(t) + y(t) + \frac{6}{6 - \frac{1}{2}\pi} y(g(t)) = 0 ,$$

then $F(t, u, v) = u + (6/(6 - (1/2)\pi))v$ and $g(t) = t - (1/2)\pi$ satisfy conditions (II_4) and (II_1). Let $q(t) = t^{1/2}$, $h(v) = v^{3/2}$ and $h_1(v) = v$. Then conditions (2.5), (2.14) and (2.15) are also satisfied. In fact, $y(t) = t \sin t$ is an oscillatory solution of (2.17) which is not bounded. Lemma 2.2 does not cover this example.

By using the techniques given in [1] and the modification in the proof of Theorem 2.1, we can prove the following theorems. We shall omit their proofs here.

THEOREM 2.3. *Let* $0 < g(t) \leqq t$. *Assume that there exist positive nondecreasing continuous functions* $h(t)$, $h_1(t)$ *and* $h_2(t)$ *for* $t \geqq a > 0$ *and that* $u \geqq v$ *implies*

$$\int^{\infty} \frac{dv}{h(v)} < \infty \quad \text{and} \quad \liminf_{v\to\infty} \left| \frac{h_1(cu)h_2(u)f(u, v)}{h(u)h_2\left(\frac{\alpha t}{g(t)} v\right)} \right| \geqq \frac{\varepsilon}{h_2\left(\frac{t}{g(t)}\right)} > 0$$

for every $c > 0$ *and for some* $\alpha > 1$ *and* $\varepsilon > 0$. *If*

$$\int^{\infty} \frac{t^{n-1}p(t)}{h_1(t)h_2\left(\frac{t}{g(t)}\right)} dt = \infty ,$$

then every extensible solution y *of* (2.1) *is either oscillatory or* $y''(t)y(t) \geqq 0$ *eventually.*

THEOREM 2.4. *Let* $0 < g(t) \leqq t$. *Assume that there exist positive nondecreasing continuous functions* $h(t)$, $h_1(t)$ *and* $h_2(t)$ *for* $t \geqq a > 0$ *and that* $u > v$ *implies*

$$\int^{\infty} \frac{dv}{h(v)} < \infty \quad \text{and} \quad \liminf_{v\to\infty} \left| \frac{h_1(cu)h_2(u)F(t, u, v)}{h(u)h_2\left(\frac{\alpha t}{g(t)} v\right)} \right| \geqq \frac{\varepsilon F(t, \beta, \beta)}{h_2\left(\frac{t}{g(t)}\right)} > 0$$

for every $c > 0$ *and for some* $\alpha > 1$, $\beta > 0$ *and* $\varepsilon > 0$. *If*

$$\int^{\infty} \frac{t^{n-1}F(t, \beta, \beta)}{h_1(t)h_2\left(\frac{t}{g(t)}\right)} dt = \infty ,$$

then every extensible solution y *of* (2.2) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

THEOREM 2.5. *Let* $g(t)$ *satisfy* (2.5) *and* $q(t) \to \infty$ *as* $t \to \infty$. *Assume that there exist a positive nondecreasing function* $h_1(t)$ *for* $t \geqq a > 0$ *and a constant* $M > 0$ *such that*

$$\liminf_{v\to\infty} \left| \frac{h_1(cv)f(u, v)}{v} \right| \geqq \varepsilon > 0$$

for every $c > 0$ *and for some* $\varepsilon > 0$. *If* (2.4) *hold and*

$$\limsup_{t\to\infty} q^{n-1}(t) \int_t^{\infty} \frac{p(s)}{h_1(g(s))} ds > \frac{(n-1)!}{\varepsilon} ,$$

then every extensible solution y *of* (2.1) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

THEOREM 2.6. *Let* $q(t)$ *satisfy* (2.5) *and* $q(t) \to \infty$ *as* $t \to \infty$. *Assume that there exist a positive nondecreasing function* $h_1(t)$ *for* $t \geqq a > 0$ *and a constant* $M > 0$ *such that*

$$\liminf_{v\to\infty}\left|\frac{h_1(cv)F(t, u, v)}{v}\right| \geqq \varepsilon F(t, \alpha, \alpha) > 0$$

for every $c > 0$ *and for some* $\alpha > 1$ *and* $\varepsilon > 0$. *If* (2.3) *hold and*

$$\limsup_{t\to\infty} q^{n-1}(t)\int_t^\infty \frac{F(s, \alpha, \alpha)}{h_1(g(s))}ds > \frac{(n-1)!}{\varepsilon} \text{ for every } \alpha > 0\,,$$

then every extensible solution y *of* (2.2) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

THEOREM 2.7. *Let* $0 < g(t) \leqq t$. *Assume that there exist positive nondecreasing continuous functions* $h_1(t)$ *and* $h_2(t)$ *for* $t \geqq a > 0$ *and that* $u \geqq v$ *implies*

$$\liminf_{v\to\infty}\left|\frac{h_1(cu)h_2(u)f(u, v)}{uh_2\left(\dfrac{\alpha t}{g(t)}v\right)}\right| \geqq \frac{\varepsilon}{h_2\left(\dfrac{t}{g(t)}\right)} > 0$$

for every $c > 0$ *and for some* $\alpha > 1$ *and* $\varepsilon > 0$. *If* (2.4) *hold and*

$$\limsup_{t\to\infty} t^{n-1}\int_t^\infty \frac{p(s)}{h_1(s)h_2\left(\dfrac{s}{g(s)}\right)}ds > \frac{2^{n-1}(n-1)!}{\varepsilon}\,,$$

then every extensible solution y *of* (2.1) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

THEOREM 2.8. *Let* $0 < g(t) \leqq t$. *Assume that there exist positive nondecreasing continuous functions* $h_1(t)$ *and* $h_2(t)$ *for* $t \geqq a > 0$ *and that* $u \geqq v$ *implies*

$$\liminf_{v\to\infty}\left|\frac{h_1(cu)h_2(u)f(t, u, v)}{uh_2\left(\dfrac{\alpha t}{g(t)}v\right)}\right| \geqq \frac{\varepsilon f(t, \beta, \beta)}{h_2\left(\dfrac{t}{g(t)}\right)} > 0$$

for every $c > 0$ *and for some* $\alpha > 1$, $\beta > 0$ *and* $\varepsilon > 0$. *If* (2.3) *hold and*

$$\limsup_{t\to\infty} t^{n-1}\int_t^\infty \frac{f(s, \beta, \beta)}{h_1(s)h_2\left(\dfrac{s}{g(s)}\right)}ds > \frac{2^{n-1}(n-1)!}{\varepsilon}\,,$$

then every extensible solution y *of* (2.2) *is either oscillatory or* $y''(t)y(t) > 0$ *eventually.*

REMARK 2.1. In a similar way, corresponding to Theorems 2.6, 2.12, 2.18 and 2.22 in [1] we can establish the same results as those of Theorems 2.1, 2.3, 2.5 and 2.7 for the equation

$$y^{(n)}(t) + \sum_{j=1}^{m} p_j(t)f_j(y(t), y(g_j(t))) = 0,\ n \geqq 2 \text{ an even integer},$$

where p_j, g_j and f_j are continuous functions, $p_j(t) \geqq 0$, $g_j(t) \to \infty$ as $t \to \infty$ and $f_j(u, v)$ has the same sign as that of u and v if $uv > 0$, $j = 1, 2, \cdots, m$.

REMARK 2.2. If $n = 2$, then the case of $y''(t)y(t) > 0$ for all large t couldn't occur. Consequently, under the assumptions in each theorem all extensible solutions of (2.1) or (2.2) with $n = 2$ are oscillatory [1, pp. 384-397].

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REFERENCES

1. K. L. Chiou, *Oscillation and nonoscillation theorems for second order functional differential equations*, J. Math. Anal. Appl., **45** (1974), 382-403.
2. I. T. Kiguradze, *The problem of oscillation of solutions of non-linear differential equations*, Differential Equations, (8) **1** (1965), 773-782.
3. G. Ladas and V. Lakshmikantham, *Oscillations caused by retarded actions*, Applicable Analysis, **4** (1974), 9-15.
4. T. S. Liu, *Classification of solutions of higher order differential equations and inequalities with deviating arguments*, J. Differential Equations, to appear.

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