

# Pacific Journal of Mathematics

**ON STARSHAPED SETS AND HELLY-TYPE THEOREMS**

JOHN DAVID BAILDON AND RUTH SILVERMAN

## ON STARSHAPED SETS AND HELLY-TYPE THEOREMS

JOHN D. BAILDON AND RUTH SILVERMAN

**Suppose an ordered pair of sets  $(S, K)$  in a linear topological space is of Helly type  $(n + 1, n)$ , i.e., for every  $n + 1$  distinct points in  $S$  there is a point in  $K$  which sees at least  $n$  of them via  $S$ . Then if  $S$  is closed,  $K$  compact, and  $n \geq 3$ , the nontrivial visibility sets in  $K$  are pairwise nondisjoint. Sufficient conditions are obtained for  $S$  to be starshaped.**

Let  $S$  be a subset of a linear topological space  $L$ . For points  $x, y$  in  $S$ , we say  $x$  sees  $y$  via  $S$  if and only if the segment  $[x, y]$  lies in  $S$ . Further, the set  $S$  is said to be *starshaped* if and only if there is some point  $p$  in  $S$  such that, for every  $x$  in  $S$ ,  $p$  sees  $x$  via  $S$ .

If  $S$  and  $K$  are subsets of  $L$ , with every point  $x$  in  $S$  associated its *visibility set*  $K(x)$ , the set of all points of  $K$  which  $x$  sees via  $S$ .

We shall say  $(S, K)$  has *Helly-type*  $(s, r)$ , where  $r$  and  $s$  are positive integers,  $r \leq s$ , if for every  $s$  distinct points in  $S$  there is a point on  $K$  seeing at least  $r$  of them via  $S$ . Clearly, if  $(S, K)$  has Helly-type  $(s, r)$ , and  $0 \leq i \leq r - 1$ , then  $(S, K)$  has Helly-type  $(s - i, r - i)$ .

In this paper we obtain a solution to a problem posed by Valentine, concerning sets of Helly type which are unions of a finite number of starshaped sets [3, Prob. 6.7, p. 178], and also obtain some related results. Breen [1] has given conditions in the plane for a simply connected set to be a union of two starshaped sets. We replace simple connectedness by the following:

For  $S$  and  $K$  subsets of a linear topological space  $L$ , we shall say the ordered pair  $(S, K)$  has the *triangle property* if the interior of every triangle having an edge on  $K$  and the other edges in  $S$  is itself a subset of  $S$ .

If  $S$  is a closed subset of a linear topological space  $L$ ,  $K$  is a compact convex subset of  $L$  of dimension  $k$  and  $(S, K)$  has the triangle property, then  $K(x)$  is compact and convex for each  $x \in S$ . If  $(S, K)$  is of Helly type  $(r, r)$ , for  $r \geq k + 1$ , then by Helly's theorem  $\bigcap \{K(x) : x \in S\} \neq \emptyset$ , and  $S$  is starshaped. However, it is possible under certain conditions to weaken the hypothesis considerably, and yet reach the same conclusion.

A collection of sets  $\mathcal{H}$  is said to have "*piercing number*"  $j$  or a *j-partition* for a positive integer  $j$ , if  $\mathcal{H}$  can be represented as a union of  $j$  collections, each with a nonvoid intersection.

The classical result on  $j$ -partitions is a theorem by H. Hadwiger and H. DeBrunner [2], which for convenience we state here as Theorem 1.

**THEOREM 1.** *For integers  $r, s$  and  $n$ , let  $J(s, r, n)$  denote the smallest integer (if one exists) for which a  $j$ -partition is admitted by each family  $\mathcal{K}$  of compact convex sets in  $R^n$  which has the  $(s, r)$  property, i.e., for every  $s$  members of  $\mathcal{K}$ , some  $r$  have a common point. Then  $J(s, r, n) = s - r + 1$  whenever  $r \leq s$  and  $nr \geq (n - 1)s + (n + 1)$ .*

**REMARKS.** When  $j = 1$  and  $r = n + 1$ , Theorem 1 reduces to Helly's theorem.

If  $S$  is a closed subset of a linear topological space,  $K$  a compact convex subset of  $S$  of dimension  $n$ , such that  $(S, K)$  has the triangle property and is of Helly type  $(s, r)$ , then for every  $x \in S$ ,  $K(x)$  is compact and convex, and the collection  $\{K(x) : x \in S\}$  has the  $(s, r)$  property.

Therefore, if  $J(s, r, n) = j$ , then the set  $S$  can be expressed as a union of  $j$  starshaped sets. However, for choices of  $s, r$  and  $n$  as small as  $s = 4$ ,  $r = 3$ ,  $n = 2$ , it is not known whether  $J(4, 3, 2)$  exists.

If  $n = 1$ , then Theorem 1 implies that  $J(s, r, 1) = s - r + 1$  if  $r \geq 2$ , so that  $J(s, 2, 1) = s - 1$ . Consequently  $S$  will be the union of  $s - 1$  starshaped sets if  $(S, K)$  has Helly-type  $(s, 2)$  and  $K$  is a compact line segment. Also, since  $J(r + 1, r, 1) = 2$  for all  $r \geq 2$ ,  $J(3, 2, 1) = J(4, 3, 1) = 2$ . Consequently if  $(S, K)$  has Helly-type  $(3, 2)$  or  $(4, 3)$ , where  $K$  is a compact line segment, then  $S$  is the union of two starshaped sets. Breen [1] proved this result for Helly-type  $(3, 2)$  without the assumption that  $K(x) \neq \emptyset$  for all  $x$  in  $S$ . We improve the  $(4, 3)$  case by showing  $S$  will be starshaped. In fact, in Theorem 4, we obtain the more general result that if  $(S, K)$  is of Helly type  $(2k + 2, 2k + 1)$  in a linear topological space, and  $K$  is of dimension  $k$ , then with a single exception  $S$  is starshaped. This result improves the prediction, from  $J(2k + 2, 2k + 1, k) = 2$ , that  $S$  would be a union of two starshaped sets. In Theorems 2 and 3, for  $(S, K)$  of Helly type  $(n + 1, n)$ , without restrictions on dimension, sufficient conditions are obtained for the visibility sets to be pairwise nondisjoint (2), or for  $S$  to be starshaped (3).

We must first prove the following lemma.

**LEMMA.** *Let  $S$  and  $K$  be a closed and a compact subset, respectively, of a linear topological space  $L$ . If there exist  $x, w$  in  $S$  such that  $K(x) \cap K(w) = \emptyset$  and  $p \in K(x)$ ,  $q \in K(w)$ , then there exist  $t_0, \tau_0$  in  $(0, 1)$  such that if  $|t| < t_0$ ,  $|\tau| < \tau_0$ , then  $K(y(t)) \cap K(z(\tau)) = \emptyset$ , where  $y(t) = tp + (1 - t)x$ , and  $z(\tau) = \tau q + (1 - \tau)w$ .*

*Proof.* We first observe that for every  $x$  in  $S$ ,  $K(x)$  is compact: recall  $K(x) = \{p \in K \cap S \mid [p, x] \subset S\}$ . Let  $p$  be a limit point of  $K(x)$ . Select a sequence  $\{p_n\}$  such that  $p_n \in K(x)$  for every  $n$  and  $p_n \rightarrow p$ . For each  $n$ , the line segment  $[p_n, x]$  is contained in  $S$ . By closure of  $S$ ,  $[p, x] \subset S$  and by closure of  $K$ ,  $p \in K$ . Therefore  $p \in$

$K(x)$ . So  $K(x)$  is a closed subset of a compact set and consequently compact.

Since  $K(w)$  and  $K(x)$  are compact and disjoint, there are disjoint open neighborhoods  $U, U'$  in  $L$ , such that  $K(x) \subset U$  and  $K(w) \subset U'$ .

We wish to prove the existence of  $t_0 > 0$  such that  $0 < t < t_0$  implies  $K(y(t)) \subset U$ . Since  $t_0$  exists trivially if  $K(x) = \{x\}$ , we may assume  $K(x) \neq \{x\}$ .

Assume no such  $t_0$  exists. Then we can find a sequence of real numbers  $\{t_n\}$ ,  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ , and a corresponding sequence of points  $\{\alpha_n\}$  in  $K \sim U$ , such that for every  $n$ ,  $\alpha_n \in K(y(t_n))$ .

By compactness of  $K \sim U$ , there is a point  $\alpha_0 \in K \sim U$  and a subsequence of  $\{\alpha_n\}$ , called for convenience  $\{\alpha_i\}$ , such that  $\alpha_i \rightarrow \alpha_0$  as  $i \rightarrow \infty$ . Now for each  $i$ ,  $\alpha_i \in K(y(t_i))$ , so the line segment from  $y(t_i)$  to  $\alpha_i$  is in  $S$ . By closure of  $S$ , the limiting line segment from  $x$  to  $\alpha_0$  is also in  $S$ . Therefore  $x$  sees  $\alpha_0$ , contradicting the hypothesis, since  $\alpha_0$ , not being in  $U$ , is clearly not in  $K(x)$ .

The same argument implies the existence of  $\tau_0 > 0$  such that for  $0 < \tau < \tau_0$ ,  $K(z(\tau)) \subset U'$ . We therefore conclude that for  $t, \tau$  sufficiently small,  $K(y(t)) \cap K(z(\tau)) = \emptyset$ .

**THEOREM 2.** *Let  $S$  and  $K$  be, respectively, a closed and a compact subset of a linear topological space  $L$ , such that  $(S, K)$  is of Helly type  $(n + 1, n)$  for some  $n \geq 3$ . Let  $\mathcal{K} = \{K(x) : x \in S, K(x) \not\subset \{x\}\}$ . Then  $\mathcal{K}$  is pairwise nondisjoint.*

*Proof.* Suppose  $\mathcal{K}$  fails to be pairwise nondisjoint and let  $K(x)$  and  $K(w)$  be members of  $\mathcal{K}$  such that  $K(x) \cap K(w) = \emptyset$ . There exist neighborhoods  $U, U'$  such that  $K(x) \subset U$ ,  $K(w) \subset U'$ , and  $U \cap U' = \emptyset$ . As in the proof of the lemma, select  $p \in K(x)$ ,  $p \neq x$ ,  $q \in K(w)$ ,  $q \neq w$ , and then  $y$  on  $(x, p)$ ,  $z$  on  $(w, q)$  such that  $K(y) \subset U$ ,  $K(z) \subset U'$ . There is no point in  $K$  seeing three of the four points  $x, y, w, z$ . Expanding the set  $\{x, y, w, z\}$  if necessary, we have a contradiction of the hypothesis of Helly type  $(n + 1, n)$  for all  $n \geq 3$ . Therefore  $\mathcal{K}$  is pairwise nondisjoint.

A special case of Theorem 2 is of sufficient interest to be stated separately.

**THEOREM 3.** *Let  $S$  and  $K$  be a closed and a compact subset respectively, of a linear topological space  $L$ , such that  $(S, K)$  is of Helly type  $(n + 1, n)$  for some  $n \geq 3$ . Let us further assume that for some  $x_0 \in S$ ,  $K(x_0) = \{p\}$ ,  $p \neq x_0$ . Then either  $S$  is starshaped relative to  $p$  or  $S$  is the union of an isolated point and a set starshaped relative to  $p$ .*

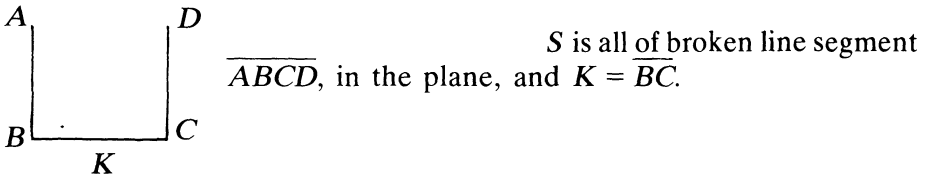
*Proof.* Suppose  $y_1$  and  $y_2$  are points in  $S \sim \{p\}$ , such that  $K(y_i) \subset \{y_i\}$ ,  $i = 1, 2$ . The set  $\{x_0, y_1, y_2\}$ , suitably expanded, lacks the

$(n + 1, n)$  property, since  $y_1$  and  $y_2$  do not see  $p$ , and  $x_0$  sees neither  $y_1$  nor  $y_2$ . Therefore there is at most one point  $y$  in  $S \sim \{p\}$  such that  $K(y) \subset \{y\}$ .

We then have, by Theorem 2, that at most one point in  $S$  does not see  $p$ . Furthermore, any such point must be isolated, by the closure of  $S$ .

REMARK. It is possible for a point  $x_0$  to be the only point with singleton visibility set. Consider the following example: Let  $S = \{(x, y) \in \mathbb{R}^2 \mid y \leq x^2, 0 \leq x \leq 1, 0 \leq y \leq 1\}$ , and  $K = \{(1, y) \mid 0 \leq y \leq 1\}$ . Let  $x_0 = (0, 0)$ . Then  $K(x_0) = \{(1, 0)\}$ . It is easily seen that  $(S, K)$  satisfies the hypothesis of Theorem 3, and that  $(0, 0)$  is the only point with the required property.

REMARK. Theorems 2 and 3 do not hold when  $(S, K)$  is of Helly type (3.2). An example is shown below.



REMARK. Theorems 2 and 3 trivially fail if the hypothesis lacks the condition that  $K(x) \neq \emptyset$ , for every  $x \in S$ .

REMARK. Let  $S$  and  $K$  be subsets of a linear topological space  $L$ , such that  $(S, K)$  is of Helly type (3, 2). If there exist points  $x, z \in S$  such that  $K(x) = \{a\}$ ,  $K(z) = \{b, c\}$ ,  $a, b, c$ , distinct, then  $S$  is a union of three starshaped sets, since an arbitrary  $w$  in  $S$  sees at least one of  $\{a, b, c\}$  via  $S$ . As we see by Breen's example [1], even with the restriction that  $S$  is a closed subset of the plane and  $K$  is a line segment we may need as many as three points to write  $S$  as a union of starshaped sets.

THEOREM 4. *Let  $S$  be a closed subset of a linear topological space  $L$ , and let  $K$  be a compact convex subset of  $S$  of finite dimension  $k$ . Suppose  $(S, K)$  has the triangle property and is of Helly type  $(2k + 2, 2k + 1)$ . Then  $S$  is the union of a starshaped set and at most one isolated point.*

*Proof.* Since the theorem is trivially true for  $k = 0$ , we assume  $k > 0$ . For arbitrary  $x \in S$ ,  $K(x)$  is compact, as was shown in the Lemma, and is also convex.

Suppose  $K(x) \neq \emptyset$  for every  $x \in S$ . If, for arbitrary  $\{x_i : x_i \in S, i = 1, 2, \dots, k + 1\}$ , the set  $\bigcap_{i=1}^{k+1} K(x_i) \neq \emptyset$ , then Helly's theorem implies  $\bigcap_{x \in S} K(x) \neq \emptyset$ , so  $S$  is starshaped. Assume  $S$  is not starshaped. Then let  $j$  be a minimal integer such that  $\bigcap_{i=1}^j K(x_i) = \emptyset$  for some collection

$\{x_i: x_i \in S, i = 1, 2, \dots, j\}$ . Then  $j \geq 2$  since  $K(x) \neq \emptyset$  for all  $x$ , and  $j \leq k + 1$  by assumption.

Consider  $(S \sim \{x_1, \dots, x_j\}, K)$ . This pair is of Helly type  $(2k + 2 - j, 2k + 2 - j)$ : for given an arbitrary collection of  $2k + 2 - j$  points from  $S \sim \{x_1, \dots, x_j\}$ , augment the collection with  $\{x_1, \dots, x_j\}$ , making a total of  $2k + 2$  points of  $S$ . By hypothesis at least  $2k + 1$  of these points must see a point of  $K$  in common. One point from the  $2k + 2$  points in  $S$  must fail to see the point in  $K$ , in fact, a point from the set  $\{x_1, \dots, x_j\}$  since otherwise the assumption that  $\bigcap_{i=1}^{2k+2} K(x_i) = \emptyset$  would be violated. Therefore all of the  $2k + 2 - j$  points from  $S \sim \{x_1, \dots, x_j\}$  see the point in question.

Since  $j \leq k + 1$ , it follows that  $2k + 2 - j \geq k + 1$ , so the pair  $(S \sim \{x_1, \dots, x_j\}, K)$  is of Helly type  $(k + 1, k + 1)$  as well, and consequently, by Helly's theorem  $S \sim \{x_1, \dots, x_j\}$  is starshaped. Then the closure of  $S \sim \{x_1, \dots, x_j\}$  is also starshaped. Our assumption that  $S$  is not starshaped implies that there is an integer  $i$ ,  $1 \leq i \leq j$ , such that  $x_i$  is not in the closure of  $S \sim \{x_1, \dots, x_j\}$ . Therefore  $x_i$  has a neighborhood containing no points of  $S \sim \{x_1, \dots, x_j\}$ , and sees no points of  $K$  via  $S$ , which contradicts that  $K(x_i) \neq \emptyset$ . Therefore  $S$  is starshaped.

On the other hand, suppose for some  $x_0 \in S$ ,  $K(x_0) = \emptyset$ . Then  $x_0$  is the only point of  $S$  with empty visibility set, and  $(S \sim \{x_0\}, K)$  is of Helly type  $(2k + 1, 2k + 1)$ . By Helly's theorem, the collection  $\{K(y): y \in S \sim \{x_0\}\}$  has a nonvoid intersection, so  $S \sim \{x_0\}$  is starshaped.  $S$  consists of the starshaped set  $S \sim \{x_0\}$  and the point  $\{x_0\}$ . Closure of  $S$  implies that  $x_0$  is isolated.

## REFERENCES

1. M. Breen, *An Example Concerning Unions of Two Starshaped Sets in the Plane*, Israel J. of Math., **17** #4 (1974), 347-349.
2. H. Hadwiger and H. DeBrunner, *Über eine Variante zum Hellyschen Satz*, Arch. Math., **8** (1957), 309-313.
3. F. A. Valentine, *Convex Sets*, McGraw-Hill, New York (1964).

Received July 2, 1975 and in revised form October 9, 1975. Partial support for the second author was provided by Pennsylvania State University through Grant PDE-OCC-EDUC-PROG IV #3412.

LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA

AND

PENNSYLVANIA STATE UNIVERSITY, WORTHINGTON SCRANTON CAMPUS







Mieczyslaw Altman, <i>Contractor directions, directional contractors and directional contractions for solving equations</i> .....	1
Michael Peter Anderson, <i>Subgroups of finite index in profinite groups</i> .....	19
Zvi Arad, <i>Abelian and nilpotent subgroups of maximal order of groups of odd order</i> .....	29
John David Baidon and Ruth Silverman, <i>On starshaped sets and Helly-type theorems</i> .....	37
John W. Baker and R. C. Lacher, <i>Some mappings which do not admit an averaging operator</i> .....	43
Joseph Barback, <i>Composite numbers and prime regressive isols</i> .....	49
David M. Boyd, <i>Composition operators on <math>H^p(A)</math></i> .....	55
Maurice Chacron, <i>Co-radical extension of PI rings</i> .....	61
Fred D. Crary, <i>Some new engulfing theorems</i> .....	65
Victor Dannon and Dany Leviatan, <i>A representation theorem for convolution transform with determining function in <math>L^p</math></i> .....	81
Mahlon M. Day, <i>Lumpy subsets in left-amenable locally compact semigroups</i> .....	87
Michael A. Gauger, <i>Some remarks on the center of the universal enveloping algebra of a classical simple Lie algebra</i> .....	93
David K. Haley, <i>Equational compactness and compact topologies in rings satisfying A.C.C</i> .....	99
Raymond Heitmann, <i>Generating ideals in Prüfer domains</i> .....	117
Gerald Norman Hile, <i>Entire solutions of linear elliptic equations with Laplacian principal part</i> .....	127
Richard Oscar Hill, <i>Moore-Postnikov towers for fibrations in which <math>\pi_1(\text{fiber})</math> is non-abelian</i> .....	141
John Rast Hubbard, <i>Approximation of compact homogeneous maps</i> .....	149
Russell L. Merris, <i>Relations among generalized matrix functions</i> .....	153
V. S. Ramamurthi and Edgar Andrews Rutter, <i>On cotorsion radicals</i> .....	163
Ralph Tyrrell Rockafellar and Roger Jean-Baptiste Robert Wets, <i>Stochastic convex programming: basic duality</i> .....	173
Alban J. Roques, <i>Local evolution systems in general Banach spaces</i> .....	197
I. Bert Russak, <i>An indirect sufficiency proof for problems with bounded state variables</i> .....	219
Richard Alexander Sanerib, Jr., <i>Ultrafilters and the basis property</i> .....	255
H. A. Seid, <i>The decomposition of multiplication operators on <math>L_p</math>-spaces</i> .....	265
Franklin D. Tall, <i>The density topology</i> .....	275
John Campbell Wells, <i>Invariant manifolds on non-linear operators</i> .....	285
James Chin-Sze Wong, <i>A characterization of topological left thick subsets in locally compact left amenable semigroups</i> .....	295