

# Pacific Journal of Mathematics

**DOUBLING CHAINS, SINGULAR ELEMENTS AND HYPER- $\mathcal{L}$   
L-GROUPS**

JORGE MARTINEZ

## DOUBLING CHAINS, SINGULAR ELEMENTS AND HYPER- $\mathcal{X}$ $l$ -GROUPS

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In a lattice-ordered group  $G$  a (descending) doubling chain is a sequence  $a_1 > a_2 > \cdots > a_n > \cdots$  of positive elements of  $G$  such that  $a_n \geq 2a_{n+1}$ . An element  $0 < s \in G$  is singular if  $0 \leq g \leq s$  implies that  $g \wedge (s - g) = 0$ . The main theorems are as follows: 1. The following two statements are equivalent: (a) every doubling chain in  $G$  is finite; (b)  $G = \bigcup_{\tau < \alpha} G^\tau$  ( $\tau$  ranging over all ordinals less than some  $\alpha$ ), where  $G^\tau$  is an  $l$ -ideal of  $G$ ,  $\sigma < \tau$  implies that  $G^\sigma \subseteq G^\tau$  and  $G^{\tau+1}/G^\tau$  is generated by its singular elements, (i.e. a Specker group, à la Conrad). 2. If  $G$  is hyper-archimedean as well then either of the above conditions is equivalent to: (c)  $G$  is hyper- $\mathcal{X}$ , i.e. every totally ordered  $l$ -homomorphic image of  $G$  is cyclic.

The purpose of this investigation was to come up with an "elementwise" definition of the abelian lattice-ordered groups (henceforth abbreviated:  $l$ -groups) having the property that each  $l$ -homomorphic image which is totally ordered is cyclic. These  $l$ -groups are called hyper- $\mathcal{X}$ , and were first introduced by the author in [5]. Thus,  $G$  is hyper- $\mathcal{X}$  if and only if  $G$  is abelian and  $G/P$  is cyclic, for each prime subgroup  $P$  of  $G$ . These  $l$ -groups are therefore hyper-archimedean, and they can in fact be characterized as those  $l$ -groups for which all the prime subgroups are maximal and have cyclic quotient; (see [3] and [5]). It should be stressed that in this characterization no assumptions need to be made with respect to commutativity. In [3] Conrad provided an example of an  $l$ -group which is hyper-archimedean and also a subdirect product of  $\mathbb{Z}$ , the additive group of integers with the usual order, yet is not hyper- $\mathcal{X}$ .

An element  $s > 0$  of the  $l$ -group  $G$  is singular provided  $0 \leq g \leq s$  implies that  $g \wedge (s - g) = 0$ . An  $S$ -group (or Specker group) is one in which each positive element is a sum of singular elements. These  $S$ -groups are well explored in [3]; the main characterization is that each  $S$ -group can be embedded as an  $l$ -subring of bounded, integer-valued functions on a set, or alternatively, as an  $l$ -subgroup of bounded, integer-valued functions generated by characteristic functions. It was observed in [7] that the  $S$ -groups form a torsion class of  $l$ -groups; that is, they are closed under taking convex  $l$ -subgroups,  $l$ -homomorphic images, and if  $G$  is any  $l$ -group, and  $\{C_\lambda \mid \lambda \in A\}$  a family of convex  $l$ -subgroups which are all  $S$ -groups then the convex  $l$ -subgroup they generate is an  $S$ -group. There is thus an associated

$S$ -radical  $\mathcal{S}(G)$  of an  $l$ -group  $G$ , and a “Loewy”-like ascending sequence  $S(G) = \mathcal{S}^1(G) \subseteq \dots \subseteq \mathcal{S}^\tau(G) \subseteq \dots$  for each ordinal  $\tau$ , so that

- (a)  $\mathcal{S}(G)$  is the largest convex  $l$ -subgroup of  $G$  which is an  $S$ -group.
- (b) For any convex  $l$ -subgroup  $A$  of  $G$ ,  $\mathcal{S}(A) = A \cap \mathcal{S}(G)$ .
- (c) If  $\alpha$  is a limit ordinal  $\mathcal{S}^\alpha(G) = \bigcup \{ \mathcal{S}^\tau(G) \mid \tau < \alpha \}$ ,
- (d) and otherwise  $\mathcal{S}^\tau(G)$  is defined by the equation:

$$\mathcal{S}^\tau(G) / \mathcal{S}^{\tau-1}(G) = \mathcal{S}(G / \mathcal{S}^{\tau-1}(G)) .$$

Then we are able to define  $\mathcal{S}^*(G) = \mathcal{S}^\tau(G)$ , where  $\tau$  is chosen so that  $\mathcal{S}^\tau(G) = \mathcal{S}^{\tau+1}(G) = \dots$ ; such a  $\tau$  exists by a simple cardinality argument.  $G$  is said to be an  $S^*$ -group if  $\mathcal{S}^*(G) = G$ .

We should observe that if  $G$  is an  $S$ -group, it can be represented as an  $l$ -group of bounded, integer-valued functions, and it is therefore hyper- $\mathcal{R}$ ; (see [3]). Examples of hyper- $\mathcal{R}$   $l$ -groups which are not  $S$ -groups are easy to construct.

In an  $l$ -group  $G$ , a (*descending*) *doubling chain* is a sequence  $s_1 > s_2 > \dots$  of positive elements of  $G$  so that  $s_n \geq 2s_{n+1}$ , for each  $n = 1, 2, \dots$ . Notice that the terms of a doubling chain may eventually be zero; in such a case it is a finite doubling chain.

We can now state our first result.

**THEOREM 1.**  *$G$  is an  $S^*$ -group if and only if every doubling chain for  $G$  is finite.*

*Proof. Necessity.* The proof proceeds by transfinite induction on the length of the Loewy sequence of  $\mathcal{S}^\tau(G)$ 's. The first thing to do is to show an  $S$ -group has this property. This is clear, because if  $G$  is an  $S$ -group, it can be represented as an  $l$ -group of bounded, integer-valued functions; and there are obviously no infinite doubling chains of such functions. Next, suppose  $G = \mathcal{S}^\alpha(G)$  and  $\mathcal{S}^\tau(G)$  has no infinite doubling chains, for each  $\tau < \alpha$ . Suppose by way of contradiction that  $a_1 > a_2 > \dots > a_n > \dots$  is an infinite doubling chain for  $G$ ; if  $\alpha$  is a limit ordinal, then  $a_1 \in \mathcal{S}^\beta(G)$  for some  $\beta < \alpha$ , and hence each  $a_n \in \mathcal{S}^\beta(G)$ , contradicting our assumption. If  $\alpha$  has a predecessor, then no  $a_n \in \mathcal{S}^{\alpha-1}(G)$ , and consequently  $a_1 + \mathcal{S}^{\alpha-1}(G) > a_2 + \mathcal{S}^{\alpha-1}(G) > \dots$  is an infinite doubling chain in the  $S$ -group  $G / \mathcal{S}^{\alpha-1}(G)$ . This again is a contradiction, and we must conclude that  $G = \mathcal{S}^\alpha(G)$  has no infinite doubling chains; this completes the proof of the necessity.

*Sufficiency.* Let us make a preliminary observation: for a given ordinal  $\tau$ , an element  $a > 0$  of an  $l$ -group  $G$  has the property that  $a \geq 2b \geq 0$  implies that  $b \in \mathcal{S}^\tau(G)$  if and only if  $a \in \mathcal{S}^\tau(G)$  or else

$a + \mathcal{S}^\tau(G)$  is a singular element of  $G/\mathcal{S}^\tau(G)$ . If  $a$  has this property and  $a \notin \mathcal{S}^\tau(G)$  we call  $a$  a  $(\tau + 1)$ -singular. (Note: for  $\tau = 0$  we set  $\mathcal{S}^\tau(G) = 0$ ; then 1-singular simply means: singular.)

Suppose now that every doubling chain of  $G$  is finite. If  $0 < g \in G$  and  $\tau$  is an ordinal, then if  $g$  is not  $(\tau + 1)$ -singular we may find an element  $0 < a_1 \in G$  such that  $a_1 \notin \mathcal{S}^\tau(G)$  and  $2a_1 \leq g$ . Inductively proceed to construct a doubling chain  $g > a_1 > a_2 > \dots > a_n > \dots$ , where  $a_n$  is the last entry outside  $\mathcal{S}^\tau(G)$ , and therefore  $(\tau + 1)$ -singular. Thus, every positive element of  $G$  exceeds a  $(\tau + 1)$ -singular element, for each ordinal  $\tau$ .

If  $G \neq \mathcal{S}^*(G)$ , we pick  $0 < g \in G \setminus \mathcal{S}^*(G)$ , and an ordinal  $\alpha$  such that  $\mathcal{S}^*(G) = \mathcal{S}^\alpha(G)$ . As we have indicated  $g \geq h$  for some  $(\alpha + 1)$ -singular element  $h$ ; that is,  $h$  is singular modulo  $\mathcal{S}^\alpha(G)$ , which is absurd. We must conclude that  $G$  is an  $S^*$ -group, and Theorem 1 is proved.

A hyper-archimedean  $l$ -group is characterized by the condition that each prime subgroup be maximal [3]. Therefore, every totally ordered  $l$ -homomorphic image of a hyper-archimedean  $l$ -group is a subgroup of the additive reals, by Hölder's theorem. Now let us prove:

**THEOREM 2.** *Suppose  $G$  is hyper-archimedean; then it is hyper- $\mathcal{X}$  if and only if every doubling chain for  $G$  is finite.*

*Proof.* Suppose  $G$  is hyper- $\mathcal{X}$ , yet  $a_1 > a_2 > \dots > a_n > \dots$  is an infinite doubling chain. The  $a_i$  are contained in an ultrafilter of the positive cone of  $G$ , and thus a minimal prime subgroup  $P$  exists so that  $a_n \notin P$  for each  $n = 1, 2, \dots$ . (Recall that an *ultrafilter* is a subset  $U$  of strictly positive elements of an  $l$ -group  $H$ , maximal with respect to the property:  $a, b \in U$  imply that  $a \wedge b \in U$ . For an account of the correspondence between ultrafilters and minimal prime subgroups we refer the reader to [1] or [2].)

Continuing then,  $a_1 + P > a_2 + P > \dots$  is an infinite descending chain for the archimedean  $o$ -group  $G/P$ ;  $G/P$  can therefore not be cyclic, and we have a contradiction.

Conversely, suppose every doubling chain of  $G$  is finite; then  $G$  is an  $S^*$ -group by Theorem 1, and it is easy to verify from this that each totally ordered quotient of  $G$  is cyclic, since the class of  $S^*$ -groups is closed under  $l$ -homomorphic images; (see [7]).

This is enough to establish Theorem 2.

**COROLLARY.** *If  $G$  is hyper-archimedean, and  $A$  is an  $l$ -ideal of  $G$  so that  $A$  and  $G/A$  are both hyper- $\mathcal{X}$ , then  $G$  is hyper- $\mathcal{X}$ .*

The following example illustrates the use of hyper-archimedeanity in Theorem 2 and the above corollary. Let  $G$  be the  $l$ -group of sequences of integers by the eventually constant sequences and  $a = (1, 2, 3, \dots)$ . This example was discussed in [6], and it was shown there that  $G$  is not hyper-archimedean. Yet  $G$  is an extension of an  $S$ -group by  $Z$ , and all its doubling chains are finite.

Finally, we state a corollary which says something about the underlying group of a hyper- $\mathcal{S}$   $l$ -group.

**COROLLARY.** *If  $G$  is a hyper- $\mathcal{S}$   $l$ -group, then  $G$  is free, quabelian group.*

*Proof.* As an  $S$ -group is a subgroup of bounded, integer-valued functions it is free abelian; this result goes back to Nöbeling [8], and it is further discussed by Hill in [4] and Conrad in [3]. If  $G$  is a hyper- $\mathcal{S}$   $l$ -group then it is an  $S^*$ -group, say  $G = \mathcal{S}^\alpha(G)$ ; we assume that  $\mathcal{S}^\tau(G)$  is free abelian for each ordinal  $\tau < \alpha$ , and that a free basis  $X_\tau$  for  $\mathcal{S}^\tau(G)$  can be picked so that  $X_\sigma = X_\tau \cap \mathcal{S}^\sigma(G)$ , if  $\sigma < \tau < \alpha$ . If  $\alpha$  is a limit ordinal, we let  $X = \bigcup \{X_\tau \mid \tau < \alpha\}$ ; it is easy to verify that  $X$  is a free basis for  $\mathcal{S}^\alpha(G)$ . Otherwise, we have that  $\mathcal{S}^{\alpha-1}(G)$  is free, and so is the  $S$ -group  $\mathcal{S}^\alpha(G)/\mathcal{S}^{\alpha-1}(G)$ ; therefore  $\mathcal{S}^\alpha(G)$  is the direct sum of  $\mathcal{S}^{\alpha-1}(G)$  and  $\mathcal{S}^\alpha(G)/\mathcal{S}^{\alpha-1}(G)$ . Clearly then  $\mathcal{S}^\alpha(G)$  is free and there is a free basis for it extending  $X_{\alpha-1}$ .

This proves the corollary; it should be noted that it is valid for any abelian  $S^*$ -group.

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Received December 30, 1974.

UNIVERSITY OF FLORIDA

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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