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The group of R-homomorphisms $\mathrm{Hom}_R(M, A)$, where M, A are modules over a ring R, is, in a natural way, a module over the endomorphism ring S of M. Under certain weak assumptions on M, the following is true: $\mathrm{Hom}_R(M, -)$ carries injective envelopes of R-modules into injective envelopes of S-modules iff M generates all its submodules. Modules of the latter type are called self-generators. For M a self-generator, $\mathrm{Hom}_R(M, -)$ has additional properties concerning chain conditions and the socle. Many of the known results in this area, in particular those for M projective, are special cases of our main theorems.

Introduction. The question of how properties of a unitary right R-module $M = M_R$ are related to properties of its endomorphism ring S has been answered completely by the Morita theorems in case M is a progenerator. Then the functors $F = \mathrm{Hom}_R(M, -): \mathfrak{M}_R \to \mathfrak{M}_S$ and $H = M \otimes_R -: {}_R\mathfrak{M} \to {}_S\mathfrak{M}$ are equivalences and hence preserve and reflect all categorical properties of objects ($\mathfrak{M}_R$ denotes the category of unitary right R-modules).

Anderson [1] determined the finitely generated and projective modules M, for which H preserves injective envelopes and called them perfect injectors. Inspired by his paper, we investigate the analogous problem for F and introduce the notion of a "perfect coinjector" along the model of [1] (without restrictions on M). When R is a Dedekind domain, we have a structure theorem for perfect coinjectors (2.1). It yields a characterization of torsion modules flat over their endomorphism ring which generalizes that for $R = Z$ in [13, Th. 2]. In particular, the perfect coinjectors coincide with those modules generating all their submodules (self-generators) fors for the special choice of R. This is false for arbitrary R, but it is true (2.4) if certain assumptions, weaker than either "projective" or "generator", are made on M (e.g., $M = MT$ where T is the trace ideal of M).

Large classes of self-generators (§3) justify a closer look: The lattices of R-submodules of $A \in \mathfrak{M}_R$ and S-submodules of $\mathrm{Hom}(M, A)$ are intimately related, and so, as a consequence, are the chain conditions and Goldie dimension of A and $\mathrm{Hom}(M, A)$. These correspondences arise as a natural continuation of Sandomierski's results in [15]. Moreover, the self-generators $M = MT$ are exactly those modules, for which F preserves the properties "simple" and "essential" just as in the optimal case, i.e. M a vector space (resulting socle-formula: 4.5).

Another application of §2 clarifies Anderson's characterization of perfect injectors by means of an equivalence of categories (§5). At one and the same time, the main result extends results of [1] and supplies additional information about the functor H.

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1. The full subcategory $\mathscr{M}$ of M-generated objects of $\mathfrak{M}_R$. A reference for standard notions and results is [2]. The following notation is observed: R is an associative (not necessarily commutative) ring with 1, $\mathfrak{M}_R$ the category of unitary right R-modules, $M = M_R$ an object of $\mathfrak{M}_R$, $S = \mathrm{Hom}_R(M, M)$ the ring of R-endomorphisms of M, $M^* = \mathrm{Hom}_R(M, R)$. Naturally, M is a left, M^* a right S-module.

The homomorphisms $(\ ,\ ): M^* \otimes_S M \to R$ with $(f, m) = f(m)$ resp. $[\ ,\ ]: M \otimes_R M^* \to S$ with $[m, f] = mf(-)$ are R-R-resp. S-S-linear, their images are denoted by T resp. Δ. As is well-known, $T = R(\Delta = S)$ means that M_R is a generator (finitely generated and projective). $T_M(A) = \Sigma\{\mathrm{Im}(f), f \in \mathrm{Hom}_R(M, A)\}$ is called the trace of M on A, $T_M(R) = T$ is called simply the trace. $\Phi: GF \to 1_{\mathfrak{M}_R}$ represents the natural transformation corresponding to the adjoined pair (G, F), where

$$F: \mathfrak{M}_R \longrightarrow \mathfrak{M}_S \quad \text{with} \quad F(A) = \mathrm{Hom}_R(M, A)$$
$$G: \mathfrak{M}_S \longrightarrow \mathfrak{M}_R \quad \text{with} \quad G(B) = B \otimes_S M$$

One observes $\mathrm{Im}(\Phi(A)) = T_M(A)$.

It is known that F preserves injective envelopes in case F is full and faithful and G is exact, the latter being true iff M_R is a generator [5] (In this statement $\mathfrak{M}_R$ may be replaced by any Grothendieck category). Noting that $\mathrm{Hom}(M, A) = \mathrm{Hom}(M, T_M(A))$, we focus attention on the full subcategory $\mathscr{M}$ of M-generated objects of $\mathfrak{M}_R$, i.e. $A \in \mathscr{M}$ iff $T_M(A) = A$ (compare [4]), with the restricted functors $F': \mathscr{M} \to \mathfrak{M}_S$, $G': \mathfrak{M}_S \to \mathscr{M}$ ($\Phi': G'F' \to 1_{\mathscr{M}}$ belongs to the adjoined pair (G', F')). As is easily checked, an injective envelope $A \to B$ of R-modules goes down to an injective envelope $T_M(A) \to T_M(B)$ in $\mathscr{M}$, hence F preserves injective envelopes if F' does. A sufficient condition for the latter: F' full and faithful, G' exact. We will interpret this in terms of equivalent conditions on M, and we will see that, in many cases, it is also necessary.

DEFINITION 1.1. 1. M is called a self-generator iff $T_M(K) = K$, for all R-submodules K of M.

2. M is called a Σ-self-generator iff $T_M(U) = U$, for all R-submodules U of M^n, $n \in N$. (compare with related concepts in [4] and [10]).

EXAMPLE 1.2. (F. Dischinger): Let

$$R = \left\{ \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & c \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} : a, b, c \in K \right\},$$

where K is a noncommutative field. Choose $\lambda, \mu \in K$ such that $\lambda\mu \neq \mu\lambda$ and let

$$I = \left\{ \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & \lambda b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : b \in K \right\}.$$

Clearly, I is a right ideal of R, and the cyclic right R-module $M = R/I$ is a selfgenerator, but not a generator. Denoting the product of μ with the unity matrix by x, we obtain $I \cap xI = 0$ and $R/I \cong R/xI$. Thus R is embedded into M^2, and consequently M is not a Σ-self-generator.

Over a commutative ring, clearly, every cyclic module is a Σ-self-generator; for further examples see §§2, 3. In view of the following two lemmas, Definition 1.1 appears as the natural choice. (Note that F' is full and faithful iff Φ' is an isomorphism.)

LEMMA 1.3. 1. *Let* $A \in \mathfrak{M}_R$. *The map* $\Phi'(A)\colon \mathrm{Hom}(M, A) \otimes_S M \to T_M(A)$ *is an isomorphism if M generates all kernels of homomorphisms* $M^n \to A$, $n \in N$.

2. [17] *The left S-module* ${}_SM$ *is flat iff M generates all kernels of homomorphisms* $M^n \to M$, $n \in N$.

Proof. 1. Let $\sum_{i=1}^n f_i(m_i) = 0$, where $f_i \in \mathrm{Hom}_R(M, A)$, $m_i \in M$. By hypothesis $(m_i)_{1 \leq i \leq n} = \sum_{j=1}^e g_j(n_j)$ for some $g_i \in \mathrm{Hom}_R(M, Ke(\bigoplus f_i))$, $n_j \in M$. Denoting the canonical projections $M^n \to M$ by pr_i, we conclude that $\sum_i f_i \otimes m_i = \sum_i f_i \otimes pr_i(\sum_j g_j(n_j)) = \sum_{i,j} f_i pr_i g_j \otimes n_j = 0$, since $\sum_i f_i pr_i g_j = 0$ for all j.

Assertion 2 is simply Lemma 19.19 of [2].

LEMMA 1.4. *The following statements are equivalent.*

(1) *M is a Σ-self-generator,*

(2) *$\mathscr{M}$ is closed with respect to R-submodules* (hence is a Grothendieck category),

(3) *F' is full and faithful, and G' is exact.*

Proof. (1) $\Leftrightarrow$ (2): One direction is clear. Conversely, let M be a Σ-self-generator, $A \in \mathscr{M}$ (i.e. there is a set I and an epimorphism $f: M^{(I)} \to A$) and A' an R-submodule of A. Because A' may be assumed finitely generated, we can choose a finite subset I' of I such that $A' \subset f(M^{(I')})$. By hypothesis, M generates $f^{-1}(A') \cap M^{(I')}$ and hence A'.

(2) $\Rightarrow$ (3): This is a special case of [5] since M is a generator for $\mathscr{M}$. It also follows directly from Lemma 1.3.

(3) $\Rightarrow$ (1): Assume that G' is exact (i.e., ${}_SM$ is flat). We claim that if $\Phi'(A)$ is an isomorphism, then M generates all kernels of homomorphisms $M^n \to A$ for $n \in N$. Let $f: M^n \to A$, let $f_i = f in_i$ where in_i is the natural injection, and let $(m_i) \in \mathrm{Ke}(f)$. Assuming that $\Phi'(A)$ is an isomorphism, $\Sigma f_i(m_i) = 0$ forces the element $\Sigma f_i \otimes m_i$ of $\mathrm{Hom}(M, A) \otimes_S M$ to be zero. Consequently, since s^M is flat, the element $\Sigma f_i \otimes m_i$ of $\Sigma f_i S \otimes M$ is zero. Thus, [3, Lemma 10] there are $g_{ij} \in S$, $n_j \in M$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ such that

$$m_i = \sum_j g_{ij}(n_j) , \qquad \text{for all } i$$

$$\sum_i f_i g_{ij} = 0 , \qquad \text{for all } j .$$

For $g_j = \Sigma in_i g_{ij}$, this means $g_j \in \mathrm{Hom}_R(M, \mathrm{Ke}(f))$ and $(m_i) = \sum_j g_j(n_j)$.

REMARKS. 1. The implications (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are independent of [5] where only Grothendieck categories are considered.

2. In our proof of (3) $\Rightarrow$ (1) we have shown that if ${}_SM$ is flat, then the converse of Lemma 1.3.1 is true.

In part (1) of the following corollary we rediscover a theorem of Pahl [12] as the special case $A = M$ a generator.

COROLLARY 1.5. *Let M be a Σ-self-generator, $f: A \to B$ a homomorphism of R-modules. Then*

1. $\mathrm{Hom}(M, A)$ *is an injective* (quasi-injective, see [2]) *S-module iff $T_M(A)$ is an M-injective* (quasi-injective) *R-module.*

2. $\mathrm{Hom}(M, f): \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B)$ *is an essential monomorphism in $\mathfrak{M}_S$ iff $f|_{T_M(A)}: T_M(A) \to T_M(B)$ is an essential monomorphism in $\mathfrak{M}_R$.*

In particular, F preserves injective envelopes.

3. *If* $\mathrm{Hom}(M, A)$ *is artinian* (noetherian) *in $\mathfrak{M}_S$, then so is*

$T_M(A)$ *in* $\mathfrak{M}_R$ (e.g. if S_S is artinian, then so is M_R).

Proof. 1.2. From 1.4. (G', F') is an adjoint pair of functors between abelian categories, G' exact, F' full and faithful. As is well-known, F' then preserves and reflects injectivity and essential extensions. Along the same line, one checks that F' (hence F) preserves and reflects quasi-injectivity.

3. $A' \mapsto \mathrm{Hom}_R(M, A')$ defines an injective map from the lattice of R-submodules of $T_M(A)$ into the lattice of S-submodules of $\mathrm{Hom}_R(M, A)$.

2. **Perfect coinjectors.** We call M a (perfect) coinjector iff $F = \mathrm{Hom}_R(M, -): \mathfrak{M}_R \to \mathfrak{M}_S$ preserves injective modules (injective envelopes). It is well-known that M is a coinjector iff M is flat as a left S-module. As we have seen, all Σ-self-generators are perfect coinjectors. We will study cases, in which this is reversible and "Σ-self-generator" may be replaced by "self-generator". First of all, the special case of R a Dedekind domain yields a structure theorem for perfect coinjectors. A resulting description of the torsion modules that are coinjectors generalizes [13, Th. 2]. For $0 \neq P \in \mathrm{Spec}\, R$, let $M_P = \{x \in M: P^n x = 0 \text{ for some } n \in N\}$.

THEOREM 2.1. *For a Dedekind domain R, the following are equivalent*:

(1) *M is a perfect coinjector,*

(2) *$F: \mathfrak{M}_R \to \mathfrak{M}_S$ preserves essential extensions,*

(3) *M is a Σ-self-generator,*

(4) *M is a self-generator,*

(5) *if M is not a torsion module, then M is a generator. If M is a torsion module, then the following holds for each primary component M_P, $0 \neq P \in \mathrm{Spec}\, R$: M_P is reduced* (i.e. does not contain a nonzero divisible submodule), *or the direct complements of the largest divisible submodule are unbounded.*

Furthermore, a torsion module is a perfect coinjector iff it is a coinjector.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial, (3) $\Rightarrow$ (1) holds for an arbitrary ring R. So does (4) $\Rightarrow$ (2): Let $A \subset B$ be an essential extension and $0 \neq f \in \mathrm{Hom}_R(M, B)$. Pick $m \in M$ with $f(m) \neq 0$ and use the fact that mR is generated by M to find $g \in \mathrm{Hom}(M, mR) \subset S$ with $0 \neq fg \in \mathrm{Hom}_R(M, A)$.

(2) $\Rightarrow$ (5): First, let M be non-torsion, i.e. R_R a submodule of M_R. The field K of quotients of R being an injective envelope of R (as an R-module), K_R is a direct summand of an injective envelope

of M. This forces $\mathrm{Hom}_R(M, K) \neq 0$ and hence $\mathrm{Hom}_R(M, R) \neq 0$ by hypothesis. But for a Dedekind integral domain R, $T \neq 0$ means $T = R$.

For M a torsion module, we may assume M P-primary and not reduced. Since $\bigcap_{n \in N} P^n M$ contains the largest divisible submodule of M, it is enough to prove $M/\bigcap_{n \in N} P^n M$ to be unbounded. Assume the contrary, $P^k(M/\bigcap_{n \in N} P^n M) = 0$ for some $k \in N$. For $r \in P^k \backslash P^{k+1}$, let $l_r \colon M \to M$ be multiplication with r. By applying (2) to the essential extension $N \subset M$, where $N = \{m \in M \colon Pm = 0\}$, one obtains $g \in S$ such that $0 \neq l_r g \in \mathrm{Hom}_R(M, N)(l_r \neq 0$, since M is not reduced and hence unbounded). In particular, this means $P^{k+1}g(M) = 0$. Consequently, $g(\bigcap_{n \in N} P^n M) = 0$, that is, g factors through $M/\cap P^n M$. From our assumption we conclude $P^k g(M) = 0$, contradicting $rg(M) \neq 0$.

(5) $\Rightarrow$ (3): We limit our attention to a primary torsion module $M = M_P$ and a cyclic R-submodule $A \cong R/P^k$ of M^n, $n \in N$. If M is bounded, then M is known to be a direct sum of cyclic R-submodules, and so $A \subset M^n$ implies the existence of a direct summand R/P^n, where $n \geqq k$.

If M is unbounded with M_1 its largest divisible submodule, then M/M_1 is also unbounded (in the case $M_1 \neq 0$ apply the hypothesis). There is no loss of generality in assuming $M_1 = 0$, since M_1 is a direct summand of M. We claim $M/\bigcap_{n \in N} P^n M$ is unbounded. If not, then $P^m M \subset \bigcap_{n \in N} P^n M$ for some m which would imply $P^m M = P^{m+i} M$, for all i. This means $P^m M$ is divisible. But with M unbounded, $P^m M \neq 0$ contradicts $M_1 = 0$. In particular, we have that $P^k(M/P^m M) \neq 0$ for some m, and hence the bounded module $M/P^m M$ contains a direct summand of the form R/P^n, $n \geqq k$. Therefore, $M/P^m M$ resp. M generates A. This completes the proof of the equivalences.

Now suppose a torsion module M to be a coinjector. Lemma 1.3 justifies the restriction $M = M_P$. In order to verify (5), let $M = M_1 \oplus M_2$ with $0 \neq M_1$ divisible and M_2 reduced, $n \in N$ arbitrary. For $r \in P^n$, consider the multiplication $l_r \colon M \to M$ with r. From 1.3 $\ker(l_r)$ is generated by M and thus by M_2, because $\mathrm{Hom}(M_1, \ker(l_r)) = 0$. Moreover, $\ker(l_r)$ contains a submodule R/P^n, since M_1 does, which forces $P^n M_2 \neq 0$. This shows the unboundedness of M_2.

REMARKS. 1. The last statement is false for nontorsion modules: Consider the $\boldsymbol{Z}$-module $\boldsymbol{Q}$.

2. A different reading of (4) $\Leftrightarrow$ (5) for the special case $R = \boldsymbol{Z}$ and M a torsion module is [8, th 2.5].

3. Our proof actually establishes the implication (4) $\Rightarrow$ (2) for all rings R.

THEOREM 2.2. *Let* $R = \boldsymbol{Z}$.

1. *Direct sums of Prüfer groups* $\mathbf{Z}(p^\infty)$, *p prime, are not coinjectors.*

2. *Direct sums of cyclic groups, especially all finitely generated or bounded groups, are perfect coinjectors.*

3. *A direct product of cyclic groups is a perfect coinjector iff it is either bounded or one of the cyclic factors is infinite.*

Proof. 1., 2. are clear, 3. is left as an exercise.

Let R be arbitrary. Example 1.2 shows that, in general, neither (2) implies (1) nor (4) implies (3). In the following we point out classes of modules M, for which the equivalence of the first four statements of 2.1 is maintained.

For an ideal I of R, Sandomierski calls an R-module A I-accessible in case $AI = A$. With this definition, M is T-accessible ("trace-accessible") if, for instance, M is a projective module, a generator or an idempotent ideal (for further examples see §3).

LEMMA 2.3. (a) *$MT = M$ iff $\Delta M = M$ iff $T_M(A) = AT$, for all $A \in \mathfrak{M}_R$.*

(b) *If $MT = M$, then T and Δ are idempotent* (i.e. $T^2 = T$, $\Delta^2 = \Delta$), *and Δ is the trace of the left S-module M* (i.e. $\Delta = \Sigma\{\mathrm{Im}(g): g \in \mathrm{Hom}_S(M, S)\}$).

Proof. In view of $m(f, n) = [m, f]n$, a) and the first part of b) are straightforward. $\Delta \subset \Sigma\{\mathrm{Im}(g): g \in \mathrm{Hom}_S(M, S)\}$ is always true, because $[-, f] \in \mathrm{Hom}_S(M, S)$ for $f \in M^*$. The other inclusion follows from $\Delta M = M$.

THEOREM 2.4. *For a module $M = MT$ or a quasi-projective module M* (see [2]), *the following are equivalent*:

(1) *M is a perfect coinjector,*

(2) *$F: \mathfrak{M}_R \to \mathfrak{M}_S$ preserves essential extensions,*

(3) *M is a Σ-self-generator,*

(4) *M is a self-generator,*

(5) *If A' is a simple essential submodule of an R-module A, then* $\mathrm{Hom}(M, A') = 0$ *implies* $\mathrm{Hom}(M, A) = 0$,

In the case $M = MT$, we may add:

(6) *T_R is a self-generator,*

(7) *R-submodules of T-accessible modules are T-accessible,*

(8) *${}_R(R/T)$ is flat.*

REMARKS. 1. For M projective, (7) $\Leftrightarrow$ (8) was proved independently in [10, Th. 2.1].

2. The proof of (5) $\Rightarrow$ (3) was inspired by [1, Th. 2.4] which is contained in the above as a special case of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (8). (Note that $M \otimes_R - \cong \mathrm{Hom}_R(M^*, -)$, and $_RM^*$ is finitely generated projective in case M_R is finitely generated projective.)

3. M quasi-projective and $M = MT$ are special cases of the following situation: There exists an M-projective module P such that $M = \Sigma\{\mathrm{Im}(f): f \in S, f$ can be factored through $P\}$. Modules of this type (as well as Σ-self-generator) are easily checked to satisfy the following two conditions, for all submodules A, B of M^n, $n \in N$:

(a) $T_M(A + B) = T_M(A) + T_M(B)$.

(b) $A \subset B$ and $T_M(B) \subset T_M(A)$ implies $T_M(B/A) = 0$. More general than 2.4, we prove the equivalence of (1)-(5) for all modules M with conditions (a), (b). (Note that they do not, in general, hold for abelian groups.)

4. For $M_R = Q_Z$ the statements (6)-(8) are true, whereas (1)-(5) are not.

Proof of 2.4. Without restrictions on M, we have established (3) $\Rightarrow$ (1) in 1.5 and (4) $\Rightarrow$ (2) in the proof of 2.1. Also for all M the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (5) are trivially true. We show (5) $\Rightarrow$ (3): Assume $x \notin T_M(xR)$ for some $x \in M^n$, $n \in N$. Choose $A \subset M^n$ maximal with respect to $T_M(xR) \subset A$, $x \notin A$. Then $A + xR/A$ is simple and essential in M^n/A. From $T_M(A + xR) = T_M(A) + T_M(xR) \subset T_M(A)$ we conclude $T_M(A + xR/A) = 0$ (compare Remark 3), hence $0 = T_M(M^n/A) = M^n/A$ by (5). This contradicts $M^n/A \neq 0$.

Now specialize to $M = MT$. In view of 1.4 and $T_M(A) = AT$, for all $A \in \mathfrak{M}_R$, conditions (3) and (7) are identical. By replacing M by $T = T^2$ in (1) $\Rightarrow$ (2), we obtain (6) $\Rightarrow$ (7). (6) $\Leftrightarrow$ (8) is easily derived from [3, p. 33].

M being a perfect coinjector as described in 2.1 and 2.4, Hom $(M, -)$ also reflects injectivity and injective envelopes in the sense of 1.5. For example, if $M = MT$ is a self-generator, then S is right injective iff M_R is quasi-injective iff M_R is T-injective.

3. **Examples of Σ-self-generators $M = MT$.** Trivially, every generator is a trace-accessible Σ-self-generator. Examples that arise for special classes of rings are listed in.

THEOREM 3.1. *Let* $M = MT$ (in particular true for M_R projective). *Then* M_R, T_R *and* $_RT$ *are trace-accessible* Σ*-self-generators if either.*

1. *R is regular,*
2. *R is commutative, and M is projective or finitely generated.*

Proof. 1. is clear (2.4). 2. If M is projective, then M is a self-generator from [6]. The assertion follows from 2.4. Let M be finitely generated. In order to verify (5) of 2.4, we regard an essential extension of R-modules $A' \subset A$ and $0 \neq f \in \mathrm{Hom}_R(M, A)$. Since $f(M)$ is finitely generated, there exists $r \in R$ with $0 \neq f(M)r \subset A'$. This means $0 \neq l_r f \in \mathrm{Hom}_R(M, A')$, where $l_r \in S$ denotes the multiplication with r.

Not every T-accessible module M over a commutative ring is a self-generator. For example, let R be the ring of all Cauchy-sequences in $\boldsymbol{Q}$ with componentwise multiplication and M the ideal of zero sequences. We observe $M = T = T^2$, whereas M is not a self-generator: Pick $a = (a_i) \in T$ with $a_i \neq 0$, for infinitely many $i \in \boldsymbol{N}$. Clearly $a \notin aT$, which means $a \notin T_M(aR)$. For R arbitrary, not even the finitely generated and projective modules are self-generators. Choose $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, where K is a field, $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \subset M$ and check $\mathrm{Hom}_R(M, A) = 0$. (Consequently M is not a perfect coinjector. However, M is a coinjector, since S is a field; compare [1]).

On the other hand, looking at M as a left S-module, we make the following simple observation that will turn out to be very useful in §5.

THEOREM 3.2. *For M a projective R-module, the left S-modules M and Δ are trace-accessible Σ-self-generators.*

Proof. The trace of ${}_SM$ coincides with Δ, and we have $\Delta M = M$ (2.3). In view of 2.4 it is enough to show $m \in \Delta m$, for all $m \in M$, which is an immediate consequence of the dual basis lemma.

Examples of modules having the considered properties on both sides simultaneously are provided by the Zelmanowitz regular modules [18] (i.e. for every $m \in M$, there is an $f \in M^*$ satisfying $m = m(f, m)$). Part 1 of the following theorem contains [10, Cor. 2.2].

THEOREM 3.3. *If M_R is* (Zelmanowitz) *regular, then the following modules are trace-accessible Σ-self-generators*:

1. *the R-modules M_R, T_R,*
2. *the S-modules ${}_SM$, ${}_S\Delta$.*

Proof. For $m \in M$, $m = m(f, m)$ is a stronger version of $m \in mT$, which means $M = MT$ is a self-generator. So 1 clearly follows from 2.4. Moreover, $m(f, m) = [m, f]m$, where $[-, f] \in \mathrm{Hom}_S(M, S)$, shows that ${}_SM$ is again regular, and the above argument may be reflected to the other side.

For examples of (Zelmanowitz) regular modules, especially non-projective ones, see [18].

4. Submodules of A_R and $\mathrm{Hom}_R(M, A)_S$. The following remarks on modules $M = MT$ form the basis for more specialized results for (finitely generated) projective modules on one hand and (T-accessible self-) generators on the other. Lemma 4.1 and the symmetry of 4.2 with respect to T and Δ show the condition $M = MT$ to be natural.

LEMMA 4.1. *$M = MT$ is true iff for every $A \in \mathfrak{M}_R$ and every S-submodule B of $\mathrm{Hom}_R(M, A)$, the S-submodule $B\Delta$ of B is essential.*

Proof. Let $M = MT$, A, B as above, and $0 \neq f \in B$. Pick $m \in M$ with $f(m) \neq 0$. By hypothesis, $m = \Sigma m_i(f_i, n_i)$, which means $0 \neq f(m) = \Sigma f(m_i(f_i, n_i)) = \Sigma f[m_i, f_i]n_i$, whence $0 \neq f[m_i, f_i] \in B\Delta$ for some i.

Conversely, we conclude $\mathrm{Hom}_R(M, M/MT) = 0$ from $\mathrm{Hom}_R(M, M/MT)\Delta = 0$.

NOTATION. For $A \in \mathfrak{M}_R$, the lattice of R-submodules resp. T-accessible R-submodules of A will be denoted by $\mathscr{U}_R(A)$ resp. $\mathscr{U}_T(A)$. For $B \in \mathfrak{M}_S$, $\mathscr{U}_S(B)$ and $\mathscr{U}_\Delta(B)$ are defined similarly.

THEOREM 4.2. *Let $M = MT$.*

1. *For every $A \in \mathfrak{M}_R$, the following are inverse lattice isomorphisms:*

$$\psi\colon \mathscr{U}_T(A) \ni X \longmapsto \mathrm{Hom}(M, X)\Delta \in \mathscr{U}_\Delta(\mathrm{Hom}(M, A))$$
$$\varphi\colon \mathscr{U}_\Delta(\mathrm{Hom}(M, A)) \ni Y \longmapsto \Sigma\{\mathrm{Im}(f)\colon f \in Y\} \in \mathscr{U}_T(A)$$

2. *Statements* (1)–(3) *resp.* (1′)–(3′) *are equivalent:*

(1) $\mathscr{U}_T(A) = \mathscr{U}_R(AT)$, *for all* $A \in \mathfrak{M}_R$	(1′) $\mathscr{U}_\Delta(B) = \mathscr{U}_S(B\Delta)$, *for all* $B \in \mathfrak{M}_S$
(2) T_R (*or* M_R) *is a self-generator*	(2′) Δ_S *is a self-generator*
(3) ${}_R(R/T)$ *is flat*	(3′) ${}_S(S/\Delta)$ *is flat*

Proof. 2. follows immediately from 2.4.

1. In view of $T^2 = T$ and $\Delta^2 = \Delta$, the maps ψ and φ are well-defined lattice homomorphisms. Moreover, $\Delta M = M$ implies $\mathrm{Im}(f) = \Sigma\{\mathrm{Im}(g)\colon g \in f\Delta\}$, for all $f \in \mathrm{Hom}_R(M, A)$. For $X \in \mathscr{U}_T(A)$, $\varphi\psi(X) = \Sigma\{\mathrm{Im}(f)\colon f \in \mathrm{Hom}(M, X)\Delta\} = \Sigma\{\mathrm{Im}(f)\colon f \in \mathrm{Hom}(M, X)\} = XT = X$.

Now let $Y \in \mathscr{U}_\Delta(\mathrm{Hom}(M, A))$. We claim $\mathrm{Hom}_R(M, X)\Delta = Y$, where $X = \Sigma\{\mathrm{Im}(f): f \in Y\}$. One inclusion is obvious. Conversely, let $h \in \mathrm{Hom}_R(M, X)$, $[m, g] \in \Delta$. From $h(m) = \Sigma f_i(m_i)$, $f_i \in Y$, $m_i \in M$, we conclude $h[m, g] = \Sigma f_i[m_i, g] \in Y\Delta = Y$, since $h[m, g]x = h(m(g, x)) = h(m)(g, x) = \Sigma f_i(m_i)(g, x) = \Sigma f_i(m_i(g, x)) = \Sigma f_i[m_i, g]x$, for all $x \in M$.

REMARKS AND COROLLARIES 4.3.

1. The lattice isomorphism in 4.2 may also be deduced from [11, Prop. 6]. (This is more complicated but reveals a more general aspect.) For $\Delta = S$ it coincides with the one established by Sandomierski [15]. The symmetric extremes $T = R$ and $\Delta = S$ even have a converse in the following sense:

$T = R$ iff $\mathscr{U}_R(A) \ni X \mapsto \mathrm{Hom}(M, X)\Delta \in \mathscr{U}_\Delta(\mathrm{Hom}(M, A)))$ is an isomorphism for all $A \in \mathfrak{M}_R$.

$\Delta = S$ iff $\mathscr{U}_T(A) \ni X \mapsto \mathrm{Hom}(M, X) \in \mathscr{U}_S(\mathrm{Hom}(M, A))$ is an isomorphism for all $A \in \mathfrak{M}_R$.

2. Let $M = MT$. We illustrate with a few examples, how chain conditions of A_R and $\mathrm{Hom}(M, A)_S$ are related (for $\Delta = S$, see [15]):

(a) AT_R is finitely generated iff $\mathrm{Hom}(M, A)\Delta_S$ is finitely generated. In particular, M_R is finitely generated iff Δ_S is finitely generated.

(b) If M_R is a self-generator and $\mathrm{Hom}(M, A)_S$ is artinian (noetherian), then AT_R is artinian (noetherian). If Δ_S is a self-generator and A_R is artinian (noetherian), then $\mathrm{Hom}(M, A)\Delta_S$ is artinian (noetherian). (An interesting case being $A = AT = M$.)

Proof. (a) Let AT_R be finitely generated, $(B_i)_{i \in I}$ a chain of proper S-submodules of $\mathrm{Hom}(M, A)\Delta$ and consider the chain $(B_i\Delta)_{i \in I}$. According to 4.2, there is a chain $(X_i)_{i \in I}$ of proper R-submodules of AT with $\psi(X_i) = B_i\Delta$. $\bigcup X_i \subsetneqq AT$ (by hypotheses) and $\bigcup X_i \in \mathscr{U}_T(A)$ yields $\bigcup(B_i\Delta) = \bigcup \psi(X_i) = \psi(\bigcup X_i) \subsetneqq \mathrm{Hom}(M, A)\Delta$, hence $\bigcup B_i \subsetneqq \mathrm{Hom}(M, A)\Delta$. The converse is proved similarly,

(b) is obvious.

3. Examples of modules M such that Δ_S is a selfgenerator (different from $\Delta = S$) are easily deduced from [16, Th. 3.5]. In fact, the following are equivalent:

(1) R is right noetherian,

(2) Δ_S is a $(\Sigma\text{-})$self-generator, for all projective modules M_R,

(3) $M_S^* = M^*\Delta_S$ is a $(\Sigma\text{-})$self-generator, for all projective modules M_R.

For R commutative noetherian, combine with 3.1 and 3.2: If M_R is projective, then all of the following modules are Σ-self-generators: M_R, T_R, M_R^*, ${}_SM$, M_S^*, ${}_S\Delta$, Δ_S. Consequently, $\mathscr{U}_R(M_R) \cong \mathscr{U}_S(\Delta_S)$, $\mathscr{U}_R(T_R) \cong \mathscr{U}_S(M_S^*)$.

Sandmierski [15] called an R-module X T-faithful iff $xT \neq 0$, for all $0 \neq x \in X$, and proved that, in case M_R is finitely generated projective, X T-faithful, then finite Goldie dimension of X_R is inherited by $\mathrm{Hom}(M, X)_S$. The latter remains true for reduced hypothesis on M and is even reversible.

COROLLARY 4.4. *Let $M = MT$, $n \in N$.*

1. *If X is T-faithful, then X_R has finite Goldie dimension n iff* $\mathrm{Hom}(M, X)_S$ *has finite Goldie dimension n.*

2. *If M is a self-generator, then, for all $X \in \mathfrak{M}_R$, XT_R has finite Goldie dimension n iff the same is true for* $\mathrm{Hom}(M, X)_S$.

Proof. 1. Let $\bigoplus_{i \in I} X_i$ be a direct sum of non-trivial R-submodules X_i of X. We conclude $X_i T \neq 0$ by hypothesis and apply ψ: $\psi(\Sigma X_i T) = \Sigma \psi(X_i T)$, where $0 \neq \psi(X_i T)_S \subset \mathrm{Hom}(M, X)_S$. The sum $\Sigma \psi(X_i T)$ is direct: $X_i T \cap \sum_{i \neq j} X_j T = 0$ implies $0 = \psi((X_i T \cap \sum_{i \neq j} X_j T) T) = (\psi(X_i T) \cap \sum_{i \neq j} \psi(X_j T))\Delta$, whence $\psi(X_i T) \cap \sum_{i \neq j} \psi(X_j T) = 0$ from 4.1.

The same method yields the converse.

2. If M is a self-generator, then XT is T-faithful for all X (4.2).

The information about the socle of $\mathrm{Hom}_R(M, A)_S$ [So(Hom (M, A))], given in the next theorem, characterizes self-generators. In the case of vector spaces, we rediscover standard results (Δ being the ideal of M-endomorphisms of finite rank). For non-trivial examples see §3. Even the computation of $\mathrm{So}(S_S)$ for M a generator (e.g. $R = \boldsymbol{Z}$, $M = \boldsymbol{Z} \oplus \boldsymbol{Z}(p^\infty)$) may be considerably simplified by 4.5. "$\subset'$" means "essential R-(resp. S-) submodule".

THEOREM 4.5. *Let $M = MT$, $A \in \mathfrak{M}_R$, X an R-submodule of AT, B an S-submodule of* $\mathrm{Hom}(M, A)$. *Then the following are equivalent* (condition (2)-(4) are understood to hold for all A, X, B).

(1) *M_R is a self-generator,*

(2) $\mathrm{Hom}_R(M, X)\Delta_S$ *is simple iff X_R is simple,*

(2′) *$B\Delta_S$ is simple iff $\Sigma\{\mathrm{Im}(f): f \in B\}_R$ is simple,*

(3) $\mathrm{Hom}(M, X) \subset' \mathrm{Hom}(M, A)$ *iff $X \subset' AT$,*

(3′) *$B \subset' \mathrm{Hom}(M, A)$ iff $\Sigma\{\mathrm{Im}(f): f \in B\} \subset' AT$,*

(4) $\mathrm{So}(\mathrm{Hom}(M, A)) = \mathrm{Hom}(M, \mathrm{So}(A))\Delta$, *and one of the following is true:*

(a) $\mathrm{So}(\mathrm{Hom}(M, A)) \subset' \mathrm{Hom}(M, A)$ *iff* $\mathrm{So}(AT) \subset' AT$,

(b) $\mathrm{So}(\mathrm{Hom}(M, A))$ *is simple iff* $\mathrm{So}(AT)$ *is simple,*

(c) $\mathrm{So}(\mathrm{Hom}(M, A)) = 0$ *iff* $\mathrm{So}(AT) = 0$.

Moreover, if (1) holds, then S is semisimple artinian iff M_R is finitely generated, projective and semisimple (this generalizes [18, Th. 4.8.]).

Proof. (1) $\Rightarrow$ (2): Let X_R be simple. From $X = XT$ (4.2) we deduce $\mathscr{U}_T(X) = \{0, X\}$, which means $\mathscr{U}_\Delta(\mathrm{Hom}(M, X)\Delta) = \{0, \mathrm{Hom}(M, X)\Delta\}$. Moreover, for any S-submodule $B \neq 0$ of $\mathrm{Hom}(M, A)\Delta$, we obtain $B\Delta \neq 0$ (4.1); that is $B = B\Delta = \mathrm{Hom}(M, X)\Delta$. Analogously check the other implication of (2) with the aid of 4.2.

(2) $\Rightarrow$ (1): We verify condition (5) of 2.4. Let A' be a simple essential R-submodule of A and $\mathrm{Hom}(M, A) \neq 0$, i.e. $AT \neq 0$. Hence $A' \subset AT$. By (2), $\mathrm{Hom}(M, A')\Delta$ is a simple S-module; in particular $\mathrm{Hom}(M, A') \neq 0$.

(1) $\Rightarrow$ (4): Let $(B_i)_{i \in I}$ be the simple S-submodules of $\mathrm{Hom}(M, A)$. We observe $B_i\Delta = B_i$ and choose $X_i \in \mathscr{U}_T(A)$ with $\psi(X_i) = B_i$ according to 4.2. By (1)$\Leftrightarrow$(2) the X_i are the simple R-submodules of AT, and we obtain: $\mathrm{So}(\mathrm{Hom}(M, A)) = \sum_{i \in I} B_i = \sum_{i \in I} \psi(X_i) = \psi(\sum_i X_i) = \psi(\mathrm{So}(AT)) = \psi(\mathrm{So}(A) \cdot T) = \mathrm{Hom}(M, \mathrm{So}(A))\Delta$.

(a), (b), (c) follow immediately from 4.2.

(4) $\Rightarrow$ (1): A, A' as in "(2) $\Rightarrow$ (1)". From $A' = \mathrm{So}(A) = \mathrm{So}(AT)$ we deduce $\mathrm{So}(\mathrm{Hom}(M, A)) = \mathrm{Hom}(M, A')\Delta$. $\mathrm{So}(AT)$ being a simple, essential submodule of AT, we conclude $\mathrm{So}(\mathrm{Hom}(M, A)) \neq 0$ from each of (a), (b), (c). Consequently, $\mathrm{Hom}(M, A') \neq 0$.

The remaining implications are proved along the same pattern. Moreover, we note: S is semisimple artinian iff $S = \mathrm{So}(S) = (\mathrm{Hom}(M, \mathrm{So}(M)))\Delta$ iff $S = \Delta$ and $\mathrm{So}(M) = M$.

5. **Perfect injectors** (compare [1]). In [1, Th. 2.4] Anderson established the equivalence of the following statements for a finitely generated projective module M_R:

(i) $M \otimes_R - : {}_R\mathfrak{M} \to {}_S\mathfrak{M}$ preserves injective envelopes (M_R is a "perfect injector"),

(ii) $M \otimes_R - : {}_R\mathfrak{M} \to {}_S\mathfrak{M}$ preserves essential extensions,

(iii) $(R/T)_R$ is flat,

From 2.4 we may add one more equivalent condition:

(iv) ${}_RT$ is a self-generator.

As we will see, the background of this result is a category equivalence between the full subcategories of ${}_R\mathfrak{M}$ resp. ${}_S\mathfrak{M}$ consisting of all T- resp. Δ-accessible objects (these will be denoted by ${}_T\mathscr{M}$ resp. ${}_\Delta\mathscr{M}$). This observation will enable us to discuss the validity of either (i) or (ii) (which are not necessarily equivalent when "finitely generated" is dropped) and other properties of the functor $M \otimes_R -$.

Throughout this section we let $M = MT$; consequently, $M \otimes_R A \in {}_\Delta\mathscr{M}$, for all $A \in {}_R\mathfrak{M}$, and $M \otimes_R TA \cong M \otimes_R A$ in case M_R is flat. The following theorem contains a variant of the Morita theorems.

THEOREM 5.1. *The following statements are equivalent:*

(1) ${}_T\mathscr{M}$ *and* ${}_\Delta\mathscr{M}$ *are closed with respect to R- resp. S-submodules, and* $M \otimes_R -$ *induces an equivalence* ${}_T\mathscr{M} \to {}_\Delta\mathscr{M}$ *with inverse* $M^* \otimes_S -$ (especially M_R is flat),

(2) ${}_RT$ *and* ${}_S\Delta$ *are self-generators,*

(3) $(R/T)_R$ *and* $(S/\Delta)_S$ *are flat.*

Before proving 5.1, we notice that, for a projective or a regular module M_R, the left S-module ${}_S\Delta$ is always a self-generator (3.2 and 3.3), hence in both of these cases (1) is true iff ${}_RT$ is a self-generator.

The following technical device contains [16, p. 358, cor.].

LEMMA 5.2. 1. *If* ${}_RT$ *(or* T_R*) is a self-generator, then* $T \cong M^* \otimes_S M$ *as R-bimodules,*

2. *If* ${}_S\Delta$ *(or* Δ_S*) is a self-generator, then* $\Delta \cong M \otimes_R M^*$ *as S-bimodules.*

Proof. 1. Let ${}_RT$ be a self-generator. We show that $(,): M^* \otimes_S M \to T$ is an isomorphism. First, since $TM^* = M^*\Delta$ and $\Delta M = M$, we have $TM^* \otimes_S M = M^* \otimes_S M \in {}_T\mathscr{M}$. Since ${}_T\mathscr{M}$ is closed with respect to R-submodules (2.4), it is sufficient to show $T\cdot \mathrm{ke}(,) = 0$. Let $\Sigma(f_i, m_i) = 0$ and $(g, n) \in T$; then $(g, n)\Sigma f_i \otimes m_i = \Sigma(g[n, f_i]) \otimes m_i = \Sigma g \otimes ([n, f_i]m_i) = g \otimes n\Sigma(f_i, m_i) = 0$. The rest follows by symmetry.

Proof of 5.1. All of 5.1 is covered by 2.4 except the fact that (2) forces the restricted functors $M \otimes_R -: {}_T\mathscr{M} \to {}_\Delta\mathscr{M}$ and $M^* \otimes_S -: {}_\Delta\mathscr{M} \to {}_T\mathscr{M}$ to be inverse equivalences. Since the inclusions $T_R \hookrightarrow R_R$ and $\Delta_S \hookrightarrow S_S$ are pure by (2), we know $TA \cong T \otimes_R A$ and $\Delta B \cong \Delta \otimes_S B$, for all $A \in {}_R\mathfrak{M}$, $B \in {}_S\mathfrak{M}$. Now let $A \in {}_T\mathscr{M}$, $B \in {}_\Delta\mathscr{M}$. By combining the above with 5.2, we obtain:

$$A = TA \cong T \underset{R}{\otimes} A \cong M^* \underset{S}{\otimes} M \underset{R}{\otimes} A$$
$$B = \Delta B \cong \Delta \underset{S}{\otimes} B \cong M \underset{R}{\otimes} M^* \underset{S}{\otimes} B .$$

From properties of the restriced functor $M \otimes_R -: {}_T\mathscr{M} \to {}_\Delta\mathscr{M}$, we easily derive information about the functor $M \otimes_R -: {}_R\mathfrak{M} \to {}_S\mathfrak{M}$.

COROLLARY 5.3. *One of the conditions of* 5.1 *being satisfied* (e.g M_R projective and $(R/T)_R$ flat), *the statements of each of the following pairs are equivalent, for all* $A, B \in {}_R\mathfrak{M}$, $f \in \mathrm{Hom}_R(A, B)$:

1. (i) $M \otimes f: M \otimes_R A \to M \otimes_R B$ *is an* (essential) *monomorphism in* ${}_S\mathfrak{M}$.

(ii) $f|_{TA}$: $TA \to TB$ *is an* (essential) *monomorphism in* ${}_R\mathfrak{M}$ (In case f is an essential monomorphism, (ii) is true.)

2. (i) ${}_SM \otimes_R A$ *is* Δ*-injective* (quasi-injective).
   (ii) ${}_RA$ *is T-injective* (quasi-injective).
3. (i) $M \otimes f: M \otimes_R A \to M \otimes_R B$ *is a projective cover in* ${}_S\mathfrak{M}$.
   (ii) $f|_{TA}$: $TA \to TB$ *is a projective cover in* ${}_R\mathfrak{M}$ (In contrast to injective envelopes, $TA \to TB$ is not necessarily a projective cover in ${}_T\mathscr{M}$ if $A \to B$ is a projective cover in ${}_R\mathfrak{M}$.)
4. (i) $M \otimes_R A$ *is artinian* (noetherian, finitely generated) *in* ${}_S\mathfrak{M}$.
   (ii) TA *is artinian* (noetherian, finitely generated) *in* ${}_S\mathfrak{M}$.

5.4. Connection with Anderson's results. If ${}_S\Delta$ is a self-generator, as is the case when M_R is projective or regular, then the statements (ii), (iii), (iv) of the beginning of this section are equivalent (for (ii) $\Rightarrow$ (iii) Anderson's proof may be adopted). However, (ii) does not imply (i): Let M be a vector-space over a field R, $\dim M_R = \infty$, then ${}_SM \cong {}_SM \otimes_R R$ is not injective (see [14]).

For the special case $\Delta = S$, the equivalence (ii) $\Leftrightarrow$ (i) follows from 5.3.

EXAMPLE 5.5. The following classes of modules M_R have the properties listed in 5.1 and 5.3, e.g. $M \otimes_R -$ preserves essential extensions:

1. All projective modules over a commutative ring R. In particular, all finitely generated projective modules over a commutative ring are perfect injectors in the sense of [1].

2. The maximal regular ideal [7] of an arbitrary ring R, considered as a right (resp. left) R-module.

Proof. 1. According to 3.1 ${}_RT$ and ${}_S\Delta$ are self-generators.

2. The maximal regular ideal is Zelmanowitz regular as a right and left R-module. Hence, all the modules ${}_RT$, T_R, ${}_S\Delta$, Δ_S are self-generators by 3.3.

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