ABELIAN AND NILPOTENT SUBGROUPS OF MAXIMAL ORDER OF GROUPS OF ODD ORDER

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Denote the maximum of the orders of all nilpotent subgroups $A$ of class at most $c$, of a finite group $G$, by $d_c(G)$. Let $A_c(G)$ be the set of all nilpotent subgroups of class at most $c$ and having order $d_c(G)$ in $G$. Let $A_\omega(G)$ denote the set of all nilpotent subgroups of maximal order of a group $G$.

The aim of this paper is to investigate the set $A_\omega(G)$ of groups $G$ of odd order and the structure of the groups $G$ with the property $A_2(G) \subseteq A_\omega(G)$. Theorem 1 gives an expression for the number of elements in $A_\omega(G)$. Theorem 2 gives criteria for the nilpotency of groups of odd order.

In this paper $G$ is a finite group, and $\pi$ is a set of primes. If $G$ is of odd order, then $G$ is solvable [6].

1. Introduction. Denote the maximum of the orders of all nilpotent subgroups $A$ of class at most $c$, of a finite group $G$, by $d_c(G)$. Let $A_c(G)$ be the set of all nilpotent subgroups of class at most $c$ and having order $d_c(G)$ in $G$. Then $J_c(G)$ is the subgroup of $G$ generated by $A_c(G)$. In particular, $J_1(G) = J(G)$ is the Thompson subgroup of $G$. Moreover, $A_\omega(G)$ is the set of all nilpotent subgroups of maximal order of a group $G$. Here $J_\omega(G)$ is the subgroup of $G$ generated by the elements of $A_\omega(G)$.

In this paper $G$ is a finite group, and $\pi$ is a set of primes. If $G$ is of odd order, then $G$ is solvable [6].

The aim of this paper is to investigate the set $A_\omega(G)$ for groups $G$ of odd order and the structure of the groups $G$ with the property $A_2(G) \subseteq A_\omega(G)$.

We shall give, in Theorem 1, an expression for the number of elements in $A_\omega(G)$. In Theorem 2 we shall state criteria for the nilpotency of groups of odd order.

For groups $G$ with the property $A_2(G) \subseteq A_\omega(G)$, we have the following:

**Theorem 3.** Let $G$ be a $\pi$-solvable group with an $S_\pi$-subgroup $K$ of $G$. Assume that $O_{\pi'}(G) = 1$ and that $A \in A_2(K) \cap A_\omega(K) \neq \emptyset$, then
(i) If either $2, 3 \not\in \pi$ or $O_2(A)$ is Abelian, then $O_p(A) = O_p(G) = O_p(K)$, for every $p \in \pi$.
(ii) If $F(G)$ is odd and if $G$ has an Abelian $S_2$-subgroup, then $J_s(K) = J_s(G) = F(G) = F(K) = A$.

Three corollaries of Theorem 3 and further information can be found in Chapter 2.

Our notation is standard and is taken mainly from [8]. In particular, $\pi(G)$ will designate the set of primes dividing $|G|$ and $G_x$ denotes an $S_x$-subgroup of $G$. For the definitions of Sylow system, system normalizers and Carter subgroups of a group $G$ see [11], Definition 11.1 p. 726 and Definition 12.1 p. 736.

2. Statements and proofs of the main theorems. The next result is needed for the proofs of the main Theorems.

**Proposition 1.** Suppose $G$ is a group. Assume that $A$ normalizes a nilpotent subgroup $B$ of $G$, and assume that at least one of the following conditions holds:

(i) $A \in A_1(G)$, and $B$ is Abelian ([3], Proposition 1).
(ii) $A \in A_1(G)$, $|A|$ is odd, and an $S_2$-subgroup of $B$ is Abelian ([3], Proposition 1).
(iii) $A \in A_2(G)$ ([7]).
(iv) $A \in A_\xi(G)$, $c \equiv 2$, $|B|$ is odd, and an $S_2$-subgroup of $A$ is Abelian ([4]).
(v) $A \in A_\xi(G)$, $|B|$ is odd and an $S_2$-subgroup of $A$ is Abelian ([4]).

Then $AB$ is nilpotent.

Define $A_\xi(G)$ to be the set $\{A_p/A \in A_\xi(G)\}$ of distinct $p$-subgroups of a group $G$, where $p$ is a prime.

**Theorem 1.** Let $G$ be a group of odd order. Then

(i) $|A_\xi(G)| = |G: N_G(A)| = \prod_{p \in \pi(A)} |A_\xi(G)|$, where $A \in A_\xi(G)$.
(ii) $|A_\xi(G)| \equiv 1 \pmod{p}$
(iii) $|A_\xi(G)|/h_p$, where $h_p = [G: N_G(G_p)]$
(iv) $G = (N_G(A)/A \in A_\xi(G))$.
(v) If $A \in A_\xi(G)$ and $A_p \subseteq G_p$ then there exists $x \in G$ such that $G_p \cap G_p^x = A_p$.

For a discussion of the number $h_p$ see [9], Theorem 9.3.1.

**Proof.** Proposition 1(v) implies that every element of $A_\xi(G)$ con-
tains $F(G)$. Therefore [13], Theorem 1 implies that $A_\infty(G)$ is a conjugate class. If $A, B \in A_\infty(G)$ then $[A_p, B_q] = 1$ for every two distinct primes $p$ and $q$ by [11], Theorem 7.18, p. 705, proving (i). Let $A_p \subseteq G_p$. We shall prove that $N_G(G_p) \subseteq N_G(A_p)$. If $x \in N_G(G_p)$ then $\langle A_p^x, A_p \rangle \subseteq G_p$ is a $p$-group. Hence, $[\langle A_p^x, A_p \rangle, A_p] = 1$. Therefore $\langle A_p^x, A_p \rangle A_p$ is nilpotent. Consequently, $A_p^x = A_p$ and $N_G(G_p) \subseteq N_G(A_p)$. Part (i) implies that $A_\infty(G)$ is a conjugate class. Hence, $|A_\infty(G)| = [G : N_G(A_p)]$. By [14], Theorem 6.2.3, $|A_\infty(G)| \equiv 1 \pmod{p}$. Thus (ii)–(iii) also hold. Let $\mathcal{F} = \{G_p/p \in \pi(G)\}$ be a sylow system of $G$. Let us denote the intersection of the normalizers of the subgroups of the given Sylow system by $N(\mathcal{F})$. By definition $N(\mathcal{F})$ is a system normalizer of $G$. Since $A_\infty(G)$ is a conjugate class of $G$ there exists $A(p) \in A_\infty(G)$ such that $[A(p)]_p \subseteq G_p$ for every $p \in \pi[A(p)]$.

As above $A = x[A(p)]_p$ is an element of $A_\infty(G)$, moreover $N_G(A) \supseteq N(\mathcal{F})$. Since $G$ is generated by the set of system normalizers of $G$ [11], we obtain that $G = \langle N_G(A)/A \in A_\infty(G) \rangle$, proving (iv). Proposition 1 implies that $A_p = O_p(N_G(A_p))$. As mentioned before $G_p \subseteq N_G(A_p)$. Therefore by Ito’s theorem [12] there exists $x \in G$ such that $G_p = AA^x = A_p$.

The author knows of no counterexample to the conjecture that if $G$ is an arbitrary group then $A_\infty(G)$ is a conjugate class.

Let $\phi$ be the class of finite solvable groups in which the system normalizers are Carter subgroups. $\phi$-groups are discussed in [5] and [11], pp. 743–751.

**Theorem 2.** Let $G$ be a group of odd order. Then

(i) If $G = BC$, where $B \in A_b(G)$, $C \in A_c(G)$, $b \equiv 1$, $c \equiv 1$, then $G$ is nilpotent.

(ii) $G$ is nilpotent if and only if $G \in \phi$ and the Carter subgroups of $G$ are elements of $A_c(G)$, for some integer $c$.

**Proof.** (i) Proposition 1 implies that $BF(G)$ and $CF(G)$ are nilpotent. Therefore, Theorem 1(i) and [13], Theorem 1 imply that there exist $x \in G$ and $A \in A_\infty(G)$ such that $B \subseteq A$ and $C \subseteq A^x$. Hence $G = AA^x$. By [11], Theorem 7.18, p. 705 $G$ is nilpotent.

(ii) Let $C$ be a Carter subgroup of $G$. Assume that $C \in A_c(G)$, for some integer $c$. Since $CF(G)$ is nilpotent by Proposition 1, there exists $A \in A_\infty(G)$ such that $C \subseteq A$. Since $N_G(C) = C$, we obtain that $C = A$. By assumption $G \in \phi$. Therefore $A$ is a system normalizer. By the definition of system normalizer, Proposition 1 implies that $G$ is nilpotent.
Remark. Theorem 2(i) is true without assuming [6].
If $G = BC$, where $B \in A_b(G)$, $C \in A_c(G)$, $b \geq 1$, $c \geq 1$, then $G$ is solvable by [14], Theorem 13.2.9.

We shall say that $G$ is a $D_\pi$-group if all the maximal $\pi$-subgroups of $G$ are conjugate $S_\pi$-subgroups of $G$. If $G$ and every normal subgroup of $G$ is a $D_\pi$-group we will call $G$ a $D_\pi^N$-group.

Let $[A, B, C]$ denotes the triple commutator $[[A, B], C]$ of three subgroups $A, B, C$ of $G$. We say that $G$ is a $\pi$-stable group if it satisfies the following condition:

Let $K$ be an arbitrary $\pi$-subgroup of $G$. Let $A$ be an arbitrary $\pi$-subgroup of $N_G(K)$. Then if $[K, A, A] = 1$, we have $AC_G(K)/C_G(K) \subseteq O_\pi(N_G(K)/C_G(K))$.

The next result is needed for the proof of Theorem 3.

**Proposition 2.** Let $G$ be a $\pi$-stable $D_\pi^N$-group. Let $K$ be an $S_\pi$-subgroup of $G$. Assume that $A_2(K) \subseteq A_\pi(K)$, $C_G(F(G)) \subseteq F(G)$ and $O_\pi^*(G) = 1$. Then we have

(i) $J_2(K) \text{ char } G$
(ii) If $|F(G)|$ is odd and $G$ has an Abelian $S_\pi$-subgroup, then $A_2(K) = A_\pi(K)$ and $J_\pi(K) = J_\pi(G)$.

**Proof.** Assume (i) is false for $G$. Let $\alpha$ be an automorphism of $G$, and choose $g \in G$ such that $K^\alpha = K^\pi$. If $J_2(K) \not\alpha G$, then

$$(J_2(K))^\pi = J_2(K^\pi) = J_2(K^\pi)^\pi = (J_2(K))^\pi = J_2(K).$$

Therefore $J_2(K) \text{ char } G$. Hence $J_2(K) \not\alpha G$.

Let $L$ be the largest normal subgroup of $G$ which normalizes $J_2(K)$. Then $K \cap L$ is an $S_\pi$-subgroup of $L$ by [11], lemma 7.2 p. 444. Since $J_2(K \cap L)$ char $K \cap L$, it follows therefore, by a generalization of the Frattini argument that $G = LN$, where $N = N_G(J_2(K \cap L))$. If $J_\pi(K) \subseteq K \cap L$, then $J_\pi(K) = J_\pi(K \cap L)$. In this case $N = N_G(J_\pi(K))$. But then $G = LN \subseteq N_G(J_\pi(K))$ and $J_\pi(K) \not\alpha G$, a contradiction. Thus, we may assume that $J_2(K) \not\subseteq L \cap K$. By Proposition 1, $F(G) \subseteq A$ for every $A \in A_2(K) \subseteq A_\pi(K)$. In particular, $[F(G), A, A] = 1$. Since $G$ is $\pi$-stable, it follows from the definition that $AC_G(F(G))/C_G(F(G)) \subseteq O_\pi(G/C_GF(G))$. By definition of $L$ and by hypothesis $C_G(F(G)) \subseteq F(G) \subseteq L$. Hence $AL/L \subseteq O_\pi(G/L)$ for every $A \in A_2(K) \subseteq A_\pi(K)$.

However, we claim that $O_\pi(G/L) = 1$. Indeed, set $O_\pi(G/L) =$
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T/L. $K \cap T$ is an $S_\pi$-subgroup of $T$, therefore $(K \cap T)L/L$ is an $S_\pi$-subgroup of $T/L$ by [11], Lemma 7.2 p. 444. Therefore $(K \cap T)T/L = T$. But $K \cap T \subseteq N_G(J_2(K))$, thus $K \cap T \subseteq L$, whence $T = L$, and $O_p(G/L) = 1$. Therefore $A \subseteq L$ for every $A \in A_s(K) \subseteq A_s(K)$. Therefore $J_2(K) \subseteq L \cap K$, a contradiction. Therefore $J_2(K)$ char $G$, proving (i).

Clearly $F(G) \subseteq F(K)$ and $|F(K)|$ is odd, since $C_G(F(G)) \subseteq F(K)$. Hence by hypothesis $C_G(F(K)) \subseteq F(G)$. Proposition 1 yields that $F(K) \subseteq A$ for every $A \in A_s(K)$. Therefore $A_s(K) = A_s(K)$ as a consequence of [13], Theorem 1. Proposition 1 implies that $F(G) \subseteq A$ for every $A \in A_s(G)$. Therefore $Z(A) \subseteq C_G(F(G)) = Z(F(G))$. By hypothesis $F(G)$ is a $\pi$-group. Hence $A$ is a $\pi$-group for every $A \in A_s(G)$. Since $G$ is a $D_\pi^N$-group and $J_2(K)$ char $G$ by part (i), we obtain:

$$J_2(K) = (J_2(K^X))/X \in G = J_2(K),$$

proving (ii).

We now obtain:

Proof of Theorem 3. (i) We use induction on the order of $G$. Let $T = O_p(G)$, $H = O_p(G)$, $G^* = AH$, and $K^* = A(K \cap H)$. Then $K \cap H$ an $S_\pi$-subgroup of $H$ and $K^*$ is an $S_\pi^*$-subgroup of $G^*$.

Suppose that $G^* \subseteq G$. Since $A \subseteq K^*$, $A \in A_s(K^*)$. By induction, $O_p(A) \subseteq O_p(G^*)$. Hence

$$[H, O_p(A)] \subseteq H \cap O_p(G^*) \subseteq O_p(H) = T.$$

Therefore, $O_p(A)/T \subseteq C_{G/T}(H/T) \subseteq H/T$, by [8], p. 228. Consequently, $O_p(A) \subseteq H$. So,

$$O_p(A) \subseteq H \cap O_p(G^*) = O_p(H) = T.$$ 

On the other hand, $T = O_p(G) \subseteq O_p(A)$ by Proposition 1. Therefore, $O_p(A) = O_p(G)$, as desired.

Suppose that $G^* = G$. Then $A_pT$ is an $S_\pi$-subgroup of $G$. By Proposition 1, $T = O_p(G) \subseteq O_p(A)$. Therefore $O_p(A)$ is an $S_\pi$-subgroup of $G$. It is well known that $G$ is $p$-strongly solvable for every $p \in \pi - \{2\}$. By definition $G$ is $p$-stable for $p = 2$. Hence $G$ is $p$-stable for every $p \in \pi$. Therefore $G = O_{p',p,p}(G)$ by Proposition 2. Hence $O_p(A) \subseteq O_{p',p,p}(G)$. By Proposition 1, $AF(O_p(G))$ is nilpotent. Hence $O_p(A) \subseteq C_G(F(O_p(G)))$. By [3], Lemma 4, $O_p(A) \subseteq O_p(G)$, for every $p \in \pi$. Therefore $O_p(A) = O_p(G)$, as desired. In particular $O_p(A) = O_p(K)$, for every $p \in \pi$, proving (i).
(ii) Define \( \sigma = \pi(F(G)) \). It is well known that \( G \) is a \( D^N_\sigma \)-group. By [1], Corollary 4.8, \( C_\sigma(F(G)) \subseteq F(G) \). Proposition 1 yields that \( F(G) \subseteq A \). Therefore \( Z(A) \subseteq C_\sigma(F(G)) \subseteq F(G) \). Hence \( A \) is a \( \sigma \)-group. By definition \( O_{\sigma}(G) = 1 \). Since by hypothesis \( G \) is \( p \)-strongly solvable, for every \( p \in \sigma \), \( G \) is \( \sigma \)-stable by [2], Lemma 3.4. Let \( R \) be an \( S_{\sigma} \)-subgroup of \( G \). Then \( J_{\sigma}(R) = J_{\sigma}(K) = J_{\sigma}(G) \) by Proposition 2. So \( J_{\sigma}(K) = J_{\sigma}(G) = F(G) = F(K) = A \), by part (i).

The author knows of no counterexample to the conjecture that if \( G \) is a group of odd order then \( O_p(A_2) \subseteq O_p(G) \), where \( p \geq 5 \) and \( A_2 \subseteq A_{\sigma}(G) \).

Theorem 3 has three corollaries.

**Corollary 1.** If the Sylow subgroups of a solvable group \( G \) are all Abelian or if \( G \) is of odd order and the Sylow subgroups of \( G \) are of class at most 2, then \( F(G) \subseteq A_{\sigma}(G) \).

**Corollary 2.** If \( P \) is an \( S_p \)-subgroup of a group \( G \), \( p \) odd, \( \text{cl}(P) \leq 2 \), and if \( N_G(P) \) has a normal \( p \)-complement, then so does \( G \).

**Proof.** Following the method of [8], Theorem 8.3.1 and using Theorem 3 we obtain Corollary 3. One must take \( N_G(P) \) instead of \( N_G(ZJ(P)) \) in the above mentioned theorem and its proof.

Corollary 2 is a known result. It can be obtained by [11], Theorem 8.1 p. 447.

We shall say that \( G \) is a \( \psi \)-group if \( A_{\sigma}(G) \) and Carter subgroups of \( G \) coincide.

**Corollary 3.** Let \( G \) be a group of odd order. Assume that \( G \) and every subgroup of \( G \) is a \( \psi \)-group, then \( G \) is nilpotent.

**Proof.** Let \( G \) be a minimal counterexample. By induction every proper subgroup of \( G \) is nilpotent. Therefore the Sylow subgroups of \( G \) are of class at most 2 by [11], Theorem 5.2, p. 281. Hence, Theorem 3 implies that \( \langle A_{\sigma}(G) \rangle = F(G) \) is a Carter subgroup of \( G \). Therefore \( G \) is nilpotent as desired.

**References**


Received September 26, 1974 and in revised form October 30, 1975.

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