

# Pacific Journal of Mathematics

**DOUBLING CHAINS, SINGULAR ELEMENTS AND HYPER- $\mathcal{L}$   
L-GROUPS**

JORGE MARTINEZ

## DOUBLING CHAINS, SINGULAR ELEMENTS AND HYPER- $\mathcal{X}$ $l$ -GROUPS

JORGE MARTINEZ

In a lattice-ordered group  $G$  a (descending) doubling chain is a sequence  $a_1 > a_2 > \cdots > a_n > \cdots$  of positive elements of  $G$  such that  $a_n \geq 2a_{n+1}$ . An element  $0 < s \in G$  is singular if  $0 \leq g \leq s$  implies that  $g \wedge (s - g) = 0$ . The main theorems are as follows: 1. The following two statements are equivalent: (a) every doubling chain in  $G$  is finite; (b)  $G = \bigcup_{\tau < \alpha} G^\tau$  ( $\tau$  ranging over all ordinals less than some  $\alpha$ ), where  $G^\tau$  is an  $l$ -ideal of  $G$ ,  $\sigma < \tau$  implies that  $G^\sigma \subseteq G^\tau$  and  $G^{\tau+1}/G^\tau$  is generated by its singular elements, (i.e. a Specker group, à la Conrad). 2. If  $G$  is hyper-archimedean as well then either of the above conditions is equivalent to: (c)  $G$  is hyper- $\mathcal{X}$ , i.e. every totally ordered  $l$ -homomorphic image of  $G$  is cyclic.

The purpose of this investigation was to come up with an "elementwise" definition of the abelian lattice-ordered groups (henceforth abbreviated:  $l$ -groups) having the property that each  $l$ -homomorphic image which is totally ordered is cyclic. These  $l$ -groups are called hyper- $\mathcal{X}$ , and were first introduced by the author in [5]. Thus,  $G$  is hyper- $\mathcal{X}$  if and only if  $G$  is abelian and  $G/P$  is cyclic, for each prime subgroup  $P$  of  $G$ . These  $l$ -groups are therefore hyper-archimedean, and they can in fact be characterized as those  $l$ -groups for which all the prime subgroups are maximal and have cyclic quotient; (see [3] and [5]). It should be stressed that in this characterization no assumptions need to be made with respect to commutativity. In [3] Conrad provided an example of an  $l$ -group which is hyper-archimedean and also a subdirect product of  $Z$ , the additive group of integers with the usual order, yet is not hyper- $\mathcal{X}$ .

An element  $s > 0$  of the  $l$ -group  $G$  is singular provided  $0 \leq g \leq s$  implies that  $g \wedge (s - g) = 0$ . An  $S$ -group (or Specker group) is one in which each positive element is a sum of singular elements. These  $S$ -groups are well explored in [3]; the main characterization is that each  $S$ -group can be embedded as an  $l$ -subring of bounded, integer-valued functions on a set, or alternatively, as an  $l$ -subgroup of bounded, integer-valued functions generated by characteristic functions. It was observed in [7] that the  $S$ -groups form a torsion class of  $l$ -groups; that is, they are closed under taking convex  $l$ -subgroups,  $l$ -homomorphic images, and if  $G$  is any  $l$ -group, and  $\{C_\lambda \mid \lambda \in A\}$  a family of convex  $l$ -subgroups which are all  $S$ -groups then the convex  $l$ -subgroup they generate is an  $S$ -group. There is thus an associated

S-radical  $\mathcal{S}(G)$  of an  $l$ -group  $G$ , and a “Loewy”-like ascending sequence  $S(G) = \mathcal{S}^1(G) \subseteq \dots \subseteq \mathcal{S}^\tau(G) \subseteq \dots$  for each ordinal  $\tau$ , so that

- (a)  $\mathcal{S}(G)$  is the largest convex  $l$ -subgroup of  $G$  which is an  $S$ -group.
- (b) For any convex  $l$ -subgroup  $A$  of  $G$ ,  $\mathcal{S}(A) = A \cap \mathcal{S}(G)$ .
- (c) If  $\alpha$  is a limit ordinal  $\mathcal{S}^\alpha(G) = \bigcup \{ \mathcal{S}^\tau(G) \mid \tau < \alpha \}$ ,
- (d) and otherwise  $\mathcal{S}^\tau(G)$  is defined by the equation:

$$\mathcal{S}^\tau(G) / \mathcal{S}^{\tau-1}(G) = \mathcal{S}(G / \mathcal{S}^{\tau-1}(G)).$$

Then we are able to define  $\mathcal{S}^*(G) = \mathcal{S}^\tau(G)$ , where  $\tau$  is chosen so that  $\mathcal{S}^\tau(G) = \mathcal{S}^{\tau+1}(G) = \dots$ ; such a  $\tau$  exists by a simple cardinality argument.  $G$  is said to be an  $S^*$ -group if  $\mathcal{S}^*(G) = G$ .

We should observe that if  $G$  is an  $S$ -group, it can be represented as an  $l$ -group of bounded, integer-valued functions, and it is therefore hyper- $\mathcal{X}$ ; (see [3]). Examples of hyper- $\mathcal{X}$   $l$ -groups which are not  $S$ -groups are easy to construct.

In an  $l$ -group  $G$ , a (descending) doubling chain is a sequence  $s_1 > s_2 > \dots$  of positive elements of  $G$  so that  $s_n \geq 2s_{n+1}$ , for each  $n = 1, 2, \dots$ . Notice that the terms of a doubling chain may eventually be zero; in such a case it is a finite doubling chain.

We can now state our first result.

**THEOREM 1.**  *$G$  is an  $S^*$ -group if and only if every doubling chain for  $G$  is finite.*

*Proof. Necessity.* The proof proceeds by transfinite induction on the length of the Loewy sequence of  $\mathcal{S}^\tau(G)$ 's. The first thing to do is to show an  $S$ -group has this property. This is clear, because if  $G$  is an  $S$ -group, it can be represented as an  $l$ -group of bounded, integer-valued functions; and there are obviously no infinite doubling chains of such functions. Next, suppose  $G = \mathcal{S}^\alpha(G)$  and  $\mathcal{S}^\tau(G)$  has no infinite doubling chains, for each  $\tau < \alpha$ . Suppose by way of contradiction that  $a_1 > a_2 > \dots > a_n > \dots$  is an infinite doubling chain for  $G$ ; if  $\alpha$  is a limit ordinal, then  $a_1 \in \mathcal{S}^\beta(G)$  for some  $\beta < \alpha$ , and hence each  $a_n \in \mathcal{S}^\beta(G)$ , contradicting our assumption. If  $\alpha$  has a predecessor, then no  $a_n \in \mathcal{S}^{\alpha-1}(G)$ , and consequently  $a_1 + \mathcal{S}^{\alpha-1}(G) > a_2 + \mathcal{S}^{\alpha-1}(G) > \dots$  is an infinite doubling chain in the  $S$ -group  $G / \mathcal{S}^{\alpha-1}(G)$ . This again is a contradiction, and we must conclude that  $G = \mathcal{S}^\alpha(G)$  has no infinite doubling chains; this completes the proof of the necessity.

*Sufficiency.* Let us make a preliminary observation: for a given ordinal  $\tau$ , an element  $a > 0$  of an  $l$ -group  $G$  has the property that  $a \geq 2b \geq 0$  implies that  $b \in \mathcal{S}^\tau(G)$  if and only if  $a \in \mathcal{S}^\tau(G)$  or else

$a + \mathcal{S}^\tau(G)$  is a singular element of  $G/\mathcal{S}^\tau(G)$ . If  $a$  has this property and  $a \notin \mathcal{S}^\tau(G)$  we call  $a$  a  $(\tau + 1)$ -singular. (Note: for  $\tau = 0$  we set  $\mathcal{S}^\tau(G) = 0$ ; then 1-singular simply means: singular.)

Suppose now that every doubling chain of  $G$  is finite. If  $0 < g \in G$  and  $\tau$  is an ordinal, then if  $g$  is not  $(\tau + 1)$ -singular we may find an element  $0 < a_1 \in G$  such that  $a_1 \notin \mathcal{S}^\tau(G)$  and  $2a_1 \leq g$ . Inductively proceed to construct a doubling chain  $g > a_1 > a_2 > \dots > a_k > \dots$ , where  $a_k$  is the last entry outside  $\mathcal{S}^\tau(G)$ , and therefore  $(\tau + 1)$ -singular. Thus, every positive element of  $G$  exceeds a  $(\tau + 1)$ -singular element, for each ordinal  $\tau$ .

If  $G \neq \mathcal{S}^*(G)$ , we pick  $0 < g \in G \setminus \mathcal{S}^*(G)$ , and an ordinal  $\alpha$  such that  $\mathcal{S}^*(G) = \mathcal{S}^\alpha(G)$ . As we have indicated  $g \geq h$  for some  $(\alpha + 1)$ -singular element  $h$ ; that is,  $h$  is singular modulo  $\mathcal{S}^\alpha(G)$ , which is absurd. We must conclude that  $G$  is an  $S^*$ -group, and Theorem 1 is proved.

A hyper-archimedean  $l$ -group is characterized by the condition that each prime subgroup be maximal [3]. Therefore, every totally ordered  $l$ -homomorphic image of a hyper-archimedean  $l$ -group is a subgroup of the additive reals, by Hölder's theorem. Now let us prove:

**THEOREM 2.** *Suppose  $G$  is hyper-archimedean; then it is hyper- $\mathcal{X}$  if and only if every doubling chain for  $G$  is finite.*

*Proof.* Suppose  $G$  is hyper- $\mathcal{X}$ , yet  $a_1 > a_2 > \dots > a_n > \dots$  is an infinite doubling chain. The  $a_i$  are contained in an ultrafilter of the positive cone of  $G$ , and thus a minimal prime subgroup  $P$  exists so that  $a_n \notin P$  for each  $n = 1, 2, \dots$ . (Recall that an *ultrafilter* is a subset  $U$  of strictly positive elements of an  $l$ -group  $H$ , maximal with respect to the property:  $a, b \in U$  imply that  $a \wedge b \in U$ . For an account of the correspondence between ultrafilters and minimal prime subgroups we refer the reader to [1] or [2].)

Continuing then,  $a_1 + P > a_2 + P > \dots$  is an infinite descending chain for the archimedean  $o$ -group  $G/P$ ;  $G/P$  can therefore not be cyclic, and we have a contradiction.

Conversely, suppose every doubling chain of  $G$  is finite; then  $G$  is an  $S^*$ -group by Theorem 1, and it is easy to verify from this that each totally ordered quotient of  $G$  is cyclic, since the class of  $S^*$ -groups is closed under  $l$ -homomorphic images; (see [7]).

This is enough to establish Theorem 2.

**COROLLARY.** *If  $G$  is hyper-archimedean, and  $A$  is an  $l$ -ideal of  $G$  so that  $A$  and  $G/A$  are both hyper- $\mathcal{X}$ , then  $G$  is hyper- $\mathcal{X}$ .*

The following example illustrates the use of hyper-archimedeanity in Theorem 2 and the above corollary. Let  $G$  be the  $l$ -group of sequences of integers by the eventually constant sequences and  $\mathbf{a} = (1, 2, 3, \dots)$ . This example was discussed in [6], and it was shown there that  $G$  is not hyper-archimedean. Yet  $G$  is an extension of an  $S$ -group by  $Z$ , and all its doubling chains are finite.

Finally, we state a corollary which says something about the underlying group of a hyper- $\mathcal{L}$   $l$ -group.

**COROLLARY.** *If  $G$  is a hyper- $\mathcal{L}$   $l$ -group, then  $G$  is free, quabelian group.*

*Proof.* As an  $S$ -group is a subgroup of bounded, integer-valued functions it is free abelian; this result goes back to Nöbeling [8], and it is further discussed by Hill in [4] and Conrad in [3]. If  $G$  is a hyper- $\mathcal{L}$   $l$ -group then it is an  $S^*$ -group, say  $G = \mathcal{S}^\alpha(G)$ ; we assume that  $\mathcal{S}^\tau(G)$  is free abelian for each ordinal  $\tau < \alpha$ , and that a free basis  $X_\tau$  for  $\mathcal{S}^\tau(G)$  can be picked so that  $X_\sigma = X_\tau \cap \mathcal{S}^\sigma(G)$ , if  $\sigma < \tau < \alpha$ . If  $\alpha$  is a limit ordinal, we let  $X = \bigcup \{X_\tau \mid \tau < \alpha\}$ ; it is easy to verify that  $X$  is a free basis for  $\mathcal{S}^\alpha(G)$ . Otherwise, we have that  $\mathcal{S}^{\alpha-1}(G)$  is free, and so is the  $S$ -group  $\mathcal{S}^\alpha(G)/\mathcal{S}^{\alpha-1}(G)$ ; therefore  $\mathcal{S}^\alpha(G)$  is the direct sum of  $\mathcal{S}^{\alpha-1}(G)$  and  $\mathcal{S}^\alpha(G)/\mathcal{S}^{\alpha-1}(G)$ . Clearly then  $\mathcal{S}^\alpha(G)$  is free and there is a free basis for it extending  $X_{\alpha-1}$ .

This proves the corollary; it should be noted that it is valid for any abelian  $S^*$ -group.

#### REFERENCES

1. P. Conrad & D. McAlister, *The completion of a lattice-ordered group*; J. Austral. Math. Soc., **9** (1969), 182-208.
2. P. Conrad, *Lattice-ordered groups*; Lecture Notes, Tulane University, New Orleans, Louisiana (1970).
3. ———, *Epi-archimedean groups*; Czech. Math. J., **24** (99), (1974), 192-218.
4. P. Hill, *Bounded sequences of integers*; preprint.
5. J. Martinez, *Archimedean-like classes of lattice-ordered groups*; Trans. Amer. Math. Soc., **186** (Dec. 1973), 33-49.
6. ———, *The hyper-archimedean kernel sequence of a lattice-ordered group*; Bull. Austral. Math. Soc., **10** (1974), 337-349.
7. ———, *Torsion theory for lattice-ordered groups*; Czech. Math. J., **25** (100) (1975), 284-299.
8. C. Nöbeling, *Verallgemeinerung einer Satzes von Herrn E. Specker*; Inventiones Math., **6** (1968), 41-55.

Received December 30, 1974.

UNIVERSITY OF FLORIDA

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

# Pacific Journal of Mathematics

Vol. 61, No. 2

December, 1975

Graham Donald Allen, Francis Joseph Narcowich and James Patrick Williams, <i>An operator version of a theorem of Kolmogorov</i> .....	305
Joel Hilary Anderson and Ciprian Foias, <i>Properties which normal operators share with normal derivations and related operators</i> .....	313
Constantin Gelu Apostol and Norberto Salinas, <i>Nilpotent approximations and quasinilpotent operators</i> .....	327
James M. Briggs, Jr., <i>Finitely generated ideals in regular <math>F</math>-algebras</i> .....	339
Frank Benjamin Cannonito and Ronald Wallace Gatterdam, <i>The word problem and power problem in 1-relator groups are primitive recursive</i> .....	351
Clifton Earle Corzatt, <i>Permutation polynomials over the rational numbers</i> .....	361
L. S. Dube, <i>An inversion of the <math>S_2</math> transform for generalized functions</i> .....	383
William Richard Emerson, <i>Averaging strongly subadditive set functions in unimodular amenable groups. I</i> .....	391
Barry J. Gardner, <i>Semi-simple radical classes of algebras and attainability of identities</i> .....	401
Irving Leonard Glicksberg, <i>Removable discontinuities of <math>A</math>-holomorphic functions</i> ...	417
Fred Halpern, <i>Transfer theorems for topological structures</i> .....	427
H. B. Hamilton, T. E. Nordahl and Takayuki Tamura, <i>Commutative cancellative semigroups without idempotents</i> .....	441
Melvin Hochster, <i>An obstruction to lifting cyclic modules</i> .....	457
Alistair H. Lachlan, <i>Theories with a finite number of models in an uncountable power are categorical</i> .....	465
Kjeld Laursen, <i>Continuity of linear maps from <math>C^*</math>-algebras</i> .....	483
Tsai Sheng Liu, <i>Oscillation of even order differential equations with deviating arguments</i> .....	493
Jorge Martinez, <i>Doubling chains, singular elements and hyper-<math>Z</math> <math>l</math>-groups</i> .....	503
Mehdi Radjabalipour and Heydar Radjavi, <i>On the geometry of numerical ranges</i> .....	507
Thomas I. Seidman, <i>The solution of singular equations, I. Linear equations in Hilbert space</i> .....	513
R. James Tomkins, <i>Properties of martingale-like sequences</i> .....	521
Alfons Van Daele, <i>A Radon Nikodým theorem for weights on von Neumann algebras</i> .....	527
Kenneth S. Williams, <i>On Euler's criterion for quintic nonresidues</i> .....	543
Manfred Wischnewsky, <i>On linear representations of affine groups. I</i> .....	551
Scott Andrew Wolpert, <i>Noncompleteness of the Weil-Petersson metric for Teichmüller space</i> .....	573
Volker Wrobel, <i>Some generalizations of Schauder's theorem in locally convex spaces</i> .....	579
Birge Huisgen-Zimmermann, <i>Endomorphism rings of self-generators</i> .....	587
Kelly Denis McKennon, <i>Corrections to: "Multipliers of type <math>(p, p)</math>"; "Multipliers of type <math>(p, p)</math> and multipliers of the group <math>L_p</math>-algebras"; "Multipliers and the group <math>L_p</math>-algebras"</i> .....	603
Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>Correction to: "<math>a^*</math>-closures to completely distributive lattice-ordered groups"</i> .....	606
Zvi Arad and George Isaac Glauberman, <i>Correction to: "A characteristic subgroup of a group of odd order"</i> .....	607
Roger W. Barnard and John Lawson Lewis, <i>Correction to: "Subordination theorems for some classes of starlike functions"</i> .....	607
David Westreich, <i>Corrections to: "Bifurcation of operator equations with unbounded linearized part"</i> .....	608