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**SOME MAPPINGS WHICH DO NOT ADMIT AN AVERAGING  
OPERATOR**

JOHN W. BAKER AND R. C. LACHER

## SOME MAPPINGS WHICH DO NOT ADMIT AN AVERAGING OPERATOR

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The problem of determining for spaces  $X$  and  $Y$  necessary and sufficient conditions such that there exists a map  $\phi$  of  $X$  onto  $Y$  which does not admit an averaging operator is considered. This corresponds to identifying the uncomplemented closed selfadjoint subalgebras of  $C(X)$  which contain  $1_X$ . Mappings  $\phi$  of  $X$  onto  $Y$  are constructed which do not admit averaging operators, for example, when  $X$  is any uncountable compact metric space and  $Y$  is any countable product of intervals. Also,  $X$  can be any space containing an open set homeomorphic to a Banach space and  $Y = X$ . These results generalize earlier work by D. Amir and S. Ditor.

If  $\phi$  is a mapping of  $X$  onto  $Y$ , the induced operator  $\phi^0$  from  $C(Y)$  to  $C(X)$  that takes  $f \in C(Y)$  to  $f \circ \phi \in C(X)$  is a multiplicative isometric isomorphism. In case  $\phi$  is a quotient map (e.g., if  $X$  and  $Y$  are compact Hausdorff spaces) then  $\phi^0(C(Y))$  consists of all functions in  $C(X)$  which are constant on each point inverse of  $\phi$ . We say  $\phi$  admits an averaging operator if there is a projection of  $C(X)$  onto  $\phi^0(C(Y))$ . It is easily seen that  $\phi$  admits an averaging operator if and only if there exists a bounded linear operator  $u$  from  $C(X)$  into  $C(Y)$  such that  $u\phi^0(f) = f$  for each  $f \in C(Y)$  (see [12], Cor. 3.2), and in this case  $u$  is called an averaging operator for  $\phi$ .

Following the appearance of the monograph by A. Pelczynski on averaging and extension operators [12], there has been much interest in the study of averaging operators (e.g., see [2], [3], [4], [5], [6], [15]). A central problem in this study, known as the complemented subalgebra problem, is to determine necessary and sufficient conditions for a map  $\phi$  from a compact Hausdorff space  $X$  onto a compact Hausdorff space  $Y$  to admit an averaging operator. Strong necessary conditions have been established in [5]. (Also, see [2] and [3].) Two closely related problems are to determine for compact Hausdorff spaces  $X$  and  $Y$  necessary and sufficient conditions that there exists a map  $\phi$  of  $X$  onto  $Y$  which (1. admits; 2. does not admit) an averaging operator. Since this corresponds to determining the complemented and uncomplemented closed selfadjoint subalgebras of  $C(X)$  which contain  $1_X$  by Stone's Theorem [14, p. 122], results of this type yield information about the structure of  $C(X)$ .

In 1968, S. Ditor established that there is a map  $\phi$  of  $[0, 1]$  onto itself

which does not admit an averaging operator (see [6] and also [5]). In [3], it was shown that if a topological space  $X$  contains an open 0-dimensional compact metric space  $K$  with  $K^{(\omega)}$  nonempty, then there is a map  $\phi$  of  $X$  onto itself which does not admit an averaging operator. The same result was also established if  $K$  is a first-countable compact subset of  $X$  and  $\text{Int}(K)^{(n)}$  contains an isolated point for each integer  $n$ . It has recently been shown [4] that if  $X$  and  $Y$  are compact metric spaces with  $|X^{(\alpha)}| \geq |Y^{(\alpha)}|$  for each ordinal number  $\alpha$ ,  $X$  is 0-dimensional, and  $Y^{(\omega)}$  is nonempty, there is a map  $\phi$  of  $X$  onto  $Y$  which does not admit an averaging operator. (Also, see [4] for other related results.)

All of the preceding results except the one by Ditor require the space  $X$  to be 0-dimensional. In this paper, we continue this study by considering Hausdorff spaces  $X$  and  $Y$  which are not necessarily 0-dimensional and establishing sufficient conditions such that there will exist a map  $\phi$  of  $X$  onto  $Y$  which does not admit an averaging operator. For example, we show that if  $X$  is locally a Banach space at some point, then there is a map  $\phi$  of  $X$  onto itself which does not admit an averaging operator (Theorem 2). The same conclusion holds if  $X = I^\alpha$  for any cardinal number  $\alpha \geq 1$  (Corollary 1.1). Another corollary is that if  $X$  is any nondispersed compact Hausdorff space and  $Y$  is any cube  $I^\alpha$ ,  $1 \leq \alpha \leq \aleph_0$ , then there exists a map  $\phi$  of  $X$  onto  $Y$  which does not admit an averaging operator (Corollary 3.1). These results generalize the previously mentioned result by Ditor and the well-known theorem by D. Amir [1] that  $C[0, 1]$  contains an uncomplemented subspace isometrically isomorphic to  $C[0, 1]$ .

The terminology used herein is standard and follows that in Dunford and Schwartz's *Linear Operators I* [9] and Dugundji's *Topology* [7]. We let  $I = [0, 1]$ .

Let  $S$  be a topological space. The cone  $K$  over  $S$  is the quotient space  $(I \times S)/R$  where  $R$  is the equivalence relation  $(0, x) \sim (0, x')$  for all  $x, x' \in S$  (see [7, p. 126]). The vertex of this cone is  $v = \{0\} \times S$  and  $S$  is identified with the base  $\{1\} \times S$ . Let  $Y = I \times K$  and  $\dot{Y} = (\{0, 1\} \times K) \cup (I \times S)$ . Frequently,  $\dot{Y}$  is the boundary of  $Y$ . The preceding assumptions about  $Y$  are satisfied by many topological spaces. For example, the closed unit ball  $K = \{x \in B \mid \|x\| \leq 1\}$  in a Banach space  $B$  is the cone on the unit sphere  $S = \{x \in B \mid \|x\| = 1\}$  and the cone on the cube  $I^\alpha$  for  $\alpha \geq 0$  is homeomorphic to  $I^{\alpha+1}$  ( $\alpha$  finite) or  $I^\alpha$  ( $\alpha$  infinite).

**THEOREM 1.** *There exists a map  $\phi$  of  $Y$  onto itself such that  $\phi(y) = y$  for each  $y \in \dot{Y}$  and  $\phi$  does not admit an averaging operator.*

*Proof.* Let  $\phi_0: I \rightarrow I$  be a monotone map such that  $\phi_0(0) = 0$  and  $\phi_0(1) = 1$ . Define a map  $\tilde{\phi}$  from  $I \times I \times S$  onto itself by

$$\tilde{\phi}(t, t', s) = (tt' + (1 - t')\phi_0(t), t', s)$$

and let

$$\phi : I \times K \rightarrow I \times K$$

be the map induced by  $\tilde{\phi}$  on the quotient space. We claim that  $\phi$  maps  $I \times (K - \{v\})$  bijectively to itself and that  $\phi|_{\dot{Y}}$  is the identity. The second statement is obvious. For the first, suppose  $(t_1, t'_1, s_1)$  and  $(t_2, t'_2, s_2)$  are two points of  $I \times I \times S$  with  $t'_1 > 0$  such that

$$\tilde{\phi}(t_1, t'_1, s_1) = \tilde{\phi}(t_2, t'_2, s_2).$$

Then  $t'_1 = t'_2$ ,  $s_1 = s_2$ , and

$$t_1 - t_2 = \frac{1 - t'_1}{t'_1} [\phi_0(t_2) - \phi_0(t_1)].$$

Thus,  $t_1 \neq t_2$  implies  $\phi_0(t_1) \neq \phi_0(t_2)$ . The claim now follows from the fact that  $\phi_0$  is monotone, for if  $t_1 < t_2$ , then  $\phi_0(t_1) < \phi_0(t_2)$ ; hence,

$$t_1 t'_1 + (1 - t'_1)\phi_0(t_1) < t_2 t'_2 + (1 - t'_2)\phi_0(t_2)$$

and  $\tilde{\phi}(t_1, t'_1, s_1) \neq \tilde{\phi}(t_2, t'_2, s_2)$ , a contradiction.

Next, define  $E : C(I) \rightarrow C(Y)$  by

$$Ef(t, x) = f(t)$$

for  $(t, x) \in I \times K$ . Then  $E$  is a linear operator with  $\|E\| = 1$  and  $RE$  is the identity operator on  $C(I)$  where  $R : C(Y) \rightarrow C(I)$  is the restriction operator with  $Rf(t) = f(t, v)$ . Moreover, since the nondegenerate point inverses of  $\phi$  all lie in  $I \times v$  (where they are of the form  $\phi_0^{-1}(t) \times v$ ) it is clear that if  $f \in C(I)$  and  $f$  is constant on each  $\phi_0^{-1}(t)$  for each  $t \in I$ , then  $E(f)$  is constant on each  $\phi^{-1}(t, x)$  for  $(t, x) \in I \times K$ . Equivalently,  $E(\phi_0^0[C(I)]) \subset \phi^0[C(Y)]$ .

Let  $\phi_0$  be a map such that  $\phi_0^0[C(I)]$  is uncomplemented in  $C(I)$ . For example, if  $\psi$  is the Cantor map from the Cantor set  $\mathcal{C}$  onto  $I$  defined by  $\psi(\sum_{i=1}^{\infty} 2\xi_i/3^i) = \sum_{i=1}^{\infty} \xi_i/2^i$ , then  $\phi_0$  can be selected to be the map of  $I$  onto itself which extends  $\psi$  and is constant on the disjoint intervals of  $I - \mathcal{C}$  (see [5, Cor. 5.8]). Then either by Corollary 5.5 in [5] or Corollary 1.4 in [2],  $\phi_0$  does not admit an averaging operator.

Suppose  $P$  is a bounded projection of  $C(Y)$  onto  $\phi^0[C(Y)]$ . Define  $P_0 : C(I) \rightarrow C(I)$  by  $P_0 = RPE$ . Then  $P_0$  is a bounded linear operator and

$$P_0[C(I)] = RPE[C(I)] \subset R\phi^0[C(Y)] \subset \phi_0^0[C(I)].$$

Moreover, if  $f \in \phi^0[C(I)]$ , then  $Ef \in \phi^0[C(Y)]$  and  $P_0(f) = RPE(f) = RE(f) = f$ ; hence,  $P_0^2 = P_0$  and  $P_0$  is a projection of  $C(I)$  onto  $\phi^0[C(I)]$ , which is a contradiction.

**COROLLARY 1.1.** *Suppose  $X = I^\alpha$  for some cardinal  $\alpha \geq 1$ . Then there exists a map  $\phi$  of  $X$  onto itself which does not admit an averaging operator.*

*Proof.*  $I^\alpha = I \times K$  where  $K$  is always a cone except when  $\alpha = 1$ , in which case the above-mentioned result of Ditor applies.

Since the next theorem is applicable to a space  $X$  which contains an open set homeomorphic to Euclidean  $n$ -space for  $n \geq 1$ , it generalizes the previously mentioned results of Amir and Ditor.

**THEOREM 2.** *Suppose  $X$  contains an open set homeomorphic to some (nonzero) Banach space. Then there exists a map  $\phi$  of  $X$  onto itself which does not admit an averaging operator.*

*Proof.* If  $B$  is a Banach space of dimension greater than one, then  $B = R \times B_1$  where  $R$  is the real line and  $B_1$  is a Banach space. Let  $K$  be the unit ball in  $B_1$ . By Theorem 1, there exists a map  $\psi$  of  $Y = I \times K$  onto itself such that  $\psi^0[C(Y)]$  is uncomplemented in  $C(Y)$  and  $\psi$  is the identity on  $Y$ . Since  $B$  may be identified with an open set in  $X$ , we define  $\phi : X \rightarrow X$  to be  $\psi$  on  $B$  and the identity otherwise. (If  $B = R$ , we simply extend the Cantor function  $\psi : I \rightarrow I$  used by Ditor to  $\phi : X \rightarrow X$ .)

Suppose  $P$  is a projection of  $C(X)$  onto  $\phi^0[C(X)]$ . Since  $Y$  is bounded in  $B$ , there is a closed neighborhood  $V$  of  $Y$  in  $B$ . Let  $Z = Y \cup (V - \text{Int } V)$  and define  $T : C(Y) \rightarrow C(Z)$  by  $Tf(x) = f(x)$  for  $x \in Y$  and  $Tf(x) = 0$  otherwise. Then  $T$  is a linear operator with  $\|Tf\| = \|f\|$  (i.e.,  $T$  is a simultaneous extension operator). By the Borsuk-Dugundji Simultaneous Extension Theorem (see [8, p. 360] or [13, p. 37]), there is a linear operator  $E : C(Z) \rightarrow C(V)$  with  $\|Ef\| = \|f\|$  and  $Ef(x) = f(x)$  for  $x \in Z$ . Let  $M = \{f \in C(V) \mid f(x) = 0 \text{ for } x \in (V - \text{Int } V)\}$  and define  $L : M \rightarrow C(X)$  by  $Lf(x) = f(x)$  for  $x \in V$  and  $Lf(x) = 0$  otherwise. Clearly,  $L$  is a simultaneous extension operator with  $\|Lf\| = \|f\|$ . Let  $R$  be the restriction operator from  $C(X)$  onto  $C(Y)$  and define  $P_0 = RPLET$ . Clearly,  $P_0$  is a linear operator on  $C(Y)$  with  $\|P_0\| = \|P\|$ . Moreover,

$$P_0[C(Y)] \subset RP[C(X)] \subset R\phi^0[C(X)] \subset \psi^0[C(Y)]$$

and if  $f \in \psi^0[C(Y)]$ , then  $LET(f) \in \phi^0[C(X)]$  and  $P_0(f) = RPLET(f) =$

$RLET(f) = f$ . Therefore  $P_0$  is a projection of  $C(Y)$  onto  $\psi^0[C(Y)]$ , which is a contradiction.

In the next theorem, we suppose  $S$  is a locally connected compact metric space,  $K$  is the cone over  $S$ , and  $Y = I \times K$ . Recall that a topological space  $X$  is called *dispersed* if  $X$  contains no perfect subsets.

**THEOREM 3.** *Suppose  $S$  is a locally connected compact metric space and  $X$  is a nondispersed compact Hausdorff space (e.g., an uncountable compact metric space). Then there exists a map  $\phi$  of  $X$  onto  $Y$  which does not admit an averaging operator.*

*Proof.* Since  $Y$  is a nonempty locally connected continuum, it follows by the Hahn-Mazurkiewicz-Sierpinski Theorem [10, p. 256] that there is a map  $\nu$  of  $I$  onto  $Y$ . Since  $X$  is nondispersed, there is a map  $\psi$  of  $X$  onto  $I$  [11, Thm. 1]. By Theorem 1, there is a map  $\pi$  of  $Y$  onto itself such that  $\pi^0[C(Y)]$  is uncomplemented in  $C(Y)$ . Let  $\phi = \pi\nu\psi$ . We show  $\phi^0[C(Y)]$  is uncomplemented in  $C(X)$ . Suppose  $P$  is a projection of  $C(X)$  onto  $\phi^0[C(Y)]$ . If  $\lambda = \nu\psi$  and  $P_0 = (\lambda^0)^{-1}P\lambda^0$ , then  $P_0$  is a linear operator from  $C(Y)$  into  $\pi^0[C(Y)]$ . Moreover, if  $f \in \pi^0[C(Y)]$ ,  $f = \pi^0g$  for some  $g \in C(Y)$  and  $P_0(f) = (\lambda^0)^{-1}P\lambda^0(\pi^0g) = (\lambda^0)^{-1}P\phi^0(g) = (\lambda^0)^{-1}\phi^0(g) = (\lambda^0)^{-1}\lambda^0(\pi^0g) = f$ . Thus  $P_0$  is projection of  $C(Y)$  onto  $\pi^0[C(Y)]$ , a contradiction.

Since the continuous image of a dispersed space is dispersed [11], we obtain the following characterization.

**COROLLARY 3.1.** *Let  $1 \leq n \leq \aleph_0$ . If  $X$  is a compact Hausdorff space, then there is a map of  $X$  onto  $I^n$  which does not admit an averaging operator if and only if  $X$  is not dispersed.*

In particular, if  $1 \leq m, n \leq \aleph_0$ , then there is a map of  $I^m$  onto  $I^n$  which does not admit an averaging operator.

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