Pacific Journal of Mathematics

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Vol. 62, No. 1 January 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 1, 1976

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Let $w(x)$, be the principal function for the set of all composite nonnegative integers and let D_w denote its canonical extension to the isols. M. Hassett proved in Comp. M. 26 (1973), as an application of a very general theorem, that if A is any T-regressive isol then $D_w(A)$ is a regressive isol that is prime. The present paper contains a number theoretic proof of the following property: if Y is any infinite multiple-free regressive isol then $D_w(Y)$ is a regressive isol that is prime.

1. Preliminaries. We will assume that the reader is familiar with the main results given in **[1]**, **[2]** and **[4]**. We use the notation of **[2]**. W will denote the set of all composite positive integers $4, 6, 8, \cdots$. The principal function of W will be denoted by $w(x)$ and D_w will denote the canonical extension of $w(x)$ to the isols. Let A be any regressive isol. Then, from results in **[1]**, the value of $D_w(A)$ is a regressive isol, and, if A is infinite then $D_w(A)$ is also infinite.

LEMMA 1. *Let A be an infinite regressive isol. Then there exist infinite regressive isols U and V such that*

$$(1) \qquad A + 2 = U + V, \quad \textit{and}$$

$$(2) \qquad D_w(A) = U + 2V.$$

Proof. We will first show that increasing recursive functions $u(x)$ and $v(x)$ can be defined such that

$$(3) \qquad x + 2 = u(x) + v(x), \quad \text{and}$$

$$(4) \qquad w(x) = u(x) + 2v(x).$$

Define $u(0) = 0$ and $v(0) = 2$. Observe that for each number x, the value of $e(x) = w(x+1) - w(x)$ is either 1 or 2. Assuming that the values of $u(x)$ and $v(x)$ have been defined, let $u(x+1) = u(x) + 1$ and $v(x+1) = v(x)$ if $e(x) = 1$, and $u(x+1) = u(x)$ and $v(x+1) = v(x) + 1$ if $e(x) = 2$. It is easy to verify that $u(x)$ and $v(x)$ are increasing recursive functions and satisfy statements (3) and (4). From (3) and (4), and the Nerode metatheorem for such statements, it follows that

(5) $$A+2=D_u(A)+D_v(A), \quad \text{and}$$

(6) $$D_w(A)=D_u(A)+2D_v(A).$$

$D_u(A)$ and $D_v(A)$ are regressive isols, since A is a regressive isol and each of the functions $u(x)$ and $v(x)$ is increasing recursive. Also, each of these isols is infinite. This fact follows by first noting that among the numbers $e(0), e(1), \cdots$ each of the values 1 and 2 will occur infinitely often. Combining this feature with the definitions of the functions $u(x)$ and $v(x)$, and with the way in which the canonical extension of a recursive function can be represented as an infinite series of isols (cf. **[1]**), it follows that both $D_u(A)$ and $D_v(A)$ are infinite. If we let $U=D_u(A)$ and $V=D_v(A)$, then the desired result of the lemma is obtained.

Let $\left[\frac{x}{2}\right]$ denote the function that is the greatest integer obtained when the number x is divided by 2. Note, $\left[\frac{x}{2}\right]$ is an increasing recursive function. Therefore its canonical extension to the isols, written as $\left[\frac{X}{2}\right]$, will map regressive isols to regressive isols.

LEMMA 2. *For all numbers d, m and n,*

$$2d=m+n \rightarrow d=\left[\frac{m}{2}\right]+\left[\frac{n+1}{2}\right].$$

LEMMA 3. *For all numbers m,*

$$\left[\frac{3m}{2}\right]=2\left[\frac{m}{2}\right]+\left[\frac{m+1}{2}\right].$$

Proofs for each of these lemmas is easy to obtain and will be omitted. Because the functions that appear in their statements are increasing recursive, it follows that each of the lemmas has an analogue that is true in the regressive isols (when the recursive functions are replaced by their respective canonical extensions to the isols). *We therefore have, for all regressive isols D, M and N,*

(7) $$2D=M+N \rightarrow D=\left[\frac{M}{2}\right]+\left[\frac{N+1}{2}\right], \quad \text{and}$$

(8) $$\left[\frac{3M}{2}\right]=2\left[\frac{M}{2}\right]+\left[\frac{M+1}{2}\right].$$

If X is an infinite regressive isol, then $\left[\frac{X}{2}\right]$ is also an infinite regressive isol. This fact follows by observing that in the infinite series representation of $\left[\frac{X}{2}\right]$ the e-difference function associated with $\left[\frac{x}{2}\right]$ will be positive infinitely often.

REMARK. Later, in the proof of the main theorem, we will apply the representations given in Lemma 1 and the features expressed in (7) and (8). In addition, we will also use the cancellation property for isols, given by Dekker and Myhill in [**5**, Theorem 40], which states that if A and B are any isols, then

$$2A \leqq 2B \rightarrow A \leqq B. \tag{9}$$

Statement (9) is applied later just in the special case that both A and B are regressive isols. We would like to show that this special case of the cancellation property may be obtained from (7). Observe first that $\left[\frac{2x}{2}\right] = x$ is an identity that is true for all numbers x. Therefore also, $\left[\frac{2X}{2}\right] = X$ is an identity that is true for all regressive isols X. Assume A and B are regressive isols and that $2A \leqq 2B$. Then there is an isol T such that $2A + T = 2B$, and, in view of [**4**, P23 and P24], we may assume that T is a regressive isol. If we now combine these facts with (7), then the cancellation property in the special case A and B are regressive isols may be obtained in the following way:

$$\begin{aligned} 2A \leqq 2B &\rightarrow 2A + T = 2B \\ &\rightarrow B = \left[\frac{2A}{2}\right] + \left[\frac{T+1}{2}\right] \\ &\rightarrow B = A + \left[\frac{T+1}{2}\right] \\ &\rightarrow A \leqq B. \end{aligned}$$

2. Multiple-free isols and the main theorem. An infinite isol Y is called *multiple-free* if for every isol B, $2B \leqq Y$ implies B is finite. Multiple-free isols were introduced and studied in [**5**]. An example of an infinite multiple-free isol that is regressive appears in [**3**]. Note that if Y is multiple-free and A is any infinite isol with $A \leqq Y$, then A is also multiple-free. Prime isols were introduced in [**5**]. Because any isol that is a predecessor of a regressive isol is also regressive, it is possible to characterize regressive isols that are prime in the following way: a regressive isol P is *prime* if it is not possible to factor P as $P=(X+2)(Y+2)$, for any regressive isols X and Y.

THEOREM. *Let A be an infinite multiple-free regressive isol. Then $D_w(A)$ is an infinite regressive isol that is prime.*

Proof. Since A is an infinite regressive isol, it follows that $D_w(A)$ is also an infinite regressive isol. Note that, from Lemma 1, (1) and (2), it follows that

$$D_w(A)+U=2(A+2), \tag{10}$$

where U is an infinite regressive isol. To show that $D_w(A)$ is prime, let us assume otherwise.

Case 1. $D_w(A)=2Y$, for some infinite regressive isol Y. If we then substitute in (10), we obtain

$$2Y+U=2(A+2). \tag{11}$$

Therefore $2Y \leqq 2(A+2)$ and also, from (9), $Y \leqq A+2$. If we now solve in (11) for U, we obtain

$$U=2(A+2-Y).$$

Hence U is an infinite regressive isol that is not multiple-free. However, from (1), U is a predecessor of $A+2$. Since A is multiple-free, $A+2$ will be also. But then U is multiple-free.

Case 2. $D_w(A)=(3+S)Y$, for some regressive isol S and infinite regressive isol Y. Beginning with the given representation of $D_w(A)$, and then successively applying the statements (10), (7), and (8), gives

$$\begin{aligned} D_w(A)=(3+S)Y &\to 2(A+2)=3Y+SY+U \\ &\to A+2=\left[\frac{3Y}{2}\right]+\left[\frac{SY+U+1}{2}\right] \\ &\to A+2=2\left[\frac{Y}{2}\right]+\left[\frac{Y+1}{2}\right]+\left[\frac{SY+U+1}{2}\right]. \end{aligned}$$

Therefore,

$$2\left[\frac{Y}{2}\right] \leqq A+2. \tag{12}$$

Since Y is an infinite regressive isol, $\left[\frac{Y}{2}\right]$ is also. From (12), we obtain a contradiction to the multiple-free property of $A + 2$.

By these two cases, we may conclude that $D_w(A)$ is an infinite prime regressive isol.

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Received June 23, 1975 and in revised form October 14, 1975.

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Pacific Journal of Mathematics

Vol. 62, No. 1 January, 1976