PROPERTIES OF MARTINGALE-LIKE SEQUENCES

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The purpose of this paper is to define a new type of stochastic sequence and to explore its properties. These new sequences of random variables, called eventual martingales, generalize the concept of a martingale.

Several known results concerning the almost sure limiting behavior of martingales are shown to remain valid for eventual martingales. In addition, eventual martingales are compared with three other martingale-like sequences.

Consider a probability space $(\Omega, \mathcal{F}, P)$. A stochastic sequence $(X_n, \mathcal{F}_n, n \geq 1)$ will be called an eventual martingale if and only if

$$ P\{E(X_\infty | \mathcal{F}_{n-1}) \neq X_{n-1} \text{ infinitely often (i.o.)}\} = 0. $$

This says, in effect, that, except on an event of probability zero, the martingale property $E(X_n | \mathcal{F}_{n-1}) = X_{n-1}$ holds for all sufficiently large $n$. In view of the Borel-Cantelli lemma, $(X_n, \mathcal{F}_n, n \geq 1)$ is an eventual martingale if $\sum_{n=1}^{\infty} P\{E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1}\} < \infty$; in particular, every martingale is an eventual martingale.

In §2, a decomposition theorem for eventual martingales will be established and used to generalize some known martingale results. Section 3 will explore the relationship among eventual martingales and three other generalizations of martingales.

Assume throughout that $\mathcal{F}_0$ is the trivial sigma-field. Let $I(A)$ denote the indicator function of an event $A \in \mathcal{F}$.

2. A decomposition theorem. Crucial to the considerations of this section is the following result.

**Theorem 1.** Let $(X_n, \mathcal{F}_n, n \geq 1)$ be an eventual martingale. Then there exist stochastic sequences $(M_n, \mathcal{F}_n, n \geq 1)$ and $(Z_n, \mathcal{F}_n, n \geq 1)$ such that (i) $X_n = M_n + Z_n$ for all $n \geq 1$, (ii) $(M_n, \mathcal{F}_n, n \geq 1)$ is a martingale, and (iii) $P\{Z_{n+1} \neq Z_n \text{ i.o.}\} = 0$.

**Proof.** Let $d_1 = X_1$ and, for $n \geq 1$, let $d_{n+1} = X_{n+1} - X_n$. If $n \geq 1$, let $M_n = \sum_{k=1}^{n} d_k I(E(d_k | \mathcal{F}_{k-1}) = 0)$ and $Z_n = X_n - M_n$. Then (i) and (ii) are obvious. Moreover, $Z_{n+1} - Z_n = d_{n+1} I(E(d_{n+1} | \mathcal{F}_n) \neq 0)$ so $[Z_{n+1} \neq Z_n] \subseteq [E(d_{n+1} | \mathcal{F}_n) \neq 0]$. Hence $0 \leq P\{Z_{n+1} \neq Z_n \text{ i.o.}\} \leq P[E(d_{n+1} | \mathcal{F}_n) \neq 0 \text{ i.o.}] = 0$ by (1).

**Remark.** Let $B = [Z_{n+1} \neq Z_n \text{ i.o.}]$. Theorem 1 (iii) says that,
except for \( \omega \in B \), each (real) sequence \( \{ Z_n(\omega) \} \) is constant from some point onward. Therefore, \( \{ Z_n \} \) is an almost surely (a.s.) convergent sequence of random variables (rv). Thus it is evident that \( X_n \) converges a.s. iff \( M_n \) converges a.s. Moreover, for any positive real sequence \( c_n \to \infty \), the sequences \( \{ X_n/c_n \} \) and \( \{ M_n/c_n \} \) have the same limiting behavior, since \( \lim_{n \to \infty} Z_n/c_n = 0 \) a.s.

These facts allow several properties of martingales to remain valid for eventual martingales, as the next theorem shows.

**Theorem 2.** Let \( (X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1) \) be an eventual martingale.

(i) (cf. Chow [4]). If \( \sum_{n=1}^\infty E|d_n|^r/n^{1+r} < \infty \) for some \( r \geq 1 \), then \( \lim_{n \to \infty} X_n/n = 0 \) a.s.

(ii) (cf. Burkholder [3]). If \( E((\sum_{n=1}^\infty d_n^2)^{1/2}) < \infty \), then \( X_n \) converges a.s.

(iii) (cf. Stout [7]). If \( |d_n| \leq M \) a.s. for some \( M < \infty \), and if
\[
\sum_{k=1}^n E(d_k^2 | \mathcal{F}_{k-1}) \to \infty \text{ a.s.},
\]
then \( \limsup_{n \to \infty} X_n/(2s_n \log \log s_n)^{1/2} = 1 \) a.s.

**Proof.** Let \( M_n - M_{n-1} = |d_n|E(d_n | \mathcal{F}_{n-1}) = 0 \) a.s. Thus the hypothesis of (i) and the theorem in [4] imply \( M_n/n \to 0 \) a.s. and, hence, \( X_n/n \to 0 \) a.s. Furthermore, under (ii), the hypothesis of Theorem 2 of [3] holds for \( (M_n, \mathcal{F}_n, n \geq 1) \) so \( M_n \) converges a.s. which is tantamount to (ii).

Finally, let \( v_n = \sum_{k=1}^n E(d_k^2 | \mathcal{F}_{k-1}) = 0 \) \( | \mathcal{F}_{k-1} \). Now \( 0 \leq P[E(d_k^2 | \mathcal{F}_{k-1}) \neq E(d_k^2 | \mathcal{F}_{k-1})= 0 | \mathcal{F}_{k-1} \) i.o.] \( \leq P[E(d_k^2 | \mathcal{F}_{k-1}) \neq 0 \) i.o.] = 0 by (1) so
\[
(2) \quad v_n/s_n \to 1 \text{ a.s.}
\]

But \( s_n \to \infty \) a.s. so \( v_n \to \infty \) a.s. Hence \( (M_n, \mathcal{F}_n, n \geq 1) \) obeys the conditions in Theorem 1 and 2 of [7] so \( \limsup_{n \to \infty} M_n/(2v^1 \log \log v^1)^{1/2} = 1 \) a.s. The remark preceding the theorem and (2) now imply (iii).

**Remark.** Let \( (X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1) \) be an eventual martingale. Writing
\[
X_n = \sum_{k=1}^n (d_k - E(d_k | \mathcal{F}_{k-1})) + \sum_{k=1}^n E(d_k | \mathcal{F}_{k-1})
\]
yields another decomposition of \( X_n \) which satisfies (i), (ii) and (iii) of Theorem 1. The next result uses this new decomposition to extend another result of Burkholder [3].

**Theorem 3.** Let \( (X_n = \sum_{k=1}^n d_k, \mathcal{F}_n, n \geq 1) \) be an eventual martingale such that \( \sup_{n \geq 1} E|X_n - \sum_{k=1}^n E(d_k | \mathcal{F}_{k-1})| < \infty \). For \( n \geq 1 \), let \( \nu_n \) be an \( \mathcal{F}_{n-1} \) measurable rv. Then \( \sum_{k=1}^n \nu_k d_k \) converges a.s. on the
event $[\sup_{n \geq 1} \nu_n] < \infty$. In particular, $X_n$ converges a.s. as $n \to \infty$.

**Proof.** By hypothesis, $(\sum_{k=1}^{n} (d_k - E(d_k | \mathcal{F}_{k-1})))$ is an $\mathcal{F}_1$-bounded martingale. So, by Theorem 1 of Burkholder [3],

$$\sum_{k=1}^{n} \nu_k (d_k - E(d_k | \mathcal{F}_{k-1}))$$

converges a.s. on $[\sup_{n \geq 1} \nu_n] < \infty$. Let $C = [E(d_n | \mathcal{F}_{n-1}) = 0 \text{ i.o.}]$; then $P(C) = 0$ by (1). Hence, if $\omega \notin C$, there exists an integer $N = N(\omega)$ such that $E(d_n | \mathcal{F}_{n-1}) = 0$ for $n \geq N$. Therefore, $\sum_{k=1}^{n} \nu_k E(d_k | \mathcal{F}_{k-1})$ converges a.s. This fact, together with (3), yields the result. Of course, the special case results when $\nu_n \equiv 1$ for all $n \geq 1$.

3. On various generalizations of martingales. Alloin [1] calls $(X_n, \mathcal{F}_n, n \geq 1)$ a progressive martingale iff $[E(X_n | \mathcal{F}_{n-1}) = X_{n-1}] \subseteq [E(X_{n+1} | \mathcal{F}_n) = X_n]$ for all $n \geq 1$ and $\lim_{n \to \infty} P[E(X_n | \mathcal{F}_{n-1}) = X_{n-1}] = 1$. Mucci [6] calls $(X_n, \mathcal{F}_n, n \geq 1)$ a martingale in the limit iff $\lim_{n \to \infty} (E(X_n | \mathcal{F}_m) - X_m) = 0$ a.s. According to Blake [2], $(X_n, \mathcal{F}_n, n \geq 1)$ is fairer with time iff $\lim_{n \to \infty} P[|E(X_n | \mathcal{F}_m) - X_m| > \varepsilon] = 0$ for all $\varepsilon > 0$.

The final theorem indicates some relationships involving these three concepts and eventual martingales.

**Theorem 4.** (i) Every progressive martingale is an eventual martingale.

(ii) Every progressive martingale is a martingale in the limit.

(iii) Every uniformly integrable eventual martingale $(X_n = \sum_{k=1}^{n} d_k, \mathcal{F}_n, n \geq 1)$ with $\sup_{n \geq 1} E[|\sum_{k=1}^{n} E(d_k | \mathcal{F}_{k-1})|] < \infty$ is fairer with time.

**Proof.** If $(X_n, \mathcal{F}_n, n \geq 1)$ is a progressive martingale, then $P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1} \text{ i.o.}] = \lim_{n \to \infty} P[\bigcup_{k=1}^{n} [E(X_k | \mathcal{F}_{k-1}) \neq X_{k-1}] = \lim_{n \to \infty} P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1}] = 0$ so (i) is true.

Let $t = \inf \{n \geq 1 : E(X_n | \mathcal{F}_{n-1}) = X_{n-1}\}$. Since $(X_n, \mathcal{F}_n, n \geq 1)$ is a progressive martingale, $t$ is a stopping rule; i.e. $t \in \{1, 2, \ldots, \infty\}$, $P[t < \infty] = 1$ and $[t = n] \in \mathcal{F}_n$ for all $n \geq 1$. Now if $t \leq m$, where $m \geq 1$, then $E(X_k | \mathcal{F}_{k-1}) = X_{k-1}$ for $k \geq m$. But $[t \leq m] \in \mathcal{F}_m$, so, for $n > m$,

$$E(X_n | \mathcal{F}_m) - X_m) = \sum_{k=m+1}^{n} E(I(t \leq m) E(X_k - X_{k-1} | \mathcal{F}_{k-1}) | \mathcal{F}_m) = 0 .$$

For each $\omega \in [t < \infty]$, there exists $m_0 = m_0(\omega)$ such that $\omega \in [t \leq m_0]$ so $E(X_n | \mathcal{F}_m) - X_m = 0$ for all $n \geq m \geq m_0$, proving (ii).

(iii) is a consequence of a result on page 162 of [5], Theorem 3 above and Theorem 1 of Mucci [6].
Remark. None of the statements in Theorem 4 have valid converses. The converses of (i) and (ii) are both shown to be false by letting $d_1, d_2, \ldots$ be independent rv with $E(d_n) = 1$, $E(d_a) = 0$ if $n \neq 3$, defining $X_n = \sum_{k=1}^{n} d_k$ and taking $\mathcal{F}_n$ to be the sigma-field generated by $d_1, d_2, \ldots, d_n$. The converse to (iii) is contradicted by the following example, the first of two examples which show that no general relationship exists between sequence fairer with time and eventual martingales.

**Example 1.** A martingale in the limit need not be an eventual martingale, even if it is uniformly integrable. Let $d_1, d_2, \ldots$ be independent rv such that $P[d_n = 1] = n^{-2}$ whereas $P[d_n = 0] = 1 - n^{-2}$ for $n \geq 1$. Let $\mathcal{F}_n$ be the sigma-field generated by $d_1, \ldots, d_n$ and set $X_n = \sum_{k=1}^{n} d_k$. Since $E(\sum_{k=1}^{\infty} |d_k|) = \sum_{k=1}^{\infty} k^{-2} < \infty$ and $|X_n| \leq \sum_{k=1}^{\infty} |d_k|$ for all $n \geq 1$, $\{X_n\}$ is uniformly integrable. Moreover, $E(X_n | \mathcal{F}_m) - X_m = \sum_{k=m+1}^{\infty} k^{-2}$ for $n > m \geq 1$ so $(X_n, \mathcal{F}_n, n - 1)$ is a martingale in the limit. But $E(X_n | \mathcal{F}_{n-1}) = X_{n-1} + n^{-2} \neq X_{n-1}$ for all $n \geq 2$, so it is not an eventual martingale.

**Example 2.** An eventual martingale need not be fairer with time and, hence, need not be a martingale in the limit. Let $U_1, U_2, \ldots$ be independent rv such that $P[U_n = -1] = 2^{-n}$ and $P[U_n = 1] = 1 - 2^{-n}$ for $n \geq 1$. Let $\mathcal{F}_n$ be the sigma-field generated by $U_1, U_2, \ldots, U_n$, $d_1 = U_1$, $d_{n+1} = 2^a U_{n+1} I(U_n = -1)$ and $X_n = \sum_{k=1}^{n} d_k$ for $n \geq 1$. For $k > 1$,

$$E(d_k | \mathcal{F}_{k-1}) = 2^{k-1} I(U_{k-1} = -1) E(U_k | \mathcal{F}_{k-1}) = (2^{k-1} - 1) I(U_{k-1} = -1).$$

Hence $\sum_{k=2}^{\infty} P[E(X_n | \mathcal{F}_{n-1}) \neq X_{n-1}] = \sum_{k=2}^{\infty} P[U_{k-1} = -1] = \sum_{k=2}^{\infty} 2^{-k+1} < \infty$. Thus $(X_n, \mathcal{F}_n, n \geq 1)$ is an eventual martingale.

But, if $m \geq 2$, $E(X_{2m} - X_m | \mathcal{F}_m) = \sum_{k=m+1}^{2m} E(E(d_k | \mathcal{F}_{k-1}) | \mathcal{F}_m) = \sum_{k=m+2}^{2m} (2^{-k} - 1) P[U_{k-1} = -1] + (2^{-m} - 1) I(U_m = -1) \geq \sum_{k=m+2}^{2m} (1 - 2^{-k+1}) > (1 - 2^{-2m+1}) > 1/2$. Hence, if $\varepsilon < 1/2$,

$$P[|E(X_{2m} | \mathcal{F}_m) - X_m| > \varepsilon] = 1$$

for all $m > 1$, so $(X_n, \mathcal{F}_m, n \geq 1)$ is not fairer with time.

**References**


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