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ON EULER'S CRITERION FOR QUINTIC NONRESIDUES

KENNETH S. WILLIAMS

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Let p be a prime $\equiv 1 \pmod{5}$. If 2 is a quintic nonresidue \pmod{p} then $2^{p-1/5} \equiv \alpha \pmod{p}$ for some fifth root of unity $\alpha_5 (\neq 1) \pmod{p}$. Emma Lehmer has given an explicit expression for α_5 in terms of a particular solution of a certain quadratic partition of p . In this paper we show how in principle the corresponding result can be obtained for any quintic nonresidue $D \pmod{p}$. Full details are given for $D = 2, 3, 5$.

1. Introduction. Let k be an integer ≥ 2 and let p be a prime $\equiv 1 \pmod{k}$. Euler's criterion states that $D^{(p-1)/k} \equiv 1 \pmod{p}$ if and only if D is a k th power residue \pmod{p} . Thus if D is not a k th power residue \pmod{p} , for some k th root of unity $\alpha_k (\neq 1) \pmod{p}$ we have $D^{(p-1)/k} \equiv \alpha_k \pmod{p}$. Clearly $\alpha_2 = -1$. For $k > 2$ Emma Lehmer [3] has proposed the problem of specifying which α_k corresponds to a given D . For $D = 2, k = 3, 4, 5, 8$, she has given explicit expressions for α_k in terms of certain quadratic partitions of p . Elsewhere the author [6] has given a complete treatment of the case $k = 3$. In this paper we treat the case $k = 5$. Full details are given for $D = 2, 3, 5$. The method used is described in §4 and can be applied to any value of D if the reader has the patience to supply the many details.

2. Two lemmas involving the domain $Z[\zeta]$. We set $\zeta = \exp(2\pi i/5)$. If Q denotes the field of rational numbers, the cyclotomic field formed by adjoining ζ to Q is denoted by $Q(\zeta)$. The domain of integers of $Q(\zeta)$ is denoted by $Z[\zeta]$. Every element of $Z[\zeta]$ can be written in the form $a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4$, where a_1, a_2, a_3, a_4 are rational integers. The domain $Z[\zeta]$ is a unique factorization domain. The element $1 - \zeta$ is a prime in $Z[\zeta]$ which divides 5. The units of $Z[\zeta]$ are given by $\pm\zeta^i(\zeta + \zeta^4)^j$, where i and j are integers with $0 \leq i \leq 4$. If α and β are associated nonzero elements, that is α/β is a unit, we write $\alpha \sim \beta$. The complex conjugate of an element $\alpha \in Z[\zeta]$ will be denoted by $\bar{\alpha} (\in Z[\zeta])$. We will need the following two results.

LEMMA 1. If $\alpha \in Z[\zeta]$ is such that $\alpha \not\equiv 0 \pmod{1 - \zeta}$ then α possesses an associate α' such that $\alpha' \equiv -1 \pmod{(1 - \zeta)^2}$.

Proof. Set $\alpha = a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4$, $b = a_1 + a_2 + a_3 + a_4$, $c =$

$a_1 + 2a_2 + 3a_3 + 4a_4$. As $\alpha \not\equiv 0 \pmod{1 - \zeta}$ we have $b \not\equiv 0 \pmod{5}$. We define d uniquely by $2^d b \equiv -1 \pmod{5}$, $0 \leq d \leq 3$. Then we have only to choose $\alpha' = \zeta^{e2^d}(\zeta + \zeta^4)^d \alpha$, as $\zeta + \zeta^4 \equiv 2 \pmod{(1 - 3)^2}$ and $\zeta^{e2^d} \equiv b \pmod{(1 - \zeta)^2}$.

LEMMA 2. *If $\alpha, \beta \in Z[\zeta]$ are such that*

- (a) $\alpha\bar{\alpha} = \beta\bar{\beta}$
- (b) $\alpha, \beta \not\equiv 0 \pmod{1 - \zeta}$,
- (c) $\alpha \equiv \beta \pmod{(1 - \zeta)^2}$,
- (d) $\alpha \sim \beta$,

then

$$\alpha = \beta .$$

Proof. By (d) we have $\alpha = \pm \zeta^i (\zeta + \zeta^4)^j \beta$, for integers i and j with $0 \leq i \leq 4$. Thus using (a) we obtain $\alpha\bar{\alpha} = (\zeta + \zeta^4)^{2j} \beta\bar{\beta} = (\zeta + \zeta^4)^{2j} \alpha\bar{\alpha}$. Now (b) guarantees that $\alpha \neq 0$, so that $\alpha\bar{\alpha} \neq 0$, and we must have $(\zeta + \zeta^4)^{2j} = 1$. As $\zeta + \zeta^4 = \frac{1}{2}(\sqrt{5} - 1) > 0$ we have $j = 0$ and so $\alpha = \pm \zeta^i \beta$, $0 \leq i \leq 4$. From (b) and (c) we have $(\pm \zeta^i - 1)\beta \equiv 0 \pmod{(1 - \zeta)^2}$, $\beta \not\equiv 0 \pmod{1 - \zeta}$, so that

$$\pm \zeta^i - 1 \equiv 0 \pmod{(1 - \zeta)^2} .$$

As $i = 0, 1, 2, 3, 4$ this can only hold with the positive sign and $i = 0$, so that $\alpha = \beta$.

3. Dickson's diophantine system. Throughout the rest of this paper p denotes a prime $\equiv 1 \pmod{5}$. Our results involve the diophantine system

$$(3.1) \quad \begin{aligned} 16p &= x^2 + 50u^2 + 50v^2 + 125w^2, & x &\equiv 1 \pmod{5}, \\ xw &= v^2 - 4uv - u^2. \end{aligned}$$

A theorem of Dickson [1] asserts that (3.1) has exactly four solutions. If (x, u, v, w) is one of these, the other three are given by $(x, -u, -v, w)$, $(x, v, -u, -w)$, $(x, -v, u, -w)$. Taking the first equation in (3.1) modulo 8 and the second one modulo 4 we can show (after a little calculation) that $x + 2u - w \equiv x + 2v + w \equiv 0 \pmod{4}$ for any solution of (3.1). This enables us to make the following definition.

DEFINITION 1. For any solution (x, u, v, w) of (3.1) we define $\psi \equiv \psi(x, u, v, w) \in Z[\zeta]$ by

$$(3.2) \quad \psi = c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 + c_4 \zeta^4 ,$$

where $c_i \equiv c_i(x, u, v, w) \in Z(1 \leq i \leq 4)$ are given by

$$(3.3) \quad \begin{aligned} 4c_1 &= -x + 2u + 4v + 5w, \\ 4c_2 &= -x + 4u - 2v - 5w, \\ 4c_3 &= -x - 4u + 2v - 5w, \\ 4c_4 &= -x - 2u - 4v + 5w. \end{aligned}$$

The properties of ψ that we shall need are given in the next lemma.

LEMMA 3. (a) $\psi\bar{\psi} = p$.
 (b) $\psi \equiv -1 \pmod{(1 - \zeta)^2}$.
 (c) If $\sigma_i (1 \leq i \leq 4)$ is the automorphism of $Q(\zeta)$ defined by $\sigma_i(\zeta) = \zeta^i$ then G.C.D. (ψ_1, ψ_2) is a prime of $Z[\zeta]$, where $\psi_i = \sigma_i(\psi) (1 \leq i \leq 4)$.

Proof. (a) As $\zeta + \zeta^4 = 1/2(-1 + \sqrt{5})$, $\zeta^2 + \zeta^3 = 1/2(-1 - \sqrt{5})$, we have from (3.2)

$$\begin{aligned} \psi\bar{\psi} &= \left\{ (c_1^2 + c_2^2 + c_3^2 + c_4^2) - \frac{1}{2}(c_1c_2 + c_2c_3 + c_3c_4 + c_1c_3 \right. \\ &\quad \left. + c_1c_4 + c_2c_4) + \frac{\sqrt{5}}{2}(c_1c_2 + c_2c_3 + c_3c_4 - c_1c_3 \right. \\ &\quad \left. - c_1c_4 - c_2c_4) \right\} \\ &= \frac{1}{16}(x^2 + 50u^2 + 50v^2 + 125w^2) - \frac{5\sqrt{5}}{8} \\ &\quad \times (v^2 - 4uv - u^2 - xw) = p. \end{aligned}$$

(b) From (3.1) and (3.3) we have

$$c_1 + c_2 + c_3 + c_4 = -x \equiv -1, \quad c_1 + 2c_2 + 3c_3 + 4c_4 \equiv 0 \pmod{5},$$

so that $\psi \equiv -1 \pmod{(1 - \zeta)^2}$.

(c) Let π be a prime dividing p . As $p \equiv 1 \pmod{5}$ we have $p = \pi_1\pi_2\pi_3\pi_4$, where $\pi_i = \sigma_i(\pi)$, $1 \leq i \leq 4$. By (a) ψ is (up to multiplication by a unit) one of $\pi_1\pi_2, \pi_1\pi_3, \pi_2\pi_4, \pi_3\pi_4$. In each case G.C.D. (ψ_1, ψ_2) is a prime.

Lemma 1 and Lemma 3(c) enable us to define a prime \mathcal{H} of $Z[\zeta]$ as follows.

DEFINITION 2. For any solution (x, u, v, w) of (3.1) we let $\mathcal{H} \equiv \mathcal{H}(x, u, v, w) \in Z[\zeta]$ be such that

$$\mathcal{H} \sim \text{G.C.D.}(\psi_1, \psi_2), \quad \mathcal{H} \equiv -1 \pmod{(1 - \zeta)^2}.$$

We remark that \mathcal{H} is not unique, indeed all such \mathcal{H} are given by

$(-1)^r(\zeta + \zeta^4)^{2r} \mathcal{H}(r \in \mathbb{Z})$. However this does not matter for our purposes. Next we give the prime decomposition of ψ using Lemma 2.

LEMMA 4. $\psi = -\mathcal{H}_1\mathcal{H}_3$.

Proof. As $\mathcal{H} \sim \text{G.C.D.}(\psi_1, \psi_2)$ we have $\mathcal{H}_1 | \psi_1$, say, $\psi_1 = \mathcal{H}_1\lambda_1$. Hence $\psi_2 = \mathcal{H}_2\lambda_2$ and as $\mathcal{H}_1 | \psi_2$ we must have $\mathcal{H}_1 | \lambda_2$, that is $\mathcal{H}_3 | \lambda_1$, say $\lambda_1 = \mathcal{H}_3\mu$. Then $\psi_1 = \mathcal{H}_1\mathcal{H}_3\mu$ and so we have

$$\begin{aligned} \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4 &= p = \psi_1\bar{\psi}_1 = (\mathcal{H}_1\mathcal{H}_3\mu)(\mathcal{H}_4\mathcal{H}_2\bar{\mu}) \\ &= \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4\mu\bar{\mu}. \end{aligned}$$

Hence we have $\mu\bar{\mu} = 1$, so that μ is a unit of $Z[\zeta]$, proving that $\psi \sim \mathcal{H}_1\mathcal{H}_3$. Clearly ψ and $-\mathcal{H}_1\mathcal{H}_3$ satisfy the conditions of Lemma 2 so that $\psi = -\mathcal{H}_1\mathcal{H}_3$.

Finally in this section we set for any solution (x, u, v, w) of (3.1):

$$(3.4) \quad \begin{aligned} &\alpha(x, u, v, w) \\ &= \frac{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x + 20u - 10v)}{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x - 20u + 10v)} \end{aligned}$$

and prove

LEMMA 5. $\alpha(x, u, v, w) \equiv \zeta \pmod{\mathcal{H}}$.

Proof. From (3.2) and $\psi_1 \equiv \psi_2 \equiv 0 \pmod{\mathcal{H}}$ we obtain modulo \mathcal{H} :

$$\begin{aligned} 5c_1 &\equiv (\zeta^2 - 1)\psi_3 + (\zeta - 1)\psi_4, \\ 5c_2 &\equiv (\zeta^4 - 1)\psi_3 + (\zeta^2 - 1)\psi_4, \\ 5c_3 &\equiv (\zeta - 1)\psi_3 + (\zeta^3 - 1)\psi_4, \\ 5c_4 &\equiv (\zeta^3 - 1)\psi_3 + (\zeta^4 - 1)\psi_4. \end{aligned}$$

Appealing to (3.3) we get

$$\begin{aligned} x &\equiv \psi_3 + \psi_4, & 25u &\equiv \alpha\psi_3 + \beta\psi_4, \\ 25v &\equiv \beta\psi_3 - \alpha\psi_4, & 25w &\equiv -\gamma\psi_3 + \gamma\psi_4, \end{aligned}$$

where

$$\begin{aligned} \alpha &= -2\zeta + \zeta^2 - \zeta^3 + 2\zeta^4, \\ \beta &= \zeta + 2\zeta^2 - 2\zeta^3 - \zeta^4, \\ \gamma &= \zeta - \zeta^2 - \zeta^3 + \zeta^4. \end{aligned}$$

It is easy to check that

$$\alpha\beta = \alpha^2 - \beta^2 = 5\gamma, \quad \gamma^2 = 5.$$

After some calculation we find that

$$25\{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x + 20u - 10v)\} \\ \equiv 4\gamma\psi_3\psi_4((2 + 2\zeta)\psi_3 + 2\zeta^3\psi_4)$$

and

$$25\{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x - 20u + 10v)\} \\ \equiv 4\gamma\psi_3\psi_4((2 + 2\zeta^4)\psi_3 + 2\zeta^2\psi_4)$$

from which the result follows immediately.

4. **Outline of method.** We start with the necessary and sufficient condition for D (without loss of generality we may take D to be a (positive) prime) to be a quintic residue (mod p) in terms of congruences (mod D) involving a solution of (3.1). These have been given for $D = 2, 3, 5, 7$ in [4] and for $D = 11, 13, 17, 19$ in [9]. Results for other values of D could be obtained using the period equation as in [9]. If D is a quintic nonresidue (mod p) this condition is used to specify a unique solution of (3.1) by means of congruences (mod D). This unique solution is specified in such a way that after using Lemma 4 we find that the corresponding \mathcal{K} satisfies $(\mathcal{K}/D)_5 = \zeta$. If $D \neq 5$ we can then appeal to Eisenstein's reciprocity law

“If $\alpha \equiv -1 \pmod{(1 - \zeta)^2}$ and a is a rational integer prime to 5 then $(\alpha/a)_5 = (a/\alpha)_5$ ”

to obtain $(D/\mathcal{K})_5 = \zeta$, so that $D^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{\mathcal{K}}$ by Lemma 5. As both $D^{(p-1)/5}$ and $\alpha(x, u, v, w)$ are rational we have $D^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{p}$ as required. If $D = 5$ we must replace the use of Eisenstein's reciprocity law by Kummer's supplement to the law of quintic reciprocity involving the prime $1 - \zeta$ [7]. Unfortunately, this requires working modulo 25 rather than modulo 5 and so involves a large number of cases. We thus give an alternative approach based on a result of Muskat [5].

5. $D = 2$. Lehmer [2] has shown that 2 is a quintic residue (mod p) if and only if $x \equiv 0 \pmod{2}$, where (x, u, v, w) is any solution of (3.1). Thus if 2 is a quintic nonresidue (mod p) we can find by Dickson's theorem a unique solution (x, u, v, w) of (3.1) such that

$$(5.1) \quad x \equiv 1 \pmod{2}, \quad u \equiv 0 \pmod{2}, \quad x + u - v \equiv 0 \pmod{4}.$$

In terms of this solution a simple calculation using (3.3) shows that $\psi \equiv \zeta^3 \pmod{2}$. Then by an examination of cases in conjunction with $\psi = -\mathcal{H}_1\mathcal{H}_3$ (Lemma 4) we find that

$$\mathcal{H} \equiv \zeta^2, \quad \zeta + \zeta^3 \quad \text{or} \quad \zeta + \zeta^2 + \zeta^3 \pmod{2},$$

so that $(\mathcal{H}/2)_5 = \zeta$. Appealing to Eisenstein's reciprocity theorem as indicated in § 4 we have reproved

THEOREM 1 (Lehmer [3]). *Let p be a prime $\equiv 1 \pmod{5}$ for which 2 is a quintic nonresidue \pmod{p} . Let (x, u, v, w) be the unique solution of (3.1) satisfying (5.1). Then we have*

$$2^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{p} .$$

6. $D = 3$. (Lehmer [2] has shown that 3 is a quintic residue \pmod{p} if and only if $u \equiv v \equiv 0 \pmod{3}$, where (x, u, v, w) is any solution of (3.1).) Thus if 3 is a quintic nonresidue \pmod{p} we can find by Dickson's theorem a unique solution (x, u, v, w) of (3.1) satisfying one of

$$(6.1) \quad \begin{aligned} (a) \quad & x \equiv 1, \quad u \equiv 1, \quad v \equiv 0, \quad w \equiv 2 \pmod{3}, \\ (b) \quad & x \equiv 2, \quad u \equiv 2, \quad v \equiv 0, \quad w \equiv 1 \pmod{3}, \\ (c) \quad & x \equiv 1, \quad u \equiv 2, \quad v \equiv 1, \quad w \equiv 1 \pmod{3}, \\ (d) \quad & x \equiv 2, \quad u \equiv 1, \quad v \equiv 2, \quad w \equiv 2 \pmod{3}. \end{aligned}$$

In terms of this solution a simple calculation using (3.3) shows that

$$\begin{aligned} \psi &\equiv -\zeta - \zeta^2 + \zeta^4 \pmod{3}, & \text{if (a) holds,} \\ \psi &\equiv \zeta + \zeta^2 - \zeta^4 \pmod{3}, & \text{if (b) holds,} \\ \psi &\equiv -\zeta^4 \pmod{3}, & \text{if (c) holds,} \\ \psi &\equiv \zeta^4 \pmod{3}, & \text{if (d) holds.} \end{aligned}$$

Then by an examination of cases $\pmod{3}$ in conjunction with Lemma 4 we find that

$$\begin{aligned} \mathcal{H} &\equiv \pm(\zeta - \zeta^2 - \zeta^4), \quad \pm(\zeta - \zeta^2 + \zeta^3 + \zeta^4) \pmod{3}, & \text{if (a) holds,} \\ \mathcal{H} &\equiv \pm(\zeta^3 - \zeta^4), \quad \pm(\zeta - \zeta^2 - \zeta^3) \pmod{3}, & \text{if (b) holds,} \\ \mathcal{H} &\equiv \pm\zeta, \quad \pm(\zeta - \zeta^3 - \zeta^4) \pmod{3}, & \text{if (c) holds,} \\ \mathcal{H} &\equiv \pm(\zeta^3 + \zeta^4), \quad \pm(\zeta + \zeta^3 + \zeta^4) \pmod{3}, & \text{if (d) holds,} \end{aligned}$$

so that in every case $(\mathcal{H}/3)_5 = \zeta$. Appealing to Eisenstein's reciprocity theorem as before we have the following result.

THEOREM 2. *Let p be a prime $\equiv 1 \pmod{5}$ for which 3 is a quintic nonresidue \pmod{p} . Let (x, u, v, w) be the unique solution of (3.1) satisfying (6.1). Then we have*

$$3^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{p} .$$

7. $D = 5$. For p a prime $\equiv 1 \pmod{5}$, g a primitive root \pmod{p} ,

h, k integers selected from 0, 1, 2, 3, 4, the cyclotomic number $(h, k)_5$ is defined to be the number of solutions (s, t) with $0 \leq s, t < (p - 1)/5$ of $g^{5s+h} + 1 \equiv g^{5t+k} \pmod{p}$. Let (x, u, v, w) be any solution of (3.1). Choose g such that $(g/\mathcal{K})_5 = \zeta$. Then it can be shown that

$$\begin{aligned} 25(0, 0)_5 &= p - 14 + 3x, \\ 100(0, 1)_5 &= 100(1, 0)_5 = 100(4, 4)_5 = 4p - 16 - 3x + 50v + 25w, \\ 100(0, 2)_5 &= 100(2, 0)_5 = 100(3, 3)_5 = 4p - 16 - 3x + 50u - 25w, \\ 100(0, 3)_5 &= 100(3, 0)_5 = 100(2, 2)_5 = 4p - 16 - 3x - 50u - 25w, \\ 100(0, 4)_5 &= 100(4, 0)_5 = 100(1, 1)_5 = 4p - 16 - 3x - 50v + 25w, \\ 100(1, 2)_5 &= 100(1, 4)_5 = 100(2, 1)_5 = 100(3, 4)_5 = 100(4, 1)_5 \\ &= 100(4, 3)_5 = 4p + 4 + 2x - 50w, \\ 100(1, 3)_5 &= 100(2, 3)_5 = 100(2, 4)_5 = 100(3, 1)_5 = 100(3, 2)_5 \\ &= 100(4, 2)_5 = 4p + 4 + 2x + 50w, \end{aligned}$$

and Muskat [5] has shown that

$$\text{ind}_g(5) \equiv (0, 4)_5 - (0, 1)_5 + 2((0, 3)_5 - (0, 2)_5) \pmod{5}$$

so that

$$\text{ind}_g(5) \equiv -2u - v \pmod{5}.$$

Thus if 5 is a quintic nonresidue \pmod{p} $2u + v \not\equiv 0 \pmod{5}$ and by Dickson's theorem there is a unique solution of (3.1) satisfying $2u + v \equiv 4 \pmod{5}$. With this solution we have $\text{ind}_g(5) \equiv 1 \pmod{5}$ and so

$$5^{(p-1)/5} \equiv g^{\text{ind}_g(5) \cdot (p-1)/5} \equiv g^{(p-1)/5} \equiv \left(\frac{g}{\mathcal{K}}\right)_5 \equiv \zeta \pmod{\mathcal{K}}.$$

Thus we have proved

THEOREM 3. *Let p be a prime $\equiv 1 \pmod{5}$ for which 5 is a quintic nonresidue \pmod{p} . Let (x, u, v, w) be the unique solution of (3.1) satisfying $2u + v \equiv 4 \pmod{5}$. Then we have*

$$5^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{p}.$$

8. EXAMPLE. We take $p = 311$. A solution of (3.1) in this case is $(-49, 7, 0, 1)$ (see for example [8]) so none of 2, 3, 5 is a quintic residue $\pmod{311}$. The unique solution given by Theorem 1 is $(-49, 0, 7, -1)$ so that

$$2^{(p-1)/5} = 2^{62} \equiv \frac{2276 - 98.46}{2276 + 98.94} \equiv \frac{-2232}{11488} \equiv \frac{-55}{-19} \equiv 52 \pmod{311}.$$

The unique solution given by Theorem 2 is $(-49, -7, 0, 1)$ so that

$$3^{(p-1)/5} = 3^{62} \equiv \frac{-2276 + 98.66}{-2276 - 98.214} \equiv \frac{4192}{-23248} \equiv \frac{149}{77} \equiv 216 \pmod{311}.$$

The unique solution given by Theorem 3 is $(-49, 7, 0, 1)$ so that

$$5^{(p-1)/5} = 5^{62} \equiv \frac{-2276 - 98.214}{-2276 + 98.66} \equiv \frac{-23248}{4192} \equiv \frac{77}{149} \equiv 36 \pmod{311}.$$

REFERENCES

1. L. E. Dickson, *Cyclotomy, higher congruences, and Waring's problem*, Amer. J. Math., **57** (1935), 391-424.
2. Emma Lehmer, *The quintic character of 2 and 3*, Duke Math. J., **18** (1951), 11-18.
3. ———, *On Euler's criterion*, J. Austral. Math. Soc., **1** (1959) 64-70.
4. ———, *On the divisors of the discriminant of the period equation*, Amer. J. Math., **90** (1968), 375-379.
5. J. B. Muskat, *On the solvability of $x^e \equiv e \pmod{p}$* , Pacific J. Math., **14** (1964), 257-260.
6. K. S. Williams, *On Euler's criterion for cubic nonresidues*, Proc. Amer. Math. Soc., **49** (1975), 277-283.
7. ———, *Explicit forms of Kummer's complementary theorems to his law of quintic reciprocity*, J. für Math., (to appear).
8. ———, *Table of solutions (x, u, v, w) of the diophantine system $16p = x^2 + 50u^2 + 50v^2 + 125w^2$, $xw = v^2 - 4uv - u^2$, $x \equiv 1 \pmod{5}$, for primes $p < 10,000$, $p \equiv 1 \pmod{5}$* , Unpublished Mathematical Tables File of American Mathematical Society (with B. Lowe).
9. ———, *Explicit criteria for quintic residuality* (submitted for publication).

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