

Pacific Journal of Mathematics

SOME GENERALIZATIONS OF SCHAUDER'S THEOREM IN LOCALLY CONVEX SPACES

VOLKER WROBEL

SOME GENERALIZATIONS OF SCHAUDER'S THEOREM IN LOCALLY CONVEX SPACES

VOLKER WROBEL

Let E_1, E_2, E_3, E_4 be four locally convex Hausdorff spaces (l.c.s.); denote by $\mathcal{L}_b(E_i, E_k)$ the set of all continuous linear operators from E_i into E_k with the topology of uniform convergence on bounded subsets of E_i . Given two linear operators $f \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{L}(E_3, E_4)$, consider the generalized adjoint operator $\text{Hom}(f, g): \mathcal{L}_b(E_2, E_3) \rightarrow \mathcal{L}_b(E_1, E_4)$ defined by $u \rightarrow \text{Hom}(f, g)u = g \circ u \circ f$. This paper deals with transformation properties of $\text{Hom}(f, g)$ and their interactions with those of f and g . This purpose may be illustrated by a result due to K. Vala which generalizes Schauder's well-known theorem concerning precompact operators and their adjoints on normed spaces: Let all spaces under consideration be normed, let f and g both be nonzero. Then $\text{Hom}(f, g)$ is a precompact operator if and only if f and g are precompact operators. In the present paper bounded and precompact operators on l.c.s. are investigated.

1. Introduction. Unless otherwise stated notation used throughout this paper will be that of Schaefer [6]. E_i ($i \in N$) will always denote a l.c.s. over the field of complex numbers C , \mathcal{P}_i denotes the set of all continuous seminorms, and $\mathcal{B}(E_i)$ a fundamental system of bounded subsets of E_i . By U_{p_i} denote the closed, convex, circled neighborhood of zero $\{x \in E_i: P_i(x) \leq 1\}$ in E_i . If B_i runs through $\mathcal{B}(E_i)$, p_k through \mathcal{P}_k , the family

$$W(B_i, U_{p_k}) = \{u \in \mathcal{L}(E_i, E_k): u(B_i) \subset U_{p_k}\}$$

is a neighborhood base of zero in $\mathcal{L}(E_i, E_k)$.

DEFINITION 1.1. A linear operator $u: E_i \rightarrow E_k$ is said to be *precompact* (*compact* or *bounded*) if there exists a neighborhood of zero U_{p_i} in E_i such that $u(U_{p_i})$ is a precompact (relatively compact or bounded) subset of E_k . A linear operator $u: E_i \rightarrow E_k$ is said to be *semi-precompact* if u maps bounded subsets of E_i into precompact subsets of E_k .

Of course bounded linear operators are continuous, whereas semi-precompact operators need not. Let $u \in \mathcal{L}(E_1, E_2)$, $w \in \mathcal{L}(E_3, E_4)$, and let $v \in \mathcal{L}(E_2, E_3)$ be a semi-precompact (precompact, compact, or bounded) operator. Then $v \circ u \in \mathcal{L}(E_1, E_3)$ and $w \circ v \in \mathcal{L}(E_2, E_4)$ are semi-precompact (precompact, compact, or bounded) operators.

A subset $\mathcal{H} \subset \mathcal{L}(E_i, E_k)$ is said to be *collectively precompact* if there exists a neighborhood of zero U_{p_i} in E_i such that $\mathcal{H}(U_{p_i}) = \{u(x): u \in \mathcal{H}, x \in U_{p_i}\}$ is a precompact subset of E_k . Let $f \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{L}(E_3, E_4)$ both be nonzero. Then the linear mapping $\text{Hom}(f, g): \mathcal{L}_b(E_2, E_3) \rightarrow \mathcal{L}_b(E_1, E_4)$ given by $u \rightarrow \text{Hom}(f, g)u = g \circ u \circ f$ is continuous, as an easy calculation shows. By setting $E_3 = E_4 = C$ and $g = \text{id}_C$ (identity map on C) $\text{Hom}(f, g)$ becomes the adjoint operator f' of f . That is why one may call $\text{Hom}(f, g)$ a *generalized adjoint operator*. In generalizing Schauder's well-known theorem Vala [7] has obtained the following result.

THEOREM 1.1. *Let E_1, E_2, E_3, E_4 denote four normed linear spaces, let $f \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{L}(E_3, E_4)$ both be nonzero. Then the mapping*

$$\text{Hom}(f, g): \mathcal{L}_b(E_2, E_3) \longrightarrow \mathcal{L}_b(E_1, E_4)$$

given by $u \rightarrow \text{Hom}(f, g)u = g \circ u \circ f$ is precompact if and only if both f and g are precompact.

We shall be interested in similar results in case of l.c.s.. For that purpose the notion of an *infrabarrelled* l.c.s. is needed. The definition given below is not entirely standard but is easily proved equivalent to the standard one (see Horváth [4, Definition 2, p. 217]). Just this definition gives more insight in why infrabarrelled spaces are involved.

DEFINITION 1.1. A l.c.s. E_1 is said to be *infrabarrelled* if each bounded subset of $\mathcal{L}_b(E_1, E_2)$ is equicontinuous for all l.c.s. E_2 .

As in normed linear spaces semi-precompact operators are precompact, the following result due to Apiola [1] and Geue [3] is a generalization of Vala's theorem.

THEOREM 1.2. *Let E_1, E_2, E_3, E_4 denote four l.c.s. and assume in addition E_2 is infrabarrelled. Let $f \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{L}(E_3, E_4)$ both be nonzero. Then*

$$\text{Hom}(f, g): \mathcal{L}_b(E_2, E_3) \longrightarrow \mathcal{L}_b(E_1, E_4)$$

given by $u \rightarrow \text{Hom}(f, g)u = g \circ u \circ f$ is a semi-precompact continuous linear operator if and only if both f and g are semi-precompact.

REMARK. If one drops the assumption of E_2 being infra-

barrelled, the implication 'Hom(f, g) semi-precompact \curvearrowright both f and g semi-precompact' remains true, whereas the converse implication in general becomes false (see Apiola [1, Theorem 3.3.]).

In this paper we shall be interested in what happens if we consider precompact or bounded linear operators instead of semi-precompact continuous linear operators. Considering the precompact situation the following result due to Floret [2, §13] is rather instructive.

THEOREM 1.3. *There exists a reflexive Fréchet space E_1 , a Banach space E_2 , and a semi-precompact operator $f \in \mathcal{L}(E_1, E_2)$ such that the adjoint operator $f': E_2' \rightarrow E_1'$ of f is precompact without f being precompact itself.*

REMARK. The theorem demonstrates that if Hom(f, g) is precompact, f in general is not. But as E_1 is a reflexive l.c.s. this result also shows that Hom(f, g) in general will not be precompact even if both f and g are.

2. Transformation properties of bounded operators. In dealing with bounded operators we have

THEOREM 2.1. *Let E_1, E_2, E_3, E_4 denote four l.c.s., and let $f \in \mathcal{L}(E_1, E_2), g \in \mathcal{L}(E_3, E_4)$. Then*

(i) *If f and g both are bounded linear operators, then Hom(f, g) is bounded.*

(ii) *Let f and g both be nonzero and assume in addition E_1 is infrabarrelled. Then Hom(f, g) is bounded if and only if both f and g are bounded operators. If Hom(f, g) is precompact, then g is a precompact operator.*

The following lemma gives more insight into the proof of the theorem.

LEMMA 2.1. *Let E_1 and E_2 denote two l.c.s., let $x_0 \in B_1 \in \mathcal{B}(E_1)$ and $p_1 \in \mathcal{S}_1$ such that $p_1(B_1) = \sup_{x \in B_1} p_1(x) = p_1(x_0) = 1$. Then we have*

$$\{u(x_0): u \in W(B_1, U_{p_2}) \text{ and } \dim u(E_1) = 1\} = U_{p_2}$$

for all $p_2 \in \mathcal{S}_2$.

Proof. The inclusion \subset is evident. For the converse inclusion let $y_0 \in U_{p_2}$ be given. By the theorem of Hahn-Banach there exists

a $\varphi \in E'_1$ such that $\varphi(x_0) = 1$ and $|\varphi(x)| \leq p_1(x)$ for all $x \in E_1$. Now the mapping $x \rightarrow u(x) = \varphi(x)y_0$ is in $\mathcal{L}(E_1, E_2)$. Furthermore $u(x_0) = y_0$ and $p_2(u(B_1)) = |\varphi(B_1)| p_2(y_0) \leq 1$, therefore $u \in W(B_1, U_{p_2})$. This completes the proof.

Proof of Theorem 2.1. For the demonstration of (i) let $p_1 \in \mathcal{P}_1$ and $p_3 \in \mathcal{P}_3$ denote two semi-norms such that $f(U_{p_1})$ and $g(U_{p_3})$ are bounded sets in E_2 and E_4 . Then we get the following factorization:

$$\begin{array}{ccccc}
 E_1 & \xrightarrow{f} & E_2 & \xrightarrow{u} & E_3 & \xrightarrow{g} & E_4 \\
 & \searrow f & \uparrow i_2 & & \searrow \hat{g} & & \uparrow i_4 \\
 & & \llbracket f(U_{p_1}) \rrbracket & & \llbracket g(U_{p_3}) \rrbracket & &
 \end{array}$$

where for a convex, circled, bounded subset $A \subset E_i$, $\llbracket A \rrbracket$ denotes the vector spaces spanned by A provided with the norm defined by the Minkowski functional of A . The mapping $\text{Hom}(i_2, \hat{g}): \mathcal{L}_b(E_2, E_3) \rightarrow \mathcal{L}_b(\llbracket f(U_{p_1}) \rrbracket, \llbracket g(U_{p_3}) \rrbracket)$ is bounded, because it is a continuous linear mapping into a normed space. Hence the composed mapping $\text{Hom}(f, g) = \text{Hom}(\hat{f}, i_4) \circ \text{Hom}(i_2, \hat{g})$ is bounded. For the demonstration of (ii) let $W(B_2, U_{p_3})$ be a neighborhood of zero in $\mathcal{L}_b(E_2, E_3)$ such that $\text{Hom}(f, g)W(B_2, U_{p_3})$ is a bounded (precompact) subset of $\mathcal{L}_b(E_1, E_4)$. As E_1 is infrabarrelled, this set is equicontinuous. Now we find a semi-norm $p_2 \in \mathcal{P}_2$ and a $x_0 \in E_1$ such that $p_2(f(x_0)) = 1$ and $p_2(B_2) \leq 1$. By taking $B_2 \cup \{f(x_0)\}$ instead of B_2 , we may assume $f(x_0) \in B_2$. Then by Lemma 2.1. we have $U_{p_3} = \{u(f(x_0)): u \in W(B_2, U_{p_3})\}$ and $\dim u(E_2) = 1$. Hence $g(U_{p_3}) = \{g \circ u \circ f(x_0): u \in W(B_2, U_{p_3}) \text{ and } \dim u(E_2) = 1\} \subset \text{Hom}(f, g)W(B_2, U_{p_3})(x_0)$ is a bounded (precompact) subset of E_4 . Thus g is a bounded (precompact) linear operator. To show that f is also bounded, let $z_0 \in E_3$ such that $g(z_0) \neq 0$. Now obviously $\mathcal{L}_b(E_2, Cz_0)$ and $\mathcal{L}_b(E_1, Cg(z_0))$ are complemented subspaces of $\mathcal{L}_b(E_2, E_3)$ and $\mathcal{L}_b(E_1, E_4)$. Furthermore $\varphi \rightarrow \varphi \otimes z_0$ and $\psi \rightarrow \psi \otimes g(z_0)$ are canonical isomorphisms from $\mathcal{L}_b(E_2, C)$ and $\mathcal{L}_b(E_1, C)$ onto $\mathcal{L}_b(E_2, Cz_0)$ and $\mathcal{L}_b(E_1, Cg(z_0))$. Thus we are in the situation of the following diagram:

$$\begin{array}{ccccc}
 \mathcal{L}_b(E_2, Cz_0) & \hookrightarrow & \mathcal{L}_b(E_2, E_3) & \xrightarrow{\text{Hom}(f, g)} & \mathcal{L}_b(E_1, E_4) \\
 \uparrow & & & & \downarrow \pi \\
 \mathcal{L}_b(E_2, C) & & & & \mathcal{L}_b(E_1, Cg(z_0)) \\
 & \searrow f' & & & \downarrow \\
 & & & & \mathcal{L}_b(E_1, C)
 \end{array}$$

The composed mapping

$$\varphi \longrightarrow \varphi \otimes z_0 \longrightarrow g \circ (\varphi \otimes z_0) \circ f \longrightarrow \varphi \circ f \otimes g(z_0) \longrightarrow \varphi \circ f$$

from $\mathcal{L}_b(E_2, C)$ into $\mathcal{L}_b(E_1, C)$ is the adjoint operator f' of f . As $\text{Hom}(f, g)$ is a bounded operator, so is f' . Hence there is a bounded subset B_2 of E_2 such that $f'(B_2^0)$ is bounded in $E_1' = \mathcal{L}_b(E_1, C)$. Because of $(f'(B_2^0))^0 = (f'')^{-1}(B_2^{00})$ we get $f'''((f'(B_2^0))^0) \subset B_2^{00}$. As E_1 is infrabarrelled E_1' induces the given topology on E_1 . Thus $f = f'''|_{E_1}$ is a bounded operator. This completes the proof.

Especially, the adjoint operator $f': E_{2b}' \rightarrow E_{1b}'$ of a bounded operator $f \in \mathcal{L}(E_1, E_2)$ is bounded, and if E_1 is infrabarrelled also the converse holds. Without E_1 being infrabarrelled, the last statement in general no longer remains true. To give an example consider the mapping $id_E: E_\sigma \rightarrow E_\sigma$ where E denotes an infinite dimensional normed linear space and σ denotes the $\sigma(E, E')$ -topology on E . Then $(id_E)' = id_E: E_b' \rightarrow E_b'$ is a bounded operator, whereas $id_E: E_\sigma \rightarrow E_\sigma$ is not.

3. Transformation properties of precompact operators. As we have learned from Theorem 1.3 there exists no precompact version of Theorem 2.1 without further assumption put on the l.c.s. or the operators involved. In general only the following holds

THEOREM 3.1. *Let E_1, E_2, E_3, E_4 denote four l.c.s., and let $f \in \mathcal{L}(E_1, E_2), g \in \mathcal{L}(E_3, E_4)$ both be bounded operators. Then*

(i) *$\text{Hom}(f, g)$ is a bounded operator. If in addition E_2 is infrabarrelled, f and g both are precompact (semi-precompact would be sufficient), then $\text{Hom}(f, g)$ is bounded and semi-precompact.*

(ii) *If g is a precompact operator, then there exists a neighborhood of zero $W(B_2, U_{p_3})$ in $\mathcal{L}_b(E_2, E_3)$ such that $\text{Hom}(f, g)W(B_2, U_{p_3})$ is a collectively precompact subset of $\mathcal{L}(E_1, E_4)$.*

Proof. Only (ii) is to be shown. For this purpose let $B_2 \in \mathcal{B}(E_2)$ and U_{p_1}, U_{p_3} be given such that $f(U_{p_1}) \subset B_2$, and $g(U_{p_3})$ is precompact. Then the following inclusions hold

$$\text{Hom}(f, g)W(B_2, U_{p_3})(U_{p_1}) \subset \{g \circ u(y) : y \in B_2, u \in W(B_2, U_{p_3})\} \subset g(U_{p_3}).$$

Since $g(U_{p_3})$ is precompact by assumption, we are done.

Now we shall give conditions making sure that $\text{Hom}(f, g)$ becomes a precompact operator. The following theorem is a generalization of a result due to Ringrose [5].

THEOREM 3.2. *Let E_1, E_2, E_3, E_4, E_5 denote five l.c.s., and let $f_1 \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{L}(E_4, E_5)$ both be precompact, let $f_2 \in \mathcal{L}(E_2, E_3)$*

be bounded. Then the mapping

$$\text{Hom}(f_2 \circ f_1, g): \mathcal{L}_b(E_3, E_4) \longrightarrow \mathcal{L}_b(E_1, E_5)$$

defined by $u \rightarrow \text{Hom}(f_2 \circ f_1, g)u = g \circ u \circ f_2 \circ f_1$ is precompact.

Proof. By E_{p_i} we denote the normed space $(E_i/p_i^{-1}(0), p_i)$. Let U_{p_1} , U_{p_2} , and U_{p_4} denote three neighborhoods of zero such that $f_1(U_{p_1})$ and $g(U_{p_4})$ both are precompact, and $f_2(U_{p_2})$ is bounded. Then we get the following factorization

$$\begin{array}{ccccccc} E_1 & \xrightarrow{f_1} & E_2 & \xrightarrow{f_2} & E_3 & \xrightarrow{u} & E_4 & \xrightarrow{g} & E_5 \\ \downarrow \pi_1 & \nearrow \tilde{f}_1 & \downarrow \pi_2 & \nearrow \tilde{f}_2 & & & \downarrow \pi_4 & \nearrow \tilde{g} & \\ E_{p_1} & \xrightarrow{\hat{f}_1} & E_{p_2} & \xrightarrow{v} & E_{p_4} & & & & \end{array}$$

Now the following mappings are continuous:

$$\begin{aligned} \text{Hom}(\hat{f}_1, \tilde{g}): \mathcal{L}_b(E_{p_2}, E_{p_4}) &\longrightarrow \mathcal{L}_b(E_{p_1}, E_5) \\ \text{Hom}(\tilde{f}_2, \pi_4): \mathcal{L}_b(E_3, E_4) &\longrightarrow \mathcal{L}_b(E_{p_2}, E_{p_4}) \end{aligned}$$

$w \rightarrow \gamma(w) = w \circ \pi_1$ from $\mathcal{L}_b(E_{p_1}, E_5)$ into $\mathcal{L}_b(E_1, E_5)$. But by Theorem 1.2 the first mapping is semi-precompact, hence precompact, as $\mathcal{L}_b(E_{p_2}, E_{p_4})$ is a normed space. Hence the composed mapping $\text{Hom}(f_2 \circ f_1, g) = \gamma \circ \text{Hom}(\hat{f}_1, \tilde{g}) \circ \text{Hom}(\tilde{f}_2, \pi_4)$ is a precompact operator. This completes the proof.

Now we shall give some applications of this theorem.

COROLLARY 1. Let E_1, E_2, E_3 denote three l.c.s., let $f_1 \in \mathcal{L}(E_1, E_2)$ be a precompact operator, and let $f_2 \in \mathcal{L}(E_2, E_3)$ be a bounded operator. Then the adjoint operator

$$(f_2 \circ f_1)' = f_1' \circ f_2': E'_{3b} \longrightarrow E'_{1b}$$

is precompact.

Epecially. Let f be a precompact endomorphism of E_1 . Then $(f')^2: E'_{1b} \rightarrow E'_{1b}$ is precompact.

A proof of this corollary is immediately obtained from Theorem 3.2 by setting $E_4 = E_5 = C$ and $g = \text{id}_C$. The corollary is originally due to Ringrose [5]. The assumptions put on f_1 may be weakened: Let E_2 be infrabarrelled and let f_1 be semi-precompact instead of being precompact. Then by Theorem 1.2 $f_1': E'_{2b} \rightarrow E'_{1b}$ is semi-precompact, and $f_2': E'_{3b} \rightarrow E'_{2b}$ is bounded by Theorem 2.1. Hence the composed operator $f_1' \circ f_2'$ is precompact. Indeed, by using the

same arguments as Ringrose in [5], one can show, that $f'_1 \circ f'_2$ is even compact.

COROLLARY 2. *Let E_1, E_2, E_3, E_4 denote four l.c.s., let $f \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{L}(E_3, E_4)$.*

(i) *Let E_1 be a Schwartz space (for a definition and further properties of these spaces see Horváth [4]), let f be bounded, and let g be precompact. The $\text{Hom}(f, g)$ is a precompact operator.*

(ii) *Let E_1 denote an infrabarrelled Schwartz space, let f and g both be nonzero. Then $\text{Hom}(f, g)$ is precompact if and only if f and g both are precompact.*

(iii) *Let E_2 be a Fréchet space, let f and g both be precompact. Then $\text{Hom}(f, g)$ is a precompact operator.*

Proof. Let $f \in \mathcal{L}(E_1, E_2)$ be a bounded operator. Then there exists a suitable neighborhood of zero U_{p_1} in E_1 such that we have the following factorization

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow \pi & \nearrow \hat{f} \\ & & E_{p_1} \end{array}$$

By Horváth [4, Proposition 3, p. 275] we may assume that π is precompact if E_1 is a Schwartz space. Thus we are in the situation of the following diagram

$$\begin{array}{ccccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{u} & E_3 & \xrightarrow{g} & E_4 \\ & \searrow \hat{f} & & \uparrow i_2 & & & \\ & & & & \llbracket f(U_{p_1}) \rrbracket & & \end{array}$$

where \hat{f} is a precompact operator. Hence $\text{Hom}(f, g) = \text{Hom}(i_2 \circ \hat{f}, g)$ is precompact by Theorem 3.2. Now assume $\text{Hom}(f, g)$ is precompact. Then by Theorem 2.1 f is bounded and g is precompact. As E_1 is a Schwartz space f is precompact by the first diagram.

To end this proof a result concerning compact subsets of a Fréchet space is needed, which can easily be deduced from Schaefer [5, Corollary 1, p. 151]:

If B is a convex, circled, compact subset of a Fréchet space E , then there exists another convex, circled, compact subset $A \subset E$ containing B such that the embedding $\llbracket B \rrbracket \hookrightarrow \llbracket A \rrbracket$ is a compact mapping.

Hence in (iii) we are in the situation of the following diagram

$$\begin{array}{ccccc}
 E_1 & \xrightarrow{f} & E_2 & \xrightarrow{u} & E_3 & \xrightarrow{g} & E_4 \\
 \downarrow f & & \uparrow i_2 & & & & \\
 \overline{[f(U_{p_1})^{E_2}]} & \xrightarrow{i} & [A] & & & &
 \end{array}$$

Where $\overline{[f(U_{p_1})^{E_2}]} \xrightarrow{i} [A]$ is compact (A a convex, circled, compact subset of E_2). Then by Theorem 3.2. $\text{Hom}(f, g) = \text{Hom}(i_2 \circ (i \circ \hat{f}), g)$ is precompact. This completes the proof.

I want to express my gratitude to the referee for his valuable and detailed suggestions concerning the first version of this paper.

REFERENCES

1. H. Apiola, *On the tensorproduct and product $\text{Hom}(f, g)$ of compact operators in locally convex topological vector spaces*, Ann. Acad. Sci. Fenn. Ser., AI No. **544** (1973), 33 pp.
2. K. Floret, *Lokalkonvexe Sequenzen mit kompakten Abbildungen*, J. reine angew. Math., **247** (1971) 155-195.
3. A. S. Geue, *Precompact and collectively semi-precompact sets of semi-precompact continuous linear operators*, Pacific J. Math., **52** (1974) 377-401.
4. J. Horváth, *Topological Vector Spaces and Distributions*, Volume I, Addison-Wesley, 1966.
5. J. R. Ringrose, *Complete continuity conditions on linear operators*, Proc. London Math. Soc., (3) **8** (1958), 343-356.
6. H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York 1971.
7. K. Vala, *On compact sets of compact operators*, Ann. Acad. Sci. Fenn. Ser. AI No. **351** (1964), 9 pp.

Received July 22, 1975 and in revised form November 13, 1975.

MATHEMATISCHES SEMINAR DER UNIVERSITÄT
 D-23 Kiel 1
 OLSHAUSENSTR. 40-60
 FED. REP. GERMANY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Graham Donald Allen, Francis Joseph Narcowich and James Patrick Williams, <i>An operator version of a theorem of Kolmogorov</i>	305
Joel Hilary Anderson and Ciprian Foias, <i>Properties which normal operators share with normal derivations and related operators</i>	313
Constantin Gelu Apostol and Norberto Salinas, <i>Nilpotent approximations and quasinilpotent operators</i>	327
James M. Briggs, Jr., <i>Finitely generated ideals in regular F-algebras</i>	339
Frank Benjamin Cannonito and Ronald Wallace Gatterdam, <i>The word problem and power problem in 1-relator groups are primitive recursive</i>	351
Clifton Earle Corzatt, <i>Permutation polynomials over the rational numbers</i>	361
L. S. Dube, <i>An inversion of the S_2 transform for generalized functions</i>	383
William Richard Emerson, <i>Averaging strongly subadditive set functions in unimodular amenable groups. I</i>	391
Barry J. Gardner, <i>Semi-simple radical classes of algebras and attainability of identities</i>	401
Irving Leonard Glicksberg, <i>Removable discontinuities of A-holomorphic functions</i>	417
Fred Halpern, <i>Transfer theorems for topological structures</i>	427
H. B. Hamilton, T. E. Nordahl and Takayuki Tamura, <i>Commutative cancellative semigroups without idempotents</i>	441
Melvin Hochster, <i>An obstruction to lifting cyclic modules</i>	457
Alistair H. Lachlan, <i>Theories with a finite number of models in an uncountable power are categorical</i>	465
Kjeld Laursen, <i>Continuity of linear maps from C^*-algebras</i>	483
Tsai Sheng Liu, <i>Oscillation of even order differential equations with deviating arguments</i>	493
Jorge Martinez, <i>Doubling chains, singular elements and hyper-Z l-groups</i>	503
Mehdi Radjabalipour and Heydar Radjavi, <i>On the geometry of numerical ranges</i>	507
Thomas I. Seidman, <i>The solution of singular equations, I. Linear equations in Hilbert space</i>	513
R. James Tomkins, <i>Properties of martingale-like sequences</i>	521
Alfons Van Daele, <i>A Radon Nikodým theorem for weights on von Neumann algebras</i>	527
Kenneth S. Williams, <i>On Euler's criterion for quintic nonresidues</i>	543
Manfred Wischnewsky, <i>On linear representations of affine groups. I</i>	551
Scott Andrew Wolpert, <i>Noncompleteness of the Weil-Petersson metric for Teichmüller space</i>	573
Volker Wrobel, <i>Some generalizations of Schauder's theorem in locally convex spaces</i>	579
Birge Huisgen-Zimmermann, <i>Endomorphism rings of self-generators</i>	587
Kelly Denis McKennon, <i>Corrections to: "Multipliers of type (p, p)"; "Multipliers of type (p, p) and multipliers of the group L_p-algebras"; "Multipliers and the group L_p-algebras"</i>	603
Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>Correction to: "a^*-closures to completely distributive lattice-ordered groups"</i>	606
Zvi Arad and George Isaac Glauberman, <i>Correction to: "A characteristic subgroup of a group of odd order"</i>	607
Roger W. Barnard and John Lawson Lewis, <i>Correction to: "Subordination theorems for some classes of starlike functions"</i>	607
David Westreich, <i>Corrections to: "Bifurcation of operator equations with unbounded linearized part"</i>	608