

Pacific Journal of Mathematics

PROPERTIES OF MARTINGALE-LIKE SEQUENCES

R. JAMES TOMKINS

PROPERTIES OF MARTINGALE-LIKE SEQUENCES

R. JAMES TOMKINS

The purpose of this paper is to define a new type of stochastic sequence and to explore its properties. These new sequences of random variables, called eventual martingales, generalize the concept of a martingale.

Several known results concerning the almost sure limiting behavior of martingales are shown to remain valid for eventual martingales. In addition, eventual martingales are compared with three other martingale-like sequences.

Consider a probability space (Ω, \mathcal{F}, P) . A stochastic sequence $(X_n, \mathcal{F}_n, n \geq 1)$ will be called an *eventual martingale* if and only if (iff)

$$(1) \quad P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1} \text{ infinitely often (i.o.)}] = 0.$$

This says, in effect, that, except on an event of probability zero, the martingale property $E(X_n | \mathcal{F}_{n-1}) = X_{n-1}$ holds for all sufficiently large n . In view of the Borel-Cantelli lemma, $(X_n, \mathcal{F}_n, n \geq 1)$ is an eventual martingale if $\sum_{n=1}^{\infty} P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1}] < \infty$; in particular, every martingale is an eventual martingale.

In §2, a decomposition theorem for eventual martingales will be established and used to generalize some known martingale results. Section 3 will explore the relationship among eventual martingales and three other generalizations of martingales.

Assume throughout that \mathcal{F}_0 is the trivial sigma-field. Let $I(A)$ denote the indicator function of an event $A \in \mathcal{F}$.

2. A decomposition theorem. Crucial to the considerations of this section is the following result.

THEOREM 1. *Let $(X_n, \mathcal{F}_n, n \geq 1)$ be an eventual martingale. Then there exist stochastic sequences $(M_n, \mathcal{F}_n, n \geq 1)$ and $(Z_n, \mathcal{F}_n, n \geq 1)$ such that (i) $X_n = M_n + Z_n$ for all $n \geq 1$, (ii) $(M_n, \mathcal{F}_n, n \geq 1)$ is a martingale, and (iii) $P[Z_{n+1} \neq Z_n \text{ i.o.}] = 0$.*

Proof. Let $d_1 = X_1$ and, for $n \geq 1$, let $d_{n+1} = X_{n+1} - X_n$. If $n \geq 1$, let $M_n = \sum_{k=1}^n d_k I(E(d_k | \mathcal{F}_{k-1}) = 0)$ and $Z_n = X_n - M_n$. Then (i) and (ii) are obvious. Moreover, $Z_{n+1} - Z_n = d_{n+1} I(E(d_{n+1} | \mathcal{F}_n) \neq 0)$ so $[Z_{n+1} \neq Z_n] \subseteq [E(d_{n+1} | \mathcal{F}_n) \neq 0]$. Hence $0 \leq P[Z_{n+1} \neq Z_n \text{ i.o.}] \leq P[E(d_{n+1} | \mathcal{F}_n) \neq 0 \text{ i.o.}] = 0$ by (1).

REMARK. Let $B = [Z_{n+1} \neq Z_n \text{ i.o.}]$. Theorem 1 (iii) says that,

except for $\omega \in B$, each (real) sequence $\{Z_n(\omega)\}$ is constant from some point onward. Therefore, $\{Z_n\}$ is an almost surely (a.s.) convergent sequence of random variables (rv). Thus it is evident that X_n converges a.s. iff M_n converges a.s. Moreover, for any positive real sequence $c_n \rightarrow \infty$, the sequences $\{X_n/c_n\}$ and $\{M_n/c_n\}$ have the same limiting behavior, since $\lim_{n \rightarrow \infty} Z_n/c_n = 0$ a.s.

These facts allow several properties of martingales to remain valid for eventual martingales, as the next theorem shows.

THEOREM 2. *Let $(X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1)$ be an eventual martingale.*

(i) (cf. Chow [4]). *If $\sum_{n=1}^{\infty} E|d_n|^{2r}/n^{1+r} < \infty$ for some $r \geq 1$, then $\lim_{n \rightarrow \infty} X_n/n = 0$ a.s.*

(ii) (cf. Burkholder [3]). *If $E((\sum_{n=1}^{\infty} d_n^2)^{1/2}) < \infty$, then X_n converges a.s.*

(iii) (cf. Stout [7]). *If $|d_n| \leq M$ a.s. for some $M < \infty$, and if $s_n^2 \equiv \sum_{k=1}^n E(d_k^2 | \mathcal{F}_{k-1}) \rightarrow \infty$ a.s., then $\limsup_{n \rightarrow \infty} X_n/(2s_n^2 \log \log s_n^2)^{1/2} = 1$ a.s.*

Proof. $|M_n - M_{n-1}| = |d_n I(E(d_n | \mathcal{F}_{n-1}) = 0)| \leq |d_n|$. Thus the hypothesis of (i) and the theorem in [4] imply $M_n/n \rightarrow 0$ a.s. and, hence, $X_n/n \rightarrow 0$ a.s. Furthermore, under (ii), the hypothesis of Theorem 2 of [3] holds for $(M_n, \mathcal{F}_n, n \geq 1)$ so M_n converges a.s. which is tantamount to (ii).

Finally, let $v_n^2 \equiv \sum_{k=1}^n E(d_k^2 I(E(d_k | \mathcal{F}_{k-1}) = 0) | \mathcal{F}_{k-1})$. Now $0 \leq P[E(d_k^2 | \mathcal{F}_{k-1}) \neq E(d_k^2 I(E(d_k | \mathcal{F}_{k-1}) = 0) | \mathcal{F}_{k-1}) \text{ i.o.}] \leq P[E(d_k | \mathcal{F}_{k-1}) \neq 0 \text{ i.o.}] = 0$ by (1) so

(2) $v_n/s_n \rightarrow 1$ a.s.

But $s_n \rightarrow \infty$ a.s. so $v_n \rightarrow \infty$ a.s. Hence $(M_n, \mathcal{F}_n, n \geq 1)$ obeys the conditions in Theorem 1 and 2 of [7] so $\limsup_{n \rightarrow \infty} M_n/(2v_n^2 \log \log v_n^2)^{1/2} = 1$ a.s. The remark preceding the theorem and (2) now imply (iii).

REMARK. Let $(X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1)$ be an eventual martingale. Writing

$$X_n = \sum_{k=1}^n (d_k - E(d_k | \mathcal{F}_{n-1})) + \sum_{k=1}^n E(d_k | \mathcal{F}_{n-1})$$

yields another decomposition of X_n which satisfies (i), (ii) and (iii) of Theorem 1. The next result uses this new decomposition to extend another result of Burkholder [3].

THEOREM 3. *Let $(X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1)$ be an eventual martingale such that $\sup_{n \geq 1} E|X_n - \sum_{k=1}^n E(d_k | \mathcal{F}_{k-1})| < \infty$. For $n \geq 1$, let ν_n be an \mathcal{F}_{n-1} -measurable rv. Then $\sum_{k=1}^n \nu_k d_k$ converges a.s. on the*

event $[\sup_{n \geq 1} |\nu_n| < \infty]$. In particular, X_n converges a.s. as $n \rightarrow \infty$.

Proof. By hypothesis, $(\sum_{k=1}^n (d_k - E(d_k | \mathcal{F}_{k-1})))$ is an \mathcal{L}_1 -bounded martingale. So, by Theorem 1 of Burkholder [3],

$$(3) \quad \sum_{k=1}^n \nu_k (d_k - E(d_k | \mathcal{F}_{k-1})) \text{ converges a.s. on } [\sup_{n \geq 1} |\nu_n| < \infty].$$

Let $C = [E(d_n | \mathcal{F}_{n-1}) \neq 0 \text{ i.o.}]$; then $P(C) = 0$ by (1). Hence, if $\omega \notin C$, there exists an integer $N = N(\omega)$ such that $E(d_n | \mathcal{F}_{n-1}) = 0$ for $n \geq N$. Therefore, $\sum_{k=1}^n \nu_k E(d_k | \mathcal{F}_{k-1})$ converges a.s. This fact, together with (3), yields the result. Of course, the special case results when $\nu_n \equiv 1$ for all $n \geq 1$.

3. On various generalizations of martingales. Alloin [1] calls $(X_n, \mathcal{F}_n, n \geq 1)$ a *progressive martingale* iff $[E(X_n | \mathcal{F}_{n-1}) = X_{n-1}] \subseteq [E(X_{n+1} | \mathcal{F}_n) = X_n]$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} P[E(X_n | \mathcal{F}_{n-1}) = X_{n-1}] = 1$. Mucci [6] calls $(X_n, \mathcal{F}_n, n \geq 1)$ a *martingale in the limit* iff $\lim_{n \geq m \rightarrow \infty} (E(X_n | \mathcal{F}_m) - X_m) = 0$ a.s. According to Blake [2], $(X_n, \mathcal{F}_n, n \geq 1)$ is *fairer with time* iff $\lim_{n \geq m \rightarrow \infty} P[|E(X_n | \mathcal{F}_m) - X_m| > \varepsilon] = 0$ for all $\varepsilon > 0$.

The final theorem indicates some relationships involving these three concepts and eventual martingales.

THEOREM 4. (i) *Every progressive martingale is an eventual martingale.*

(ii) *Every progressive martingale is a martingale in the limit.*

(iii) *Every uniformly integrable eventual martingale $(X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1)$ with $\sup_{n \geq 1} E|\sum_{k=1}^n E(d_k | \mathcal{F}_{k-1})| < \infty$ is fairer with time.*

Proof. If $(X_n, \mathcal{F}_n, n \geq 1)$ is a progressive martingale, then $P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1} \text{ i.o.}] = \lim_{n \rightarrow \infty} P\{\bigcup_{k=n}^{\infty} [E(X_k | \mathcal{F}_{k-1}) \neq X_{k-1}]\} = \lim_{n \rightarrow \infty} P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1}] = 0$ so (i) is true.

Let $t = \inf\{n \geq 1: E(X_n | \mathcal{F}_{n-1}) = X_{n-1}\}$. Since $(X_n, \mathcal{F}_n, n \geq 1)$ is a progressive martingale, t is a stopping rule; i.e. $t \in \{1, 2, \dots, \infty\}$, $P[t < \infty] = 1$ and $[t = n] \in \mathcal{F}_n$ for all $n \geq 1$. Now if $t \leq m$, where $m \geq 1$, then $E(X_k | \mathcal{F}_{k-1}) = X_{k-1}$ for $k \geq m$. But $[t \leq m] \in \mathcal{F}_m$, so, for

$$\begin{aligned} n > m, \quad & (E(X_n | \mathcal{F}_m) - X_m)I(t \leq m) \\ &= \sum_{k=m+1}^n E(I(t \leq m)E(X_k - X_{k-1} | \mathcal{F}_{k-1}) | \mathcal{F}_m) = 0. \end{aligned}$$

For each $\omega \in [t < \infty]$, there exists $m_0 = m_0(\omega)$ such that $\omega \in [t \leq m_0]$ so $E(X_n | \mathcal{F}_m) - X_m = 0$ for all $n \geq m \geq m_0$, proving (ii).

(iii) is a consequence of a result on page 162 of [5], Theorem 3 above and Theorem 1 of Mucci [6].

REMARK. None of the statements in Theorem 4 have valid converses. The converses of (i) and (ii) are both shown to be false by letting d_1, d_2, \dots be independent rv with $E(d_3) = 1$, $E(d_n) = 0$ if $n \neq 3$, defining $X_n = \sum_{k=1}^n d_k$ and taking \mathcal{F}_n to be the sigma-field generated by d_1, d_2, \dots, d_n . The converse to (iii) is contradicted by the following example, the first of two examples which show that no general relationship exists between sequence fairer with time and eventual martingales.

EXAMPLE 1. A martingale in the limit need not be an eventual martingale, even if it is uniformly integrable. Let d_1, d_2, \dots be independent rv such that $P[d_n = 1] = n^{-2}$ whereas $P[d_n = 0] = 1 - n^{-2}$ for $n \geq 1$. Let \mathcal{F}_n be the sigma-field generated by d_1, \dots, d_n and set $X_n = \sum_{k=1}^n d_k$. Since $E(\sum_{k=1}^{\infty} |d_k|) = \sum_{k=1}^{\infty} k^{-2} < \infty$ and $|X_n| \leq \sum_{k=1}^{\infty} |d_k|$ for all $n \geq 1$, $\{X_n\}$ is uniformly integrable. Moreover, $E(X_n | \mathcal{F}_m) - X_m = \sum_{k=m+1}^n k^{-2}$ for $n > m \geq 1$ so $(X_n, \mathcal{F}_n, n-1)$ is a martingale in the limit. But $E(X_n | \mathcal{F}_{n-1}) = X_{n-1} + n^{-2} \neq X_{n-1}$ for all $n \geq 2$, so it is not an eventual martingale.

EXAMPLE 2. An eventual martingale need not be fairer with time and, hence, need not be a martingale in the limit. Let U_1, U_2, \dots be independent rv such that $P[U_n = -1] = 2^{-n}$ and $P[U_n = 1] = 1 - 2^{-n}$ for $n \geq 1$. Let \mathcal{F}_n be the sigma-field generated by $U_1, U_2, \dots, U_n, d_1 = U_1, d_{n+1} = 2^n U_{n+1} I(U_n = -1)$ and $X_n = \sum_{k=1}^n d_k$ for $n \geq 1$. For $k > 1$,

$$\begin{aligned} E(d_k | \mathcal{F}_{k-1}) &= 2^{k-1} I(U_{k-1} = -1) E(U_k | \mathcal{F}_{k-1}) \\ &= (2^{k-1} - 1) I(U_{k-1} = -1). \end{aligned}$$

Hence $\sum_{k=2}^{\infty} P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1}] = \sum_{k=2}^{\infty} P[U_{k-1} = -1] = \sum_{k=2}^{\infty} 2^{-k+1} < \infty$. Thus $(X_n, \mathcal{F}_n, n \geq 1)$ is an eventual martingale.

But, if $m \geq 2$, $E(X_{2m} - X_m | \mathcal{F}_m) = \sum_{k=m+1}^{2m} E(E(d_k | \mathcal{F}_{k-1}) | \mathcal{F}_m) = \sum_{k=m+2}^{2m} (2^{k-1} - 1) P[U_{k-1} = -1] + (2^m - 1) I(U_m = -1) \geq \sum_{k=m+2}^{2m} (1 - 2^{-k+1}) > (1 - 2^{-2m+1}) > 1/2$. Hence, if $\varepsilon < 1/2$,

$$P[|E(X_{2m} | \mathcal{F}_m) - X_m| > \varepsilon] = 1$$

for all $m > 1$, so $(X_n, \mathcal{F}_n, n \geq 1)$ is not fairer with time.

REFERENCES

1. C. Alloin, *Martingales progressives*, Cahiers Centre Études Recherche Opér., **12** (1970), 201-210.
2. L. H. Blake, *A generalization of martingales and two consequent convergence theorems*, Pacific J. Math., **35** (1970), 279-283.
3. D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist., **37** (1966), 1494-1504.

4. Y. S. Chow, *On a strong law of large numbers for martingales*, Ann. Math. Statist., **38** (1967), 610.
5. M. Loève, *Probability Theory*, Van Nostrand (1963).
6. A. G. Mucci, *Limits for martingale-like sequences*, Pacific J. Math., **48** (1973), 197-202.
7. W. F. Stout, *A martingale analogue of Kolmogorov's law of the iterated logarithm*, Z. Wahrscheinlichkeitstheorie verw. Geb., **15** (1970), 270-290.

Received January 23, 1975. This research was supported by National Research Council of Canada Grant A7588.

UNIVERSITY OF REGINA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

| | |
|---|-----|
| Graham Donald Allen, Francis Joseph Narcowich and James Patrick Williams, <i>An operator version of a theorem of Kolmogorov</i> | 305 |
| Joel Hilary Anderson and Ciprian Foias, <i>Properties which normal operators share with normal derivations and related operators</i> | 313 |
| Constantin Gelu Apostol and Norberto Salinas, <i>Nilpotent approximations and quasinilpotent operators</i> | 327 |
| James M. Briggs, Jr., <i>Finitely generated ideals in regular F-algebras</i> | 339 |
| Frank Benjamin Cannonito and Ronald Wallace Gatterdam, <i>The word problem and power problem in 1-relator groups are primitive recursive</i> | 351 |
| Clifton Earle Corzatt, <i>Permutation polynomials over the rational numbers</i> | 361 |
| L. S. Dube, <i>An inversion of the S_2 transform for generalized functions</i> | 383 |
| William Richard Emerson, <i>Averaging strongly subadditive set functions in unimodular amenable groups. I</i> | 391 |
| Barry J. Gardner, <i>Semi-simple radical classes of algebras and attainability of identities</i> | 401 |
| Irving Leonard Glicksberg, <i>Removable discontinuities of A-holomorphic functions</i> ... | 417 |
| Fred Halpern, <i>Transfer theorems for topological structures</i> | 427 |
| H. B. Hamilton, T. E. Nordahl and Takayuki Tamura, <i>Commutative cancellative semigroups without idempotents</i> | 441 |
| Melvin Hochster, <i>An obstruction to lifting cyclic modules</i> | 457 |
| Alistair H. Lachlan, <i>Theories with a finite number of models in an uncountable power are categorical</i> | 465 |
| Kjeld Laursen, <i>Continuity of linear maps from C^*-algebras</i> | 483 |
| Tsai Sheng Liu, <i>Oscillation of even order differential equations with deviating arguments</i> | 493 |
| Jorge Martinez, <i>Doubling chains, singular elements and hyper-Z l-groups</i> | 503 |
| Mehdi Radjabalipour and Heydar Radjavi, <i>On the geometry of numerical ranges</i> | 507 |
| Thomas I. Seidman, <i>The solution of singular equations, I. Linear equations in Hilbert space</i> | 513 |
| R. James Tomkins, <i>Properties of martingale-like sequences</i> | 521 |
| Alfons Van Daele, <i>A Radon Nikodým theorem for weights on von Neumann algebras</i> | 527 |
| Kenneth S. Williams, <i>On Euler's criterion for quintic nonresidues</i> | 543 |
| Manfred Wischnewsky, <i>On linear representations of affine groups. I</i> | 551 |
| Scott Andrew Wolpert, <i>Noncompleteness of the Weil-Petersson metric for Teichmüller space</i> | 573 |
| Volker Wrobel, <i>Some generalizations of Schauder's theorem in locally convex spaces</i> | 579 |
| Birge Huisgen-Zimmermann, <i>Endomorphism rings of self-generators</i> | 587 |
| Kelly Denis McKennon, <i>Corrections to: "Multipliers of type (p, p)"; "Multipliers of type (p, p) and multipliers of the group L_p-algebras"; "Multipliers and the group L_p-algebras"</i> | 603 |
| Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>Correction to: "a^*-closures to completely distributive lattice-ordered groups"</i> | 606 |
| Zvi Arad and George Isaac Glauberman, <i>Correction to: "A characteristic subgroup of a group of odd order"</i> | 607 |
| Roger W. Barnard and John Lawson Lewis, <i>Correction to: "Subordination theorems for some classes of starlike functions"</i> | 607 |
| David Westreich, <i>Corrections to: "Bifurcation of operator equations with unbounded linearized part"</i> | 608 |