ON EULER'S CRITERION FOR QUINTIC NONRESIDUES

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Let $p$ be a prime $\equiv 1 \pmod{5}$. If $2$ is a quintic nonresidue $(\mod p)$ then $2^{p-1/5} \equiv \alpha \pmod{p}$ for some fifth root of unity $\alpha \neq 1 \pmod{p}$. Emma Lehmer has given an explicit expression for $\alpha$ in terms of a particular solution of a certain quadratic partition of $p$. In this paper we show how in principle the corresponding result can be obtained for any quintic nonresidue $D \pmod{p}$. Full details are given for $D = 2, 3, 5$.

1. Introduction. Let $k$ be an integer $\geq 2$ and let $p$ be a prime $\equiv 1 \pmod{k}$. Euler’s criterion states that $D^{(p-1)/k} \equiv 1 \pmod{p}$ if and only if $D$ is a $k$th power residue $(\mod p)$. Thus if $D$ is not a $k$th power residue $(\mod p)$, for some $k$th root of unity $\alpha_k \pmod{p}$ we have $D^{(p-1)/k} \equiv \alpha_k \pmod{p}$. Clearly $\alpha_2 = -1$. For $k > 2$ Emma Lehmer [3] has proposed the problem of specifying which $\alpha_k$ corresponds to a given $D$. For $D = 2, k = 3, 4, 5, 8$, she has given explicit expressions for $\alpha_k$ in terms of certain quadratic partitions of $p$. Elsewhere the author [6] has given a complete treatment of the case $k = 3$. In this paper we treat the case $k = 5$. Full details are given for $D = 2, 3, 5$. The method used is described in §4 and can be applied to any value of $D$ if the reader has the patience to supply the many details.

2. Two lemmas involving the domain $\mathbb{Z}[\zeta]$. We set $\zeta = \exp(2\pi i/5)$. If $Q$ denotes the field of rational numbers, the cyclotomic field formed by adjoining $\zeta$ to $Q$ is denoted by $Q(\zeta)$. The domain of integers of $Q(\zeta)$ is denoted by $\mathbb{Z}[\zeta]$. Every element of $\mathbb{Z}[\zeta]$ can be written in the form $a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4$, where $a_1, a_2, a_3, a_4$ are rational integers. The domain $\mathbb{Z}[\zeta]$ is a unique factorization domain. The element $1 - \zeta$ is a prime in $\mathbb{Z}[\zeta]$ which divides 5. The units of $\mathbb{Z}[\zeta]$ are given by $\pm \zeta^i(\zeta + \zeta^j)^4$, where $i$ and $j$ are integers with $0 \leq i \leq 4$. If $\alpha$ and $\beta$ are associated nonzero elements, that is $\alpha/\beta$ is a unit, we write $\alpha \sim \beta$. The complex conjugate of an element $\alpha \in \mathbb{Z}[\zeta]$ will be denoted by $\bar{\alpha} \in \mathbb{Z}[\zeta])$. We will need the following two results.

**Lemma 1.** If $\alpha \in \mathbb{Z}[\zeta]$ is such that $\alpha \neq 0 \pmod{1 - \zeta}$ then $\alpha$ possesses an associate $\alpha'$ such that $\alpha' \equiv -1 \pmod{1 - \zeta^i}$.

**Proof.** Set $\alpha = a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4$, $b = a_1 + a_2 + a_3 + a_4$, $c =$
\[ a_1 + 2a_2 + 3a_3 + 4a_4. \] As \( \alpha \not\equiv 0 \pmod{1 - \zeta} \) we have \( b \not\equiv 0 \pmod{5} \).

We define \( \eta \) uniquely by \( 2^\eta \equiv -1 \pmod{5} \), \( 0 \leq \eta \leq 3 \). Then we have only to choose \( \alpha' = \zeta^{\eta} (\zeta + \zeta')^j \alpha \), as \( \zeta + \zeta' \equiv 2 \pmod{1 - \zeta} \) and \( \zeta^{\eta} \equiv b \pmod{1 - \zeta} \).

**Lemma 2.** If \( \alpha, \beta \in Z[\zeta] \) are such that
(a) \( \alpha \beta = \beta \beta \)
(b) \( \alpha, \beta \not\equiv 0 \pmod{1 - \zeta} \),
(c) \( \alpha \equiv \beta \pmod{1 - \zeta} \),
(d) \( \alpha \sim \beta \),

then

\[ \alpha = \beta. \]

**Proof.** By (d) we have \( \alpha = \pm \zeta^i (\zeta + \zeta')^j \beta \), for integers \( i \) and \( j \) with \( 0 \leq i \leq 4 \). Thus using (a) we obtain \( \alpha \beta = (\zeta + \zeta')^j \beta \beta = (\zeta + \zeta')^i \alpha \beta \). Now (b) guarantees that \( \alpha \not\equiv 0 \), so that \( \alpha \beta \not\equiv 0 \), and we must have \( (\zeta + \zeta')^i = 1 \). As \( \zeta + \zeta' = \frac{1}{2}(\sqrt{5} - 1) > 0 \) we have \( j = 0 \) and so \( \alpha = \pm \zeta^i \beta \), \( 0 \leq i \leq 4 \). From (b) and (c) we have \( (\pm \zeta^i - 1)\beta \equiv 0 \pmod{1 - \zeta} \), \( \beta \not\equiv 0 \pmod{1 - \zeta} \), so that

\[ \pm \zeta^i - 1 \equiv 0 \pmod{1 - \zeta}. \]

As \( i = 0, 1, 2, 3, 4 \) this can only hold with the positive sign and \( i = 0 \), so that \( \alpha = \beta \).

3. Dickson's diophantine system. Throughout the rest of this paper \( p \) denotes a prime \( \equiv 1 \pmod{5} \). Our results involve the diophantine system

\[
16p = x^5 + 50u^5 + 50v^5 + 125w^5, \quad x \equiv 1 \pmod{5},
\]

\[
xw = v^5 - 4uv - u^5.
\]

A theorem of Dickson [1] asserts that (3.1) has exactly four solutions. If \( (x, u, v, w) \) is one of these, the other three are given by \( (x, -u, -v, w), (x, v, -u, -w), (x, -v, u, -w) \). Taking the first equation in (3.1) modulo 8 and the second one modulo 4 we can show (after a little calculation) that \( x + 2u - w \equiv x + 2v + w \equiv 0 \pmod{4} \) for any solution of (3.1). This enables us to make the following definition.

**Definition 1.** For any solution \( (x, u, v, w) \) of (3.1) we define \( \psi = \psi(x, u, v, w) \in Z[\zeta] \) by

\[
\psi = c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 + c_4 \zeta^4,
\]

where \( c_i \equiv c_i(x, u, v, w) \in Z(1 \leq i \leq 4) \) are given by
\[ 4c_1 = -x + 2u + 4v + 5w, \]
\[ 4c_2 = -x + 4u - 2v - 5w, \]
\[ 4c_3 = -x - 4u + 2v - 5w, \]
\[ 4c_4 = -x - 2u - 4v + 5w. \]

(3.3)

The properties of \( \psi \) that we shall need are given in the next lemma.

**Lemma 3.** (a) \( \psi \bar{\psi} = p \).

(b) \( \psi \equiv -1 \pmod{(1 - \zeta)^2} \).

(c) If \( \sigma_i(1 \leq i \leq 4) \) is the automorphism of \( \mathbb{Q}(\zeta) \) defined by \( \sigma_i(\zeta) = \zeta^i \) then G.C.D. \( (\psi_1, \psi_2) \) is a prime of \( \mathbb{Z}[\zeta] \), where \( \psi_i = \sigma_i(\psi)(1 \leq i \leq 4) \).

**Proof.** (a) As \( \zeta + \zeta^4 = 1/2(-1 + \sqrt{5}) \), \( \zeta^2 + \zeta^3 = 1/2(-1 - \sqrt{5}) \), we have from (3.2)

\[
\psi \bar{\psi} = \left\{ (c_1^2 + c_2^2 + c_3^2 + c_4^2) - \frac{1}{2}(c_1c_2 + c_2c_3 + c_3c_4 + c_4c_1) + c_1c_4 + c_2c_3 + \frac{\sqrt{5}}{2}(c_2c_3 + c_3c_4 + c_4c_1 - c_1c_3) - c_1c_4 - c_2c_3 \right\}
\]

\[
= \frac{1}{16}(x^2 + 50u^2 + 50v^2 + 125w^2) - \frac{5\sqrt{5}}{8}
\]
\[
\times (v^3 - 4uv - u^3 - xw) = p .
\]

(b) From (3.1) and (3.3) we have

\[ c_1 + c_2 + c_3 + c_4 = -x \equiv -1, c_1 + 2c_2 + 3c_3 + 4c_4 \equiv 0 \pmod{5} , \]

so that \( \psi \equiv -1 \pmod{(1 - \zeta)^2} \).

(c) Let \( \pi \) be a prime dividing \( p \). As \( p \equiv 1 \pmod{5} \) we have \( p = \pi_1\pi_2\pi_3\pi_4 \), where \( \pi_i = \sigma_i(\pi), 1 \leq i \leq 4 \). By (a) \( \psi \) is (up to multiplication by a unit) one of \( \pi_1\pi_2, \pi_1\pi_3, \pi_1\pi_4, \pi_2\pi_4 \). In each case G.C.D. \( (\psi_1, \psi_2) \) is a prime.

Lemma 1 and Lemma 3(c) enable us to define a prime \( \mathcal{H} \) of \( \mathbb{Z}[\zeta] \) as follows.

**Definition 2.** For any solution \( (x, u, v, w) \) of (3.1) we let \( \mathcal{H} = \mathcal{H}(x, u, v, w) \in \mathbb{Z}[\zeta] \) be such that

\[ \mathcal{H} \sim \text{G.C.D.} (\psi_1, \psi_2), \mathcal{H} \equiv -1 \pmod{(1 - \zeta)^2} . \]

We remark that \( \mathcal{H} \) is not unique, indeed all such \( \mathcal{H} \) are given by
\((-1)^r(\zeta + \zeta^4)^r K(r \in \mathbb{Z})\). However this does not matter for our purposes. Next we give the prime decomposition of \(\psi\) using Lemma 2.

**Lemma 4.** \(\psi = -K_1K_3\).

**Proof.** As \(K \sim \text{G.C.D.} (\psi_1, \psi_2)\) we have \(K_1|\psi_1\), say, \(\psi_1 = K_1\lambda_1\). Hence \(\psi_2 = K_2\lambda_2\) and as \(K_1|\psi_2\) we must have \(K_1|\lambda_2\), that is \(K_1|\lambda_n\), say \(\lambda_1 = K_3\mu\). Then \(\psi_1 = K_1K_3\mu\) and so we have

\[
K_1K_2K_3K_4 = p = \psi_1\psi_1 = (K_1\lambda_1\mu)(K_1\lambda_1\mu) = K_1K_2K_3K_4\mu^2.
\]

Hence we have \(\mu\mu = 1\), so that \(\mu\) is a unit of \(\mathbb{Z}[\zeta]\), proving that \(\psi \sim K_1K_3\). Clearly \(\psi\) and \(-K_1K_3\) satisfy the conditions of Lemma 2 so that \(\psi = -K_1K_3\).

Finally in this section we set for any solution \((x, u, v, w)\) of (3.1):

\[
\alpha(x, u, v, w) = w(125w^2 - x^2) + 2(xw + 5uv)(25w - x + 20u - 10v)
\]

\[
= \frac{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x - 20u + 10v)}{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x - 20u + 10v)}
\]

and prove

**Lemma 5.** \(\alpha(x, u, v, w) \equiv \zeta (\text{mod }K).

**Proof.** From (3.2) and \(\psi_1 = \psi_2 = 0 (\text{mod } K)\) we obtain modulo \(K\):

\[
5c_1 \equiv (\zeta^2 - 1)\psi_3 + (\zeta - 1)\psi_4, \\
5c_2 \equiv (\zeta^4 - 1)\psi_3 + (\zeta^3 - 1)\psi_4, \\
5c_3 \equiv (\zeta - 1)\psi_3 + (\zeta^3 - 1)\psi_4, \\
5c_4 \equiv (\zeta^3 - 1)\psi_3 + (\zeta^4 - 1)\psi_4.
\]

Appealing to (3.3) we get

\[
x \equiv \psi_3 + \psi_4, \quad 25u \equiv \alpha\psi_3 + \beta\psi_4, \\
25v \equiv \beta\psi_3 - \alpha\psi_4, \quad 25w \equiv -\gamma\psi_3 + \gamma\psi_4,
\]

where

\[
\alpha = -2\zeta + \zeta^2 - \zeta^3 + 2\zeta^4, \\
\beta = \zeta + 2\zeta^2 - 2\zeta^3 - \zeta^4, \\
\gamma = \zeta - \zeta^2 - \zeta^3 + \zeta^4.
\]

It is easy to check that

\[
\alpha\beta = \alpha^2 - \beta^2 = 5\gamma, \quad \gamma^2 = 5.
\]
After some calculation we find that
\[
25\{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x + 20u - 10v)\}
= 4\zeta_3\psi_4((2 + 2\zeta_2)\psi_3 + 2\zeta_3^2\psi_4)
\]
and
\[
25\{w(125w^2 - x^2) + 2(xw + 5uv)(25w - x - 20u + 10v)\}
= 4\zeta_3\psi_4((2 + 2\zeta_2)\psi_3 + 2\zeta_3^2\psi_4)
\]
from which the result follows immediately.

4. Outline of method. We start with the necessary and sufficient condition for \( D \) (without loss of generality we may take \( D \) to be a (positive) prime) to be a quintic residue (mod \( p \)) in terms of congruences (mod \( D \)) involving a solution of (3.1). These have been given for \( D = 2, 3, 5, 7 \) in [4] and for \( D = 11, 13, 17, 19 \) in [9]. Results for other values of \( D \) could be obtained using the period equation as in [9]. If \( D \) is a quintic nonresidue (mod \( p \)) this condition is used to specify a unique solution of (3.1) by means of congruences (mod \( D \)). This unique solution is specified in such a way that after using Lemma 4 we find that the corresponding \( \mathcal{N} \) satisfies \((\mathcal{N}|D)_3 = \zeta \). If \( D \neq 5 \) we can then appeal to Eisenstein's reciprocity law

If \( \alpha \equiv -1 \text{(mod } 1 - \zeta)^5 \) and \( \alpha \) is a rational integer prime to 5
to obtain \((D/\mathcal{N})_3 = \zeta\), so that \( D^{(p-1)/5} \equiv \alpha(x, u, v, w) \text{(mod } \mathcal{N}) \) by Lemma 5. As both \( D^{(p-1)/5} \) and \( \alpha(x, u, v, w) \) are rational we have \( D^{(p-1)/5} \equiv \alpha(x, u, v, w) \text{(mod } p) \) as required. If \( D = 5 \) we must replace the use of Eisenstein's reciprocity law by Kummer's supplement to the law of quintic reciprocity involving the prime \( 1 - \zeta \) [7]. Unfortunately, this requires working modulo 25 rather than modulo 5 and so involves a large number of cases. We thus give an alternative approach based on a result of Muskat [5].

5. \( D = 2 \). Lehmer [2] has shown that 2 is a quintic residue (mod \( p \)) if and only if \( x \equiv 0 \text{(mod } 2) \), where \( (x, u, v, w) \) is any solution of (3.1). Thus if 2 is a quintic nonresidue (mod \( p \)) we can find by Dickson's theorem a unique solution \( (x, u, v, w) \) of (3.1) such that

\[
(5.1) \quad x \equiv 1 \text{(mod } 2 \), \quad u \equiv 0 \text{(mod } 2 \), \quad x + u - v \equiv 0 \text{(mod } 4 \).
\]

In terms of this solution a simple calculation using (3.3) shows that \( \psi \equiv \zeta^3 \text{(mod } 2) \). Then by an examination of cases in conjunction with \( \psi = -\mathcal{N}_1\mathcal{N}_2 \) (Lemma 4) we find that

\[
\mathcal{N} \equiv \zeta^3, \quad \zeta + \zeta^3 \text{ or } \zeta + \zeta^3 + \zeta^3 \text{(mod } 2),
\]
so that \((N/2)_5 = \zeta\). Appealing to Eisenstein’s reciprocity theorem as indicated in §4 we have reproved

**Theorem 1** (Lehmer [3]). Let \(p\) be a prime \(\equiv 1 \pmod{5}\) for which 2 is a quintic nonresidue \((\pmod{p})\). Let \((x, u, v, w)\) be the unique solution of (3.1) satisfying (5.1). Then we have

\[
2^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{p}.
\]

6. \(D = 3\). (Lehmer [2] has shown that 3 is a quintic residue \((\pmod{p})\) if and only if \(u \equiv v \equiv 0 \pmod{3}\), where \((x, u, v, w)\) is any solution of (3.1).) Thus if 3 is a quintic nonresidue \((\pmod{p})\) we can find by Dickson’s theorem a unique solution \((x, u, v, w)\) of (3.1) satisfying one of

\[
\begin{align*}
(a) & \quad x \equiv 1, \quad u \equiv 1, \quad v \equiv 0, \quad w \equiv 2 \pmod{3}, \\
(b) & \quad x \equiv 2, \quad u \equiv 2, \quad v \equiv 0, \quad w \equiv 1 \pmod{3}, \\
(c) & \quad x \equiv 1, \quad u \equiv 2, \quad v \equiv 1, \quad w \equiv 1 \pmod{3}, \\
(d) & \quad x \equiv 2, \quad u \equiv 1, \quad v \equiv 2, \quad w \equiv 2 \pmod{3}.
\end{align*}
\]

(6.1)

In terms of this solution a simple calculation using (3.3) shows that

\[
\begin{align*}
\psi & \equiv -\zeta - \zeta^2 + \zeta^4 \pmod{3}, \quad \text{if (a) holds}, \\
\psi & \equiv \zeta + \zeta^2 - \zeta^4 \quad \pmod{3}, \quad \text{if (b) holds}, \\
\psi & \equiv -\zeta^4 \quad \pmod{3}, \quad \text{if (c) holds}, \\
\psi & \equiv \zeta^4 \quad \pmod{3}, \quad \text{if (d) holds}.
\end{align*}
\]

Then by an examination of cases \((\pmod{3})\) in conjunction with Lemma 4 we find that

\[
\begin{align*}
\mathcal{K} & \equiv \pm(\zeta - \zeta^2 - \zeta^4), \quad \pm(\zeta - \zeta^2 + \zeta^3 + \zeta^4) \pmod{3}, \quad \text{if (a) holds}, \\
\mathcal{K} & \equiv \pm(\zeta^2 - \zeta^4), \quad \pm(\zeta - \zeta^2 - \zeta^3) \pmod{3}, \quad \text{if (b) holds}, \\
\mathcal{K} & \equiv \pm \zeta, \quad \pm(\zeta - \zeta^2 - \zeta^3) \pmod{3}, \quad \text{if (c) holds}, \\
\mathcal{K} & \equiv \pm(\zeta^3 + \zeta^4), \quad \pm(\zeta + \zeta^3 + \zeta^4) \pmod{3}, \quad \text{if (d) holds},
\end{align*}
\]

so that in every case \((\mathcal{K}/3)_5 = \zeta\). Appealing to Eisenstein’s reciprocity theorem as before we have the following result.

**Theorem 2.** Let \(p\) be a prime \(\equiv 1 \pmod{5}\) for which 3 is a quintic nonresidue \((\pmod{p})\). Let \((x, u, v, w)\) be the unique solution of (3.1) satisfying (6.1). Then we have

\[
3^{(p-1)/5} \equiv \alpha(x, u, v, w) \pmod{p}.
\]

7. \(D = 5\). For \(p\) a prime \(\equiv 1 \pmod{5}\), \(g\) a primitive root \((\pmod{p})\),
h, k integers selected from 0, 1, 2, 3, 4, the cyclotomic number \((h, k)\) is defined to be the number of solutions \((s, t)\) with \(0 \leq s, t < (p - 1)/5\) of \(g^{s+t} + 1 \equiv g^{s+t} \pmod{p}\). Let \((x, u, v, w)\) be any solution of (3.1). Choose \(g\) such that \((g/\hat{\mathcal{X}})_5 = \zeta\). Then it can be shown that

\[
25(0, 0)_5 = p - 14 + 3x , \\
100(0, 1)_5 = 100(1, 0)_5 = 100(4, 4)_5 = 4p - 16 - 3x + 50v + 25w , \\
100(0, 2)_5 = 100(2, 0)_5 = 100(3, 3)_5 = 4p - 16 - 3x + 50u - 25w , \\
100(0, 3)_5 = 100(3, 0)_5 = 100(2, 2)_5 = 4p - 16 - 3x - 50u - 25w , \\
100(0, 4)_5 = 100(4, 0)_5 = 100(1, 1)_5 = 4p - 16 - 3x - 50v + 25w , \\
100(1, 2)_5 = 100(1, 4)_5 = 100(2, 1)_5 = 100(3, 4)_5 = 100(4, 1)_5 = 100(4, 3)_5 = 4p + 4 + 2x - 50v , \\
100(1, 3)_5 = 100(2, 3)_5 = 100(2, 4)_5 = 100(3, 1)_5 = 100(3, 2)_5 = 100(4, 2)_5 = 4p + 4 + 2x + 50w ,
\]

and Muskat [5] has shown that

\[
\text{ind}_r (5) \equiv (0, 4)_5 - (0, 1)_5 + 2((0, 3)_5 - (0, 2)_5)) \pmod{5}
\]

so that

\[
\text{ind}_r (5) \equiv -2u - v \pmod{5} .
\]

Thus if 5 is a quintic nonresidue \((mod p)\) \(2u + v \equiv 0 \pmod{5}\) and by Dickson's theorem there is a unique solution of (3.1) satisfying \(2u + v \equiv 4 \pmod{5}\). With this solution we have \(\text{ind}_r (5) \equiv 1 \pmod{5}\) and so

\[
5^{(p-1)/5} \equiv g^{\text{ind}_r (5) \cdot (p-1)/5} \equiv g^{(p-1)/5} \equiv \left( \frac{g}{\hat{\mathcal{X}}} \right)_5 \equiv \zeta \pmod{\hat{\mathcal{X}}} .
\]

Thus we have proved

**THEOREM 3.** Let \(p\) be a prime \(\equiv 1 \pmod{5}\) for which 5 is a quintic nonresidue \((mod p)\). Let \((x, u, v, w)\) be the unique solution of (3.1) satisfying \(2u + v \equiv 4 \pmod{5}\). Then we have

\[
5^{(p-1)/5} \equiv \alpha (x, u, v, w) \pmod{p} .
\]

**8. Example.** We take \(p = 311\). A solution of (3.1) in this case is \((-49, 7, 0, 1)\) (see for example [8]) so none of 2, 3, 5 is a quintic residue \((mod 311)\). The unique solution given by Theorem 1 is \((-49, 0, 7, -1)\) so that

\[
2^{(p-1)/5} = 2^{68} = \frac{2276 - 98.46}{2276 + 98.94} = \frac{-2232}{11488} = -\frac{55}{-19} = 52 \pmod{311} .
\]
The unique solution given by Theorem 2 is \((-49, -7, 0, 1)\) so that
\[
3^{(p-1)/5} = 3^{e_2} = \frac{-2276 + 98.66}{-2276 - 98.214} = \frac{4192}{-23248} = \frac{149}{77} = 216 \pmod{311}.
\]

The unique solution given by Theorem 3 is \((-49, 7, 0, 1)\) so that
\[
5^{(p-1)/5} = 5^{e_2} = \frac{-2276 - 98.66}{-2276 + 98.214} = -\frac{23248}{4192} = \frac{77}{149} = 36 \pmod{311}.
\]

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Received June 4, 1974 and in revised form August 8, 1975. Research supported by National Research Council of Canada Grant No. A-7233.

**Carleton University**
Pacific Journal of Mathematics
Vol. 61, No. 2 December, 1975

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