ABELIAN AND NILPOTENT SUBGROUPS OF MAXIMAL ORDER OF GROUPS OF ODD ORDER

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Denote the maximum of the orders of all nilpotent subgroups \( A \) of class at most \( c \), of a finite group \( G \), by \( d_c(G) \). Let \( A_c(G) \) be the set of all nilpotent subgroups of class at most \( c \) and having order \( d_c(G) \) in \( G \). Let \( A_\infty(G) \) denote the set of all nilpotent subgroups of maximal order of a group \( G \).

The aim of this paper is to investigate the set \( A_\infty(G) \) of groups \( G \) of odd order and the structure of the groups \( G \) with the property \( A_2(G) \subseteq A_\infty(G) \). Theorem 1 gives an expression for the number of elements in \( A_\infty(G) \). Theorem 2 gives criteria for the nilpotency of groups of odd order.

In this paper \( G \) is a finite group, and \( \pi \) is a set of primes. If \( G \) is of odd order, then \( G \) is solvable [6].

1. Introduction. Denote the maximum of the orders of all nilpotent subgroups \( A \) of class at most \( c \), of a finite group \( G \), by \( d_c(G) \). Let \( A_c(G) \) be the set of all nilpotent subgroups of class at most \( c \) and having order \( d_c(G) \) in \( G \). Then \( J_c(G) \) is the subgroup of \( G \) generated by \( A_c(G) \). In particular, \( J_1(G) = J(G) \) is the Thompson subgroup of \( G \). Moreover, \( A_\infty(G) \) is the set of all nilpotent subgroups of maximal order of a group \( G \). Here \( J_\infty(G) \) is the subgroup of \( G \) generated by the elements of \( A_\infty(G) \).

In this paper \( G \) is a finite group, and \( \pi \) is a set of primes. If \( G \) is of odd order, then \( G \) is solvable [6].

The aim of this paper is to investigate the set \( A_\infty(G) \) for groups \( G \) of odd order and the structure of the groups \( G \) with the property \( A_2(G) \subseteq A_\infty(G) \).

We shall give, in Theorem 1, an expression for the number of elements in \( A_\infty(G) \). In Theorem 2 we shall state criteria for the nilpotency of groups of odd order.

For groups \( G \) with the property \( A_2(G) \subseteq A_\infty(G) \), we have the following:

**Theorem 3.** Let \( G \) be a \( \pi \)-solvable group with an \( S_\pi \)-subgroup \( K \) of \( G \). Assume that \( O_\infty(G) = 1 \) and that \( A \in A_2(K) \cap A_\infty(K) \neq \emptyset \), then
(i) If either 2, 3 \not\in \pi \text{ or } O_2(A) \text{ is Abelian, then } O_p(A) = O_p(G) = O_p(K), \text{ for every } p \in \pi.

(ii) If \(F(G)\) is odd and if \(G\) has an Abelian \(S_2\)-subgroup, then \(J_s(K) = J_s(G) = F(G) = F(K) = A\).

Three corollaries of Theorem 3 and further information can be found in Chapter 2.

Our notation is standard and is taken mainly from [8]. In particular, \(\pi(G)\) will designate the set of primes dividing \(|G|\) and \(G_\pi\) denotes an \(S_\pi\)-subgroup of \(G\). For the definitions of Sylow system, system normalizers and Carter subgroups of a group \(G\) see [11], Definition 11.1 p. 726 and Definition 12.1 p. 736.

2. Statements and proofs of the main theorems. The next result is needed for the proofs of the main Theorems.

**Proposition 1.** Suppose \(G\) is a group. Assume that \(A\) normalizes a nilpotent subgroup \(B\) of \(G\), and assume that at least one of the following conditions holds:

(i) \(A \in A_1(G)\), and \(B\) is Abelian ([3], Proposition 1).

(ii) \(A \in A_1(G)\), \(|A|\) is odd, and an \(S_2\)-subgroup of \(B\) is Abelian ([3], Proposition 1).

(iii) \(A \in A_2(G)\) ([7]).

(iv) \(A \in A_c(G)\), \(c \geq 2\), \(|B|\) is odd, and an \(S_2\)-subgroup of \(A\) is Abelian ([4]).

(v) \(A \in A_\infty(G)\), \(|B|\) is odd and an \(S_2\)-subgroup of \(A\) is Abelian ([4]).

Then \(AB\) is nilpotent.

Define \(A_{\pi}(G)\) to be the set \(\{A_p/A \in A_\pi(G)\}\) of distinct \(p\)-subgroups of a group \(G\), where \(p\) is a prime.

**Theorem 1.** Let \(G\) be a group of odd order. Then

(i) \(|(A_\pi(G))| = [G : N_G(A)] = \prod_{p \in \pi(G)}|A_p(G)|\), where \(A \in A_\pi(G)\).

(ii) \(|A_p(G)| \equiv 1 \pmod{p}\).

(iii) \(|A_p(G)| / h_p\), where \(h_p = [G : N_G(G_p)]\)

(iv) \(G = \langle N_G(A)/A \in A_\pi(G) \rangle\).

(v) If \(A \in A_\pi(G)\) and \(A_p \subseteq G_p\) then there exists \(x \in G\) such that \(G_p \cap G^x_p = A_p\).

For a discussion of the number \(h_p\) see [9], Theorem 9.3.1.

**Proof.** Proposition 1(v) implies that every element of \(A_\pi(G)\) con-
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... contains $F(G)$. Therefore [13], Theorem 1 implies that $A_{\infty}(G)$ is a conjugate class. If $A, B \in A_{\infty}(G)$ then $[A_p, B_q] = 1$ for every two distinct primes $p$ and $q$ by [11], Theorem 7.18, p. 705, proving (i). Let $A_p \subseteq G_p$. We shall prove that $N_G(G_p) \subseteq N_G(A_p)$. If $x \in N_G(G_p)$ then $(A_p, x) \subseteq G_p$ is a $p$-group. Hence, $[(A_p, x), A_p] = 1$. Therefore $(A_p, x)A_p$ is nilpotent. Consequently, $A_p = A_p$ and $N_G(G_p) \subseteq N_G(A_p)$.

Part (i) implies that $A_{\infty}(G)$ is a conjugate class. Hence, $|A_{\infty}(G)| = |G : N_G(A_p)|$. By [14], Theorem 6.2.3, $|A_{\infty}(G)| = 1 (\text{mod } p)$. Thus (ii)–(iii) also hold. Let $\mathcal{F} = \{G_p/p \in \pi(G)\}$ be a sylow system of $G$. Let us denote the intersection of the normalizers of the subgroups of the given Sylow system by $N(\mathcal{F})$. By definition $N(\mathcal{F})$ is a system normalizer of $G$. Since $A_{\infty}(G)$ is a conjugate class of $G$ there exists $A(p) \in A_{\infty}(G)$ such that $[A(p)]_p \subseteq G_p$, for every $p \in \pi[A(p)]$.

As above $A = x[A(p)]_p$ is an element of $A_{\infty}(G)$, moreover $N_G(A) \supseteq N(\mathcal{F})$. Since $G$ is generated by the set of system normalizers of $G$ [11], we obtain that $G = \langle N_G(A)/A \in A_{\infty}(G) \rangle$, proving (iv). Proposition 1 implies that $A_p = O_p(N_G(A_p))$. As mentioned before $G_p \subseteq N_G(A_p)$. Therefore by Ito’s theorem [12] there exists $x \in G$ such that $G_p \cap G_p^x = A_p$.

The author knows of no counterexample to the conjecture that if $G$ is an arbitrary group then $A_{\infty}(G)$ is a conjugate class.

Let $\phi$ be the class of finite solvable groups in which the system normalizers are Carter subgroups. $\phi$-groups are discussed in [5] and [11], pp. 743–751.

THEOREM 2. Let $G$ be a group of odd order. Then

(i) If $G = BC$, where $B \in A_b(G)$, $C \in A_c(G)$, $b \geq 1$, $c \geq 1$, then $G$ is nilpotent.

(ii) $G$ is nilpotent if and only if $G \in \phi$ and the Carter subgroups of $G$ are elements of $A_c(G)$, for some integer $c$.

Proof. (i) Proposition 1 implies that $BF(G)$ and $CF(G)$ are nilpotent. Therefore, Theorem 1(i) and [13], Theorem 1 imply that there exist $x \in G$ and $A \in A_{\infty}(G)$ such that $B \subseteq A$ and $C \subseteq A^x$. Hence $G = AA^x$. By [11], Theorem 7.18, p. 705 $G$ is nilpotent.

(ii) Let $C$ be a Carter subgroup of $G$. Assume that $C \in A_c(G)$, for some integer $c$. Since $CF(G)$ is nilpotent by Proposition 1, there exists $A \in A_{\infty}(G)$ such that $C \subseteq A$. Since $N_G(C) = C$, we obtain that $C = A$. By assumption $G \in \phi$. Therefore $A$ is a system normalizer. By the definition of system normalizer, Proposition 1 implies that $G$ is nilpotent.
REMARK. Theorem 2(i) is true without assuming [6].

If \( G = BC \), where \( B \in A_b(G) \), \( C \in A_c(G) \), \( b \geq 1 \), \( c \geq 1 \), then \( G \) is solvable by [14], Theorem 13.2.9.

We shall say that \( G \) is a \( D_\pi \)-group if all the maximal \( \pi \)-subgroups of \( G \) are conjugate \( S_\pi \)-subgroups of \( G \). If \( G \) and every normal subgroup of \( G \) is a \( D_\pi \)-group we will call \( G \) a \( D^N_\pi \)-group.

Let \([A, B, C]\) denotes the triple commutator \([ [A, B], C]\) of three subgroups \( A, B, C \) of \( G \). We say that \( G \) is a \( \pi \)-stable group if it satisfies the following condition:

Let \( K \) be an arbitrary \( \pi \)-subgroup of \( G \). Let \( A \) be an arbitrary \( \pi \)-subgroup of \( N_G(K) \). Then if \([ K, A, A ] = 1 \), we have \( AC_\sigma(K)/C_\sigma(K) \subseteq O_\pi(N_G(K)/C_G(K)) \).

The next result is needed for the proof of Theorem 3.

**Proposition 2.** Let \( G \) be a \( \pi \)-stable \( D^N_\pi \)-group. Let \( K \) be an \( S_\pi \)-subgroup of \( G \). Assume that \( A_2(K) \subseteq A_\pi(K) \), \( C_\sigma(F(G)) \subseteq F(G) \) and \( O_\pi^-(G) = 1 \). Then we have

(i) \( J_2(K) \) char \( G \)
(ii) If \( |F(G)| \) is odd and \( G \) has an Abelian \( S_\pi \)-subgroup, then \( A_2(K) = A_\pi(K) \) and \( J_\pi(K) = J_\pi(G) \).

**Proof.** Assume (i) is false for \( G \). Let \( \alpha \) be an automorphism of \( \hat{G} \), and choose \( g \in G \) such that \( K^\alpha = K^\xi \). If \( J_2(K) \triangleleft G \), then

\[
(J_2(K))^\alpha = J_2(K^\alpha) = J_2(K^\xi) = (J_2(K))^\xi = J_2(K)
\]

Therefore \( J_2(K) \) char \( G \). Hence \( J_2(K) \not\triangleleft G \).

Let \( L \) be the largest normal subgroup of \( G \) which normalizes \( J_2(K) \). Then \( K \cap L \) is an \( S_\pi \)-subgroup of \( L \) by [11], lemma 7.2 p. 444. Since \( J_2(K \cap L) \) char \( K \cap L \), it follows therefore, by a generalization of the Frattini argument that \( G = LN \), where \( N = N_\sigma(J_2(K \cap L)) \). If \( J_2(K) \subseteq K \cap L \), then \( J_2(K) = J_2(K \cap L) \). In this case \( N = N_\sigma(J_2(K)) \). But then \( G = LN \subseteq N_\sigma(J_2(K)) \) and \( J_2(K) \not\triangleleft G \), a contradiction. Thus, we may assume that \( J_2(K) \not\subseteq L \cap K \). By Proposition 1, \( F(G) \subseteq A \) for every \( A \in A_2(K) \subseteq A_\pi(K) \). In particular, \([ F(G), A, A ] = 1 \). Since \( G \) is \( \pi \)-stable, it follows from the definition that \( AC_\sigma(F(G))/C_\sigma(F(G)) \subseteq O_\pi(G/C_G(F(G))) \). By definition of \( L \) and by hypothesis \( C_\sigma(F(G)) \subseteq F(G) \subseteq L \). Hence \( AL/L \subseteq O_\pi(G/L) \) for every \( A \in A_2(K) \subseteq A_\pi(K) \).

However, we claim that \( O_\pi(G/L) = 1 \). Indeed, set \( O_\pi(G/L) = \)
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... conclude

Clearly $F(G) \subseteq F(K)$ and $|F(K)|$ is odd, since $C_G(F(G)) \subseteq F(K)$. Hence by hypothesis $C_G(F(K)) \subseteq F(K)$. Proposition 1 yields that $F(K) \subseteq A$ for every $A \in A_x(K)$. Therefore $A_x(K) = A_x(K)$ as a consequence of [13], Theorem 1. Proposition 1 implies that $F(G) \subseteq A$ for every $A \in A_x(G)$. Therefore $Z(A) \subseteq C_G(F(G)) = Z(F(G))$. By hypothesis $F(G)$ is a $\pi$-group. Hence $A$ is a $\pi$-group for every $A \in A_x(G)$. Since $G$ is a $D_x^*$-group and $J_x(K)$ char $G$ by part (i), we obtain:

$$J_x(G) = \langle J_x(K^X) / X \in G \rangle = J_x(K),$$

proving (ii).

We now obtain:

Proof of Theorem 3. (i) We use induction on the order of $G$. Let $T = O_p(G)$, $H = O_{pp}(G)$, $G^* = AH$, and $K^* = A(K \cap H)$. Then $K \cap H$ is an $S_\pi$-subgroup of $H$ and $K^*$ is an $S_\pi$-subgroup of $G^*$.

Suppose that $G^* \subseteq G$. Since $A \subseteq K^*$, $A \in A_x(K^*)$. By induction, $O_p(A) \subseteq O_p(G^*)$. Hence

$$[H, O_p(A)] \subseteq H \cap O_p(G^*) \subseteq O_p(H) = T.$$

Therefore, $O_p(A)T/T \subseteq C_{G/T}(H/T) \subseteq H/T$, by [8], p. 228. Consequently, $O_p(A) \subseteq H$. So,

$$O_p(A) \subseteq H \cap O_p(G^*) = O_p(H) = T.$$

On the other hand, $T = O_p(G) \subseteq O_p(A)$ by Proposition 1. Therefore, $O_p(A) = O_p(G)$, as desired.

Suppose that $G^* = G$. Then $A_pT$ is an $S_p$-subgroup of $G$. By Proposition 1, $T = O_p(G) \subseteq O_p(A)$. Therefore $O_p(A)$ is an $S_p$-subgroup of $G$. It is well known that $G$ is $p$-strongly solvable for every $p \in \pi - \{2\}$. By definition $G$ is $p$-stable for $p = 2$. Hence $G$ is $p$-stable for every $p \in \pi$. Therefore $G = O_{p^*}(G)$ by Proposition 2. Hence $O_p(A) \subseteq O_{p^*}(G)$. By Proposition 1, $AF(O_p(G))$ is nilpotent. Hence $O_p(A) \subseteq C_G(F(O_p(G)))$. By [3], Lemma 4, $O_p(A) \subseteq O_p(G)$, for every $p \in \pi$. Therefore $O_p(A) = O_p(G)$, as desired. In particular $O_p(A) = O_p(K)$, for every $p \in \pi$, proving (i).
(ii) Define $\sigma = \pi(F(G))$. It is well known that $G$ is a $D_\sigma^N$-group. By [1], Corollary 4.8, $C_G(F(G)) \subseteq F(G)$. Proposition 1 yields that $F(G) \subseteq A$. Therefore $Z(A) \subseteq C_G(F(G)) \subseteq F(G)$. Hence $A$ is a $\sigma$-group. By definition $O_\sigma(G) = 1$. Since by hypothesis $G$ is $p$-strongly solvable, for every $p \in \sigma$, $G$ is $\sigma$-stable by [2], Lemma 3.4. Let $R$ be an $S_\sigma$-subgroup of $G$. Then $J_\sigma(R) = J_\sigma(K) = J_\sigma(G)$ by Proposition 2. So $J_\sigma(K) = J_\sigma(G) = F(G) = F(K) = A$, by part (i).

The author knows of no counterexample to the conjecture that if $G$ is a group of odd order then $O_p(A_2) \subseteq O_p(G)$, where $p \geq 5$ and $A_2 \subseteq A_2(G)$.

Theorem 3 has three corollaries.

**Corollary 1.** If the Sylow subgroups of a solvable group $G$ are all Abelian or if $G$ is of odd order and the Sylow subgroups of $G$ are of class at most 2, then $F(G) \in A_\sigma(G)$.

**Corollary 2.** If $P$ is an $S_\sigma$-subgroup of a group $G$, $p$ odd, $\text{cl}(P) \leq 2$, and if $N_G(P)$ has a normal $p$-complement, then so does $G$.

**Proof.** Following the method of [8], Theorem 8.3.1 and using Theorem 3 we obtain Corollary 3. One must take $N_G(P)$ instead of $N_G(ZJ(P))$ in the above mentioned theorem and its proof.

Corollary 2 is a known result. It can be obtained by [11], Theorem 8.1 p. 447.

We shall say that $G$ is a $\psi$-group if $A_\sigma(G)$ and Carter subgroups of $G$ coincide.

**Corollary 3.** Let $G$ be a group of odd order. Assume that $G$ and every subgroup of $G$ is a $\psi$-group, then $G$ is nilpotent.

**Proof.** Let $G$ be a minimal counterexample. By induction every proper subgroup of $G$ is nilpotent. Therefore the Sylow subgroups of $G$ are of class at most 2 by [11], Theorem 5.2, p. 281. Hence, Theorem 3 implies that $(A_\sigma(G)) = F(G)$ is a Carter subgroup of $G$. Therefore $G$ is nilpotent as desired.

**References**


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