SOME MAPPINGS WHICH DO NOT ADMIT AN AVERAGING OPERATOR

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The problem of determining for spaces $X$ and $Y$ necessary and sufficient conditions such that there exists a map $\phi$ of $X$ onto $Y$ which does not admit an averaging operator is considered. This corresponds to identifying the uncomplemented closed selfadjoint subalgebras of $C(X)$ which contain $1_x$. Mappings $\phi$ of $X$ onto $Y$ are constructed which do not admit averaging operators, for example, when $X$ is any uncountable compact metric space and $Y$ is any countable product of intervals. Also, $X$ can be any space containing an open set homeomorphic to a Banach space and $Y = X$. These results generalize earlier work by D. Amir and S. Ditor.

If $\phi$ is a mapping of $X$ onto $Y$, the induced operator $\phi^0$ from $C(Y)$ to $C(X)$ that takes $f \in C(Y)$ to $f \circ \phi \in C(X)$ is a multiplicative isometric isomorphism. In case $\phi$ is a quotient map (e.g., if $X$ and $Y$ are compact Hausdorff spaces) then $\phi^0(C(Y))$ consists of all functions in $C(X)$ which are constant on each point inverse of $\phi$. We say $\phi$ admits an averaging operator if there is a projection of $C(X)$ onto $\phi^0(C(Y))$. It is easily seen that $\phi$ admits an averaging operator if and only if there exists a bounded linear operator $u$ from $C(X)$ into $C(Y)$ such that $u \phi^0(f) = f$ for each $f \in C(Y)$ (see [12], Cor. 3.2), and in this case $u$ is called an averaging operator for $\phi$.

Following the appearance of the monograph by A. Pelczynski on averaging and extension operators [12], there has been much interest in the study of averaging operators (e.g., see [2], [3], [4], [5], [6], [15]). A central problem in this study, known as the complemented subalgebra problem, is to determine necessary and sufficient conditions for a map $\phi$ from a compact Hausdorff space $X$ onto a compact Hausdorff space $Y$ to admit an averaging operator. Strong necessary conditions have been established in [5]. (Also, see [2] and [3].) Two closely related problems are to determine for compact Hausdorff spaces $X$ and $Y$ necessary and sufficient conditions that there exists a map $\phi$ of $X$ onto $Y$ which (1. admits; 2. does not admit) an averaging operator. Since this corresponds to determining the complemented and uncomplemented closed selfadjoint subalgebras of $C(X)$ which contain $1_x$ by Stone’s Theorem [14, p. 122], results of this type yield information about the structure of $C(X)$.

In 1968, S. Ditor established that there is a map $\phi$ of $[0, 1]$ onto itself
which does not admit an averaging operator (see [6] and also [5]). In [3], it was shown that if a topological space $X$ contains an open 0-dimensional compact metric space $K$ with $K^{(\infty)}$ nonempty, then there is a map $\phi$ of $X$ onto itself which does not admit an averaging operator. The same result was also established if $K$ is a first-countable compact subset of $X$ and $\text{Int}(K)\alpha \eta$ contains an isolated point for each integer $n$. It has recently been shown [4] that if $X$ and $Y$ are compact metric spaces with $|X^{(\alpha)}| \geq |Y^{(\alpha)}|$ for each ordinal number $\alpha$, $X$ is 0-dimensional, and $Y^{(\omega)}$ is nonempty, there is a map $\phi$ of $X$ onto $Y$ which does not admit an averaging operator. (Also, see [4] for other related results.)

All of the preceding results except the one by Ditor require the space $X$ to be 0-dimensional. In this paper, we continue this study by considering Hausdorff spaces $X$ and $Y$ which are not necessarily 0-dimensional and establishing sufficient conditions such that there will exist a map $\phi$ of $X$ onto $Y$ which does not admit an averaging operator. For example, we show that if $X$ is locally a Banach space at some point, then there is a map $\phi$ of $X$ onto itself which does not admit an averaging operator (Theorem 2). The same conclusion holds if $X = I^\alpha$ for any cardinal number $\alpha \geq 1$ (Corollary 1.1). Another corollary is that if $X$ is any nondispersed compact Hausdorff space and $Y$ is any cube $I^\alpha$, $1 \leq \alpha \leq \aleph_0$, then there exists a map $\phi$ of $X$ onto $Y$ which does not admit an averaging operator (Corollary 3.1). These results generalize the previously mentioned result by Ditor and the well-known theorem by D. Amir [1] that $C[0, 1]$ contains an uncomplemented subspace isometrically isomorphic to $C[0, 1]$.

The terminology used herein is standard and follows that in Dunford and Schwartz’s Linear Operators I [9] and Dugundji’s Topology [7]. We let $I = [0, 1]$.

Let $S$ be a topological space. The cone $K$ over $S$ is the quotient space $(I \times S)/R$ where $R$ is the equivalence relation $(0, x) \sim (0, x')$ for all $x, x' \in S$ (see [7, p. 126]). The vertex of this cone is $v = \{0\} \times S$ and $S$ is identified with the base $\{1\} \times S$. Let $Y = I \times K$ and $\hat{Y} = (\{0, 1\} \times K) \cup (I \times S)$. Frequently, $\hat{Y}$ is the boundary of $Y$. The preceding assumptions about $Y$ are satisfied by many topological spaces. For example, the closed unit ball $K = \{x \in B \mid \|x\| \leq 1\}$ in a Banach space $B$ is the cone on the unit sphere $S = \{x \in B \mid \|x\| = 1\}$ and the cone on the cube $I^\alpha$ for $\alpha \geq 0$ is homeomorphic to $I^{\alpha+1}$ ($\alpha$ finite) or $I^\alpha$ ($\alpha$ infinite).

**Theorem 1.** There exists a map $\phi$ of $Y$ onto itself such that $\phi(y) = y$ for each $y \in \hat{Y}$ and $\phi$ does not admit an averaging operator.

**Proof.** Let $\phi_0 : I \to I$ be a monotone map such that $\phi_0(0) = 0$ and $\phi_0(1) = 1$. Define a map $\tilde{\phi}$ from $I \times I \times S$ onto itself by
and let
\[ \phi : I \times K \rightarrow I \times K \]
be the map induced by \( \bar{\phi} \) on the quotient space. We claim that \( \phi \) maps \( I \times (K - \{v\}) \) bijectively to itself and that \( \phi|\bar{Y} \) is the identity. The second statement is obvious. For the first, suppose \((t_1, t_1', s_1)\) and \((t_2, t_2', s_2)\) are two points of \( I \times I \times S \) with \( t_1' > 0 \) such that
\[ \bar{\phi}(t_1, t_1', s_1) = \bar{\phi}(t_2, t_2', s_2). \]
Then \( t_1' = t_2' \), \( s_1 = s_2 \), and
\[ t_1 - t_2 = \frac{1 - t_1'}{t_1'} [\phi_0(t_2) - \phi_0(t_1)]. \]
Thus, \( t_1 \neq t_2 \) implies \( \phi_0(t_1) \neq \phi_0(t_2) \). The claim now follows from the fact that \( \phi_0 \) is monotone, for if \( t_1 < t_2 \), then \( \phi_0(t_1) < \phi_0(t_2) \); hence,
\[ t_1 t_1' + (1 - t_1')\phi_0(t_1) < t_2 t_2' + (1 - t_2')\phi_0(t_2) \]
and \( \bar{\phi}(t_1, t_1', s_1) \neq \bar{\phi}(t_2, t_2', s_2) \), a contradiction.

Next, define \( E : C(I) \rightarrow C(Y) \) by
\[ Ef(t, x) = f(t) \]
for \((t, x) \in I \times K\). Then \( E \) is a linear operator with \( \|E\| = 1 \) and \( RE \) is the identity operator on \( C(I) \) where \( R : C(Y) \rightarrow C(I) \) is the restriction operator with \( Rf(t) = f(t, v) \). Moreover, since the nondegenerate point inverses of \( \phi \) all lie in \( I \times v \) (where they are of the form \( \phi^{-1}(t) \times v \)) it is clear that if \( f \in C(I) \) and \( f \) is constant on each \( \phi^{-1}(t) \) for each \( t \in I \), then \( E(f) \) is constant on each \( \phi^{-1}(t, x) \) for \((t, x) \in I \times K\). Equivalently, \( E(\phi_0[C(I)]) \subset \phi_0[C(Y)] \).

Let \( \phi_0 \) be a map such that \( \phi_0[C(I)] \) is uncomplemented in \( C(I) \). For example, if \( \psi \) is the Cantor map from the Cantor set \( \mathcal{C} \) onto \( I \) defined by \( \psi(\Sigma_{i=1}^n 2\xi_i/3^i) = \Sigma_{i=1}^n \xi_i/2^i \), then \( \phi_0 \) can be selected to be the map of \( I \) onto itself which extends \( \psi \) and is constant on the disjoint intervals of \( I - \mathcal{C} \) (see [5, Cor. 5.8]). Then either by Corollary 5.5 in [5] or Corollary 1.4 in [2], \( \phi_0 \) does not admit an averaging operator.

Suppose \( P \) is a bounded projection of \( C(Y) \) onto \( \phi_0[C(Y)] \). Define \( P_0 : C(I) \rightarrow C(I) \) by \( P_0 = RPE \). Then \( P_0 \) is a bounded linear operator and
\[ P_0[C(I)] = RPE[C(I)] \subset R\phi_0[C(Y)] \subset \phi_0[C(I)]. \]
Moreover, if \( f \in \phi_0^[[C(I)]] \), then \( Ef \in \phi^[[C(Y)]] \) and \( p_0(f) = \text{RPE}(f) = \text{RE}(f) = f \); hence, \( p_0 = p_0 \) and \( p_0 \) is a projection of \( C(I) \) onto \( \phi_0^[[C(I)]] \), which is a contradiction.

**Corollary 1.1.** Suppose \( X = I^\alpha \) for some cardinal \( \alpha \geq 1 \). Then there exists a map \( \phi \) of \( X \) onto itself which does not admit an averaging operator.

**Proof.** \( I^\alpha = I \times K \) where \( K \) is always a cone except when \( \alpha = 1 \), in which case the above-mentioned result of Ditor applies.

Since the next theorem is applicable to a space \( X \) which contains an open set homeomorphic to Euclidean \( n \)-space for \( n \geq 1 \), it generalizes the previously mentioned results of Amir and Ditor.

**Theorem 2.** Suppose \( X \) contains an open set homeomorphic to some (nonzero) Banach space. Then there exists a map \( \phi \) of \( X \) onto itself which does not admit an averaging operator.

**Proof.** If \( B \) is a Banach space of dimension greater than one, then \( B = R \times B_1 \) where \( R \) is the real line and \( B_1 \) is a Banach space. Let \( K \) be the unit ball in \( B_1 \). By Theorem 1, there exists a map \( \psi \) of \( Y = I \times K \) onto itself such that \( \psi^[[C(Y)]] \) is uncomplemented in \( C(Y) \) and \( \psi \) is the identity on \( Y \). Since \( B \) may be identified with an open set in \( X \), we define \( \phi : X \to X \) to be \( \psi \) on \( B \) and the identity otherwise. (If \( B = R \), we simply extend the Cantor function \( \psi : I \to I \) used by Ditor to \( \phi : X \to X \).

Suppose \( P \) is a projection of \( C(X) \) onto \( \phi_0^[[C(X)]] \). Since \( Y \) is bounded in \( B \), there is a closed neighborhood \( V \) of \( Y \) in \( B \). Let \( Z = Y \cup (V - \text{Int } V) \) and define \( T : C(Y) \to C(Z) \) by \( Tf(x) = f(x) \) for \( x \in Y \) and \( Tf(x) = 0 \) otherwise. Then \( T \) is a linear operator with \( \|Tf\| = \|f\| \) (i.e., \( T \) is a simultaneous extension operator). By the Borsuk-Dugundji Simultaneous Extension Theorem (see [8, p. 360] or [13, p. 37]), there is a linear operator \( E : C(Z) \to C(V) \) with \( \|Ef\| = \|f\| \) and \( Ef(x) = f(x) \) for \( x \in Z \). Let \( M = \{f \in C(V) | f(x) = 0 \text{ for } x \in (V - \text{Int } V) \} \) and define \( L : M \to C(X) \) by \( Lf(x) = f(x) \) for \( x \in V \) and \( Lf(x) = 0 \) otherwise. Clearly, \( L \) is a simultaneous extension operator with \( \|Lf\| = \|f\| \). Let \( R \) be the restriction operator from \( C(X) \) onto \( C(Y) \) and define \( p_0 = \text{RPLET} \). Clearly, \( p_0 \) is a linear operator on \( C(Y) \) with \( \|p_0\| = \|P\| \). Moreover,

\[
P_0[C(Y)] \subset \text{RPE}[C(X)] \subset R\phi^[[C(X)]] \subset \phi_0^[[C(Y)]]
\]

and if \( f \in \psi_0^[[C(Y)]] \), then \( \text{LET}(f) \in \phi_0^[[C(X)]] \) and \( p_0(f) = \text{RPLET}(f) = \)
RLET(f) = f. Therefore $P_0$ is a projection of $C(Y)$ onto $\psi^0[C(Y)]$, which is a contradiction.

In the next theorem, we suppose $S$ is a locally connected compact metric space, $K$ is the cone over $S$, and $Y = I \times K$. Recall that a topological space $X$ is called dispersed if $X$ contains no perfect subsets.

**Theorem 3.** Suppose $S$ is a locally connected compact metric space and $X$ is a nondispersed compact Hausdorff space (e.g., an uncountable compact metric space). Then there exists a map $\phi$ of $X$ onto $Y$ which does not admit an averaging operator.

**Proof.** Since $Y$ is a nonempty locally connected continuum, it follows by the Hahn-Mazurkiewicz-Sierpinski Theorem [10, p. 256] that there is a map $\nu$ of $I$ onto $Y$. Since $X$ is nondispersed, there is a map $\psi$ of $X$ onto $I$ [11, Thm. 1]. By Theorem 1, there is a map $\pi$ of $Y$ onto itself such that $\pi^0[C(Y)]$ is uncomplemented in $C(Y)$. Let $\phi = \pi \psi$. We show $\phi^0[C(Y)]$ is uncomplemented in $C(X)$. Suppose $P$ is a projection of $C(X)$ onto $\phi^0[C(Y)]$. If $\lambda = \nu \psi$ and $P_0 = (\lambda^0)^{-1}P\lambda^0$, then $P_0$ is a linear operator from $C(Y)$ into $\pi^0[C(Y)]$. Moreover, if $f \in \pi^0[C(Y)]$, $f = \pi^0 g$ for some $g \in C(Y)$ and $P_0(f) = (\lambda^0)^{-1}P\lambda^0(\pi^0 g) = (\lambda^0)^{-1}P\phi^0(g) = (\lambda^0)^{-1}\phi^0(g) = (\lambda^0)^{-1}\lambda^0(\pi^0 g) = f$. Thus $P_0$ is projection of $C(Y)$ onto $\pi^0[C(Y)]$, a contradiction.

Since the continuous image of a dispersed space is dispersed [11], we obtain the following characterization.

**Corollary 3.1.** Let $1 \leq n \leq N$. If $X$ is a compact Hausdorff space, then there is a map of $X$ onto $I^n$ which does not admit an averaging operator if and only if $X$ is not dispersed.

In particular, if $1 \leq m$, $n \leq N$, then there is a map of $I^m$ onto $I^n$ which does not admit an averaging operator.

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