AN INDIRECT SUFFICIENCY PROOF FOR PROBLEMS WITH BOUNDED STATE VARIABLES

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A set of sufficient conditions is obtained for problems involving constraints of the form $\psi^\alpha(t, x) \leqq 0$ $\alpha = 1, \cdots, m$. The method of proof is indirect. It is shown by essentially strengthening the first and second order necessary results previously obtained by the author for problems of this type, that a proper strong relative minimum is obtained.

1. Introduction. Consider the class of arcs

$$a: x^i(t), \qquad p^k(t), \qquad b^\sigma, \qquad t^0 \leqq t \leqq t^1$$

$$i = 1, \cdots, N \qquad k = 1, \cdots, K \qquad \sigma = 1, \cdots, r$$

where[1] elements $(t, x(t), p(t))$ and b lie respectively in open sets R in txp space and B in b space. The terms x^i are called state variables while the terms p^k, b^σ are called control variables and control parameters respectively. We require these arcs to satisfy the constraints

(1-1) $\quad \dot{x}^i(t) = f^i(t, x(t), p(t)), \qquad i = 1, \cdots, N \qquad$ a.e. in $[t^0, t^1]$

(1-2) $\quad \psi^\alpha(t, x(t)) \leqq 0 \qquad \alpha = 1, \cdots, m \qquad [t^0, t^1]$

(1-3) $\quad I_\gamma(a) \leqq 0 \qquad \gamma = 1, \cdots, p' \qquad I_\gamma(a) = 0 \qquad \gamma = p'+1, \cdots, p$

(1-4) $\qquad\qquad\qquad x^i(t^s) = X^{is}(b) \qquad s = 0, 1$

(1-5) where $\quad I_\gamma(a) = g_\gamma(b) + \displaystyle\int_{t^0}^{t^1} L_\gamma(t, x(t), p(t)) dt \qquad \gamma = 1, \cdots, p$

and are interested in minimizing the functional

(1-6) $\qquad\qquad I_0(a) = g_0(b) + \displaystyle\int_{t^0}^{t^1} L_0(t, x(t), p(t)) dt.$

Let C be the class of arcs described above with $x(t)$ absolutely continuous, $p(t)$, $f(t, x(t), p(t))$ and $L_\gamma(t, x(t), p(t))$ $\gamma = 0, 1, \cdots, p$ integrable on $[t^0, t^1]$. It is desired to minimize $I_0(a)$ on the class C.

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[1] Unless otherwise noted, the indices i, k, σ, α will have the respective ranges $i = 1, \cdots, N$, $k = 1, \cdots, K$, $\sigma = 1, \cdots, r$ and $\alpha = 1, \cdots, m$.

In [2] and [3] the author establishes first order necessary conditions for this problem. By essentially strengthening those conditions, and extending a technique devised originally by Hestenes and used by Pennisi in [1], a set of sufficient conditions for a proper strong relative minimum is obtained without the use of field theory and an invariant integral.

2. Assumptions. Using the problem defined above, the functions will be assumed to possess the following continuity properties; the functions ψ^α will be class C^3 while the functions f^i, X^{is}, g_γ, L_γ, L_0, and g_0 will be of class C^2. Also, an arc will be called admissible if it is in the class C.

Next, define the functions[2] $\phi^\alpha = \psi_t^\alpha + \psi_{x^i}^\alpha f^i$ $\alpha = 1, \cdots, m$. These functions act as derivatives of ψ^α along admissible arcs. Also define the set R_0 of points (txp) in R satisfying

(2–1) $\psi^\alpha \leqq 0$

(2–2) $\phi^\alpha \geqq 0$ for all α with $\psi^\alpha = 0$ or $\phi^\alpha \leqq 0$ for all α with $\psi^\alpha = 0$.

We shall be concerned with a particular admissible arc a_0

$$a_0\colon\ x_0^i(t), \qquad p_0^k(t), \qquad b_0^\alpha, \qquad t^0 \leqq t \leqq t^1$$

and shall make some assumptions about the arc a_0. In order to state these we first make the following definitions:

The set S^α is the set of t such that $\psi^\alpha(t, x_0(t)) = 0$.

For each t, the symbol $\Gamma(t)$ denotes the set of indices α such that $\psi^\alpha(t, x_0(t)) = 0$.

We will have need to talk of the quantities $z_i(t)$, $\mu_\alpha(t)$, λ_ρ, K^α where $z_i(t)$ are of class C^1, $\mu_\alpha(t)$ are absolutely continuous functions with continuous derivatives $\dot\mu_\alpha(t)$ and K^α, λ_ρ $(\rho = 1, \cdots, p + N)$ are constants.

For each t, the set $\Delta(t)$ is the set of α indices such that $\dot\mu_\alpha(t) \neq 0$.

The functions $\bar\psi^\alpha$ and $\bar\phi^\alpha$ are defined as

(3) $\bar\psi^\alpha \equiv \psi^\alpha / [1 + (\psi^\alpha)^2]^{\frac{1}{2}}$ $\bar\phi^\alpha \equiv \phi^\alpha / [1 + (\phi^\alpha)^2]^{\frac{1}{2}}$

Our assumptions concerning a_0 are as follows:
(i) $p_0(t)$ is continuous in $[t^0, t^1]$.
(ii)   For each t the set $\Gamma(t) - \Delta(t)$ contains at most one index.
(iii)   The matrix $(\phi_{p^k}^\alpha)$ has rank m along a_0.
(iv)   There exist the quantities $z_i(t)$, $\mu_\alpha(t)$, K^α, λ_ρ.
$(\rho = 1, \cdots, p + N)$ as referred to above, satisfying

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[2] For a function $M(t, x, p, b)$ the notations M_{x^i}, M_{p^k}, M_{b^α}, M_t will denote first partial derivatives with respect to the indicated variable. Also repeated indices will be summed.

(4-1) $\lambda_\gamma \geqq 0$ with $\lambda_\gamma = 0$ if $I_\gamma(a_0) < 0$ $1 \leqq \gamma \leqq p'$,

$$K^\alpha \geqq 0, \qquad \lambda_{p+i} = K^\alpha \psi_{x^i}^\alpha(t^0)$$

$$\mu_\alpha(t^1) = 0 \quad \text{if} \quad \psi^\alpha(t^1) < 0$$

$$\mu_\alpha(t^0) \leqq K^\alpha \quad \text{with} \quad \mu_\alpha(t^0) = K^\alpha \quad \text{if} \quad \psi^\alpha(t^0) < 0$$

and also such that the functions $\mu_\alpha(t)$ are nonincreasing nonnegative functions which are constant on intervals where[3] $\psi^\alpha(t) < 0$.   Notice that this last statement means

(4-2) $$\int_{t^0}^{t^1} \dot\mu_\alpha(t)\psi^\alpha(t)dt = 0.$$

Using the terms of (iv) define the functions

(5) $\mathcal{G}(b) \equiv g_0(b) + \lambda_\gamma g_\gamma(b) + \lambda_{p+i}X^{i0}(b)$ $\gamma = 1, \cdots, p$

(6-1) $H(t, x, p, z(t), \mu(t)) \equiv z_i(t)f^i - L_0 - \lambda_\gamma L_\gamma - \mu_\alpha(t)\phi^\alpha$

(6-2) $G(t, x, p, z(t), \mu(t)) \equiv -H(t, x, p, z(t), \mu(t)) - \dot z_i(t)x^i$

and the Weierstrass E function for G,

(7) $E_G(t, x, p, q, z(t), \mu(t)) \equiv G(t, x, q, z(t), \mu(t)) - G(t, x, p, z(t), \mu(t))$

$$- (q^k - p^k)G_{p^k}(t, x, p, z(t), \mu(t)).$$

Then the following statements are true:
      (va)   The arc a_0 satisfies the transversality relation

(8) $$d\mathcal{G} + [z_i(t^s)dX^{is}]_{s=0}^{s=1} = 0$$

(where e.g. $d\mathcal{G}$ means $\mathcal{G}_{b^\sigma}(b_0)db^\sigma$) for arbitrary vectors db.
      (vb)   The relations

(9) $\dot z_i = -H_{x^i}$ $\dot x^i = H_{z^i}$ $H_{p^k} = 0$

hold along a_0.   Note that (9) implies that

(10) $G_{x^i} = 0$ $G_{p^k} = 0$

along a_0.

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[3] Henceforth unless otherwise stated, a function $M(t, x, p, b)$ evaluated along a_0 at $(t, x_0(t), p_0(t), b_0)$ will be denoted by $M(t)$ or if just a function of b, it will be denoted by $M(b_0)$ or just M.

(vc)   There is a positive constant b and a neighborhood D_1 of a_0 relative to R_0 such that[4]

(11) $E_G(t, x, p, q, z(t), \mu(t)) + \dot{\mu}_\alpha(t)\psi^\alpha(t, x)$

$$\geqq b[E_L(p, q) + \max(\dot{\mu}_\alpha(t)\bar{\psi}^\alpha(t, x), |\dot{\mu}_\alpha(t)\bar{\phi}^\alpha(t, x, q)|)]$$

and with E_γ as the E function for L_γ, also

(12) $E_G(t, x, p, q, z(t), \mu(t)) + \dot{\mu}_\alpha(t)\psi^\alpha(t, x) \geqq b|E_\gamma(t, x, p, q)|$ $\gamma = 1, \cdots, p$

when (t, x, p) is in D_1, (t, x, q) is in R_0 and $\phi^\alpha(t, x, p) = 0$ if $\dot{\mu}_\alpha(t) \neq 0$.   The function $E_L(p, q)$ is the E function for the function

(13) $$L(p) \equiv (1 + p^k p^k)^{\frac{1}{2}}$$

which is the integrand of the length integral.   Thus

(14) $$E_L(p, q) \equiv L(q) - L(p) - (q^k - p^k)L_{p^k}(p)$$

which is the same as

$$E_L(p, q) = L(q) - \frac{1 + (p^k q^k)}{L(p)}$$

(vi)   There is a neighborhood of a_0 in tx space and a positive constant ρ such that the Lipschitz condition

(16) $$|f(t, x, p) - f(t, z, q))| < \rho[|x - z|^2 + |p - q|^2]^{\frac{1}{2}}$$

holds for all points (t, x, p) and (t, z, q) in that neighborhood.

We note that the majority of these assumptions about a_0 are either just the necessary conditions for a solution to our problem or the assumptions used in proving these necessary conditions as shown in [2] and [3].

In particular, the only assumptions listed which do not come under those headings[5] are:

(a)   the existence of $\dot{\mu}_\alpha(t)$ on S^α sets, and

(b)   the assumptions (ii), (vc) and (vi).

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[4] We shall often omit the argument t when referring to the functions $\mu_\alpha(t)$ and $\dot{\mu}_\alpha(t)$ of (4). However we shall always understand the terms μ_α and $\dot{\mu}_\alpha$ to refer to those functions.

[5] A note soon to appear will modify the results of [3] to include the last stated property in (4–1) as a necessary condition.

3. Statement of the main result and definition of an admissible variation. We next define what we shall mean by an admissible variation of the arc a_0.

A set of functions

$$(17) \qquad \delta a: \delta x^i(t), \qquad \delta p^k(t), \qquad \delta b^\sigma, \qquad t^0 \le t \le t^1$$

with $\delta x(t)$ absolutely continuous and $\delta p(t)$, $\delta \dot{x}(t)$ square integrable on $[t^0, t^1]$ will be called a variation of the arc a_0 and for any function $M(t, x, p, b)$ which possesses first partial derivatives along a_0 we call

$$(18) \qquad \delta M(t) = M_{x^i}(t)\delta x^i(t) + M_{p^k}(t)\delta p^k(t) + M_{b^\sigma}(t)\delta b^\sigma$$

the variation of M along a_0 due to the variation δa.

Next define the functionals of any arc a as:

$$(19) \qquad \begin{aligned} J_\gamma(a) &= I_\gamma(a), \qquad \gamma = 0, 1, \cdots, p \\ J_{p+i}(a) &= x^i(t^1) - X^{i1}(b) \\ J_{p+N+i}(a) &= X^{i0}(b) - x^i(t^0) \end{aligned} \left.\rule{0pt}{2.5em}\right\} \quad i = 1, \cdots, N,$$

and the variations of these functionals due to the variation δa as:

$$(20) \qquad \begin{aligned} J'_\gamma(a_0, \delta a) &= \delta g_\gamma(b_0) + \int_{t^0}^{t^1} \delta L_\gamma(t)dt \qquad \gamma = 0, 1, \cdots, p \\ J'_{p+i}(a_0, \delta a) &= \delta x^i(t^1) - \delta X^{i1}(b_0) \\ J'_{p+i}(a_0, \delta a) &= \delta X^{i0}(b_0) - \delta x^i(t^0) \end{aligned} \left.\rule{0pt}{2.5em}\right\} \quad i = 1, \cdots, N.$$

Also let γ_k be those indices $1 \le \gamma \le p'$ for which

$$(21) \qquad I_\gamma(a_0) < 0 \qquad \gamma = \gamma_k.$$

Then, we know by (4) that

$$(22) \qquad \lambda_{\gamma_k} = 0.$$

With these definitions in mind, we call a variation admissible if the following are true:

$$(23\text{-}1) \qquad \delta \dot{x}^i(t) = \delta f^i(t) \qquad \text{a.e. on} \quad [t^0, t^1]$$

$$(23\text{-}2) \qquad \delta \psi^\alpha(t) \le 0 \quad \text{on} \quad S^\alpha$$

(23-3) $\delta\psi^\alpha(t^1) = 0$ if $\mu_\alpha(t^1) \neq 0$

(23-4) $\dot\mu_\alpha(t)\delta\psi^\alpha(t) = 0$ a.e.   on $[t^0, t^1]$ (α not summed)

(23-5) $\delta\psi^\alpha(t^0) = 0$ if $\mu_\alpha(t^0) \neq K^\alpha$

(23-6) $J'_\gamma(a_0, \delta a) = 0$ if $\lambda_\gamma \neq 0$ $1 \leqq \gamma \leqq p'$

(23-7) $J'_\gamma(a_0, \delta a) \leqq 0$ if $\lambda_\gamma = 0$ $\gamma \neq \gamma_k$ $1 \leqq \gamma \leqq p'$

(23-8) $J'_\rho(a_0, \delta a) = 0$ $p' < \rho \leqq p + 2N.$

For each such admissible variation the second variation

$$(24) \quad J_2(a_0, \delta a) = [(z^i(t^s)X^{is}_{b^\sigma b^\tau})^{s=1}_{s=0} + \mathcal{G}_{b^\sigma b^\tau} + K^\alpha\psi^\alpha_{x^i_x}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]\delta b^\sigma\delta b^\tau$$

$$+ \int_{t^0}^{t^1} [G_{x^ix^j}\delta x^i\delta x^j + 2G_{x^ip^k}\delta x^i\delta p^k + G_{p^hp^k}\delta p^h\delta p^k]dt$$

$$i,j = 1, \cdots, N \qquad \sigma, \tau = 1, \cdots, r \qquad \alpha = 1, \cdots, m \qquad h, k = 1, \cdots, K,$$

(where $\mathcal{G}$, G are the functions of (5) and (6-2) respectively, and K^α are the constants referred to in assumption (iv)) is well defined.

The Theorem to be proven in this paper is:

THEOREM 3.1. *Let a_0 be an admissible arc which satisfies assumptions* (i) *through* (vi) *and suppose that $J_2(a_0, \delta a) > 0$ for every non-null admissible variation. Then there is a neighborhood F of a_0 in txb space such that the inequality $I_0(a) > I_0(a_0)$ holds for every admissible arc a in F which is different from a_0.*

It is noted that [2] proves as a second order necessary condition that $J_2(a_0, \delta a) \geqq 0$ for all such admissible variations δa as described above[6]. Thus the hypotheses of the theorem is only a strengthened necessary condition.

Henceforth unless otherwise stated, our arc a_0 will be assumed to satisfy the conditions (i) through (vi) and we shall not explicitly state this each time we refer to a_0.

4. Convergent sequences of admissible arcs.

We proceed in a manner similar to [1].   Consider a sequence[7] $\{a_k\}$ of admissible arcs which converges uniformly to a_0 in *txb* space.

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[6] Actually in [2] the condition (23-5) is replaced by $\delta\psi^\alpha(t^0) = 0$ if $\mu_\alpha(t^0) \neq 0$. However a note to appear soon will extend that proof to the condition which we use.

[7] Subscripts attached to the symbols x, p, b will denote association with an arc with that subscript. Thus $p_k(t)$ is the value of control along an arc a_k.

Using the function $L(p)$ defined in (13) we first define a quantity which will act as part of the square of a norm in arc space.   Let

$$(25) \qquad K(a, a_0) \equiv \int_{t^0}^{t^1} [L(p(t) - p_0(t)) - 1] dt$$

where p, p_0 are the respective controls along a, a_0.   Our first major result is

THEOREM 4.1.   *If* $\{a_k\}$ *is a sequence of admissible arcs which converges uniformly to* a_0 *in txb space and also satisfies* $\limsup_{k \to \infty} I_0(a_k) \leqq I_0(a_0)$, *then* $\lim_{k \to \infty} K(a_k, a_0) = 0$ *and there is a subsequence* $\{a_{k_r}\}$ *of* $\{a_k\}$ *such that* $p_{k_r}(t)$ *converges to* $p_0(t)$ [a. unif.] *on* $[t^0, t^1]$.

The proof of this theorem will be based on a number of definitions and lemmas which we proceed to list.   The first of these is:

LEMMA 4.1.   *There exist functions* $P^k(t, x)$ $k = 1, \cdots, K$ *which are of class* C^2 *with respect to* x, C^0 *in* t *and satisfy* $\phi^\alpha(t, x, P(t, x)) - \phi^\alpha(t) = 0$, $\alpha = 1, \cdots, m$ $t^0 \leqq t \leqq t^1$, $|x - x_0(t)| < \xi$ *for some positive constant* ξ.   *In addition,* $P^k(t, x_0(t)) = p_0^k(t)$ *on* $[t^0, t^1]$.

Proof.   The proof of this follows from the continuity properties of ψ^α, the definitions of ϕ^α and the use of Lemma 10.1 of [3].

Next with the functions $P(t, x)$ of Lemma 4.1 we shall be able to break up the functionals of our problem in a convenient manner.
As a first example of this, we consider the functional $J_T(a)$ defined by:

$$(26\text{-}1) \qquad J_T(a) = [z^{\,i}(t^s) X^{is}(b)]_{s=0}^{s=1} + g_0(b) + \lambda_\gamma g_\gamma(b) + \int_{t^0}^{t^1} G(t, x, p) dt.$$

Notice that by (1–5), (1–6), (6–1), and (6–2) we have (with arguments along a), that

$$(26\text{-}2) \qquad I_0(a) = J_T(a) - \lambda_\gamma I_\gamma(a) - \int_{t^0}^{t^1} \mu_\alpha(t) \phi^\alpha(t, x, p) dt.$$

Next, we write:

$$(27\text{-}1)\ J_T^*(a) = [z^{\,i}(t^s) X^{is}(b_0)]_{s=0}^{s=1} + g_0(b_0) + \lambda_\gamma g_\gamma(b_0)$$
$$+ [[z^{\,i}(t^s) X_{b^\sigma}^{is}]_{s=0}^{s=1} + g_{0_b\sigma} + \lambda_\gamma g_{\gamma_b\sigma}][b^\sigma - b_0^\sigma]$$
$$+ \int_{t^0}^{t^1} [G(t, x, P) + [p^k - P^k] G_{p^k}(t, x, P)] dt.$$

$$(27\text{-}2)\, E^*_T(a) = \int_{t^0}^{t^1} E_G(t, x, P, p)\, dt$$

$$\equiv \int_{t^0}^{t^1} [G(t, x, p) - G(t, x, P) - [p^k - P^k]G_{p^k}(t, x, P)]\, dt$$

$$(27\text{-}3)\, B^*_T(a) = z'(t^s)[X^{is}(b) - X^{is}(b_0) - X^{is}_{b^\sigma}[b^\sigma - b^\sigma_0]]^{s=1}_{s=0} + g_0(b) + \lambda_\gamma g_\gamma(b)$$

$$- g_0(b_0) - \lambda_\gamma g_\gamma(b_0) - [g_{0 b^\sigma} + \lambda_\gamma g_{\gamma b^\sigma}][b^\sigma - b^\sigma_0]$$

where: (i) all derivatives with respect to b are at b_0; (ii) the arguments x, p, P, b are evaluated along a with $P = P(t, x)$ and (iii) the functions E_G, G are the function of (6-2), (7) where for conciseness of notation we have deleted the arguments $z(t), \mu(t)$ but understand them to still be present. This convention in writing the arguments of E_G, G will be used throughout this paper.

It is convenient here also to define another functional of an arc a.

$$(28) \qquad E_T(a) \equiv E^*_T(a) + \int_{t^0}^{t^1} \dot{\mu}_\alpha(t)\psi^\alpha(t, x)\, dt$$

where E^*_T is from (27-2), $\dot{\mu}_\alpha(t)$ are the derivatives appearing in the assumption (iv) and the arguments t, x are on a. With these definitions, we see that

$$(29) \qquad J_T(a) = J^*_T(a) + B^*_T(a) + E^*_T(a).$$

Next, let $V(t, x, p)$ be any function of class C^2. Then we make the following definitions:

(i)   For an arc a set

$$V(a) = \int_{t^0}^{t^1} V(t, x, p)\, dt$$

(where the arguments are along a) and

(ii)   we shall say that V is E_T dominated near a_0 on R_0 if there is a positive constant c and a neighborhood R_1 of a_0 relative to R_0 such that

$$(30\text{-}1) \qquad E_G(t, x, p, q) + \dot{\mu}_\alpha(t)\psi^\alpha(t, x) \geqq c\,|E_V(t, x, p, q)|$$

whenever t, x, p is in R_1, t, x, q is in R_0, and $\phi^\alpha(t, x, p) = 0$ if $\dot{\mu}_\alpha(t) \neq 0$ and where

$$(30\text{-}2)\, E_V(t, x, p, q) \equiv V(t, x, q) - V(t, x, p) - (p^k - q^k)V_{p^k}(t, x, p)$$

We shall further restrict the neighborhood R_1 and constant c if necessary so that

$$(31) \quad E_G(t, x, p, q) + \dot{\mu}_\alpha(t)\psi^\alpha(t, x) \geqq cE_L(p - P(t, x), q - P(t, x))$$

(where E_L is the function introduced in (14)) whenever t, x, p is in R_1 and t, x, q is in R_0 and $\phi^\alpha(t, x, p) = 0$ if $\dot{\mu}_\alpha(t) \neq 0$. The proof of this is analagous to [1] Page 30, using our assumption (vc).

We next prove:

LEMMA 4.2. *Given $\epsilon' > 0$ then there is a neighborhood $\mathcal{F}$ of a_0 in txb space such that*

$$(32) \qquad -\epsilon' + (1 - \epsilon')E_T(a) < I_0(a) - I_0(a_0)$$

for each admissible arc a in $\mathcal{F}$.

Proof. By the definition of the functions ϕ^α we see that along any admissible arc a

$$(33) \qquad \int_{t^0}^{t^1} \mu_\alpha \phi^\alpha dt = \int_{t^0}^{t^1} \frac{d}{dt} [\mu_\alpha \psi^\alpha] dt - \int_{t^0}^{t^1} \dot{\mu}_\alpha \psi^\alpha dt$$

with arguments along[8] a. Then by (26-2), (29), (28) and (33), we have for a, an admissable arc that

$$(34) \quad I_0(a) = J_T^*(a) + \mu_\alpha(t^0)\psi^\alpha(t^0, x(t^0)) + E_T(a) + B_T^*(a) - \lambda_\gamma I_\gamma(a)$$
$$- \mu_\alpha(t^1)\psi^\alpha(t^1, x(t^1))$$

(where $x(t^0)$, $x(t^1)$ are along a).

Then with Δ denoting[9] the change in a quantity evaluated from a_0 to a, so that e.g.

$$(35) \qquad \Delta I_0(a) \equiv I_0(a) - I_0(a_0)$$

we have

$$(36) \Delta I_0(a) = \Delta J_T^*(a) + \mu_\alpha(t^0)\Delta\psi^\alpha(t^0, x(t^0)) + E_T(a) + B_T^*(a) - \lambda_\gamma\Delta I_\gamma(a)$$
$$- \mu_\alpha(t^1)\Delta\psi^\alpha(t^1, x(t^1))$$

where in (36) we have recognized, because of (4-2) together with the

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[8] We shall often omit arguments in this fashion whenever the context makes clear what those arguments are.

[9] Henceforth we shall use this notation frequently, thus as another example $\Delta x_\gamma(t)$ will mean $x_\gamma(t) - x_0(t)$ where $x_\gamma(t)$ is the value of state on an arc a_γ.

definition of E_T^* in (27–2) and the construction of the functions P of Lemma 4.1, that $E_T(a_0) = 0$ and also, by (27–3), that $B_T^*(a_0) = 0$.

Now in (31) set $p^k = P^k$ so that

$$(37) \quad c[L(q - P(t, x)) - 1] = cE_L(0, q - P(t, x)) \leqq E_G(t, x, P(t, x), q)$$
$$+ \dot{\mu}_\alpha(t)\psi^\alpha(t, x)$$

which is true for t, x in some neighborhood $\bar{\bar{\mathscr{F}}}$ (in tx space) reduced if necessary from the projection of R_1 of (31) and also for t, x, q in R_0.

Next with $p(t)$ having the value associated with the arc a, set q equal to $p(t)$ so that we get

$$(38) \quad |p(t) - P(t, x)| < L(p(t) - P(t, x))$$
$$\leqq c^{-1}[1 + E_G(t, x, P(t, x), p(t)) + \dot{\mu}_\alpha(t)\psi^\alpha(t, x)]$$

for t, x in the neighborhood $\bar{\bar{\mathscr{F}}}$ of (37), where in (38) we have increased c^{-1} to be greater than 1 if it was not already so.

Now consider the integral part of $\Delta J_T^*(a)$, that is the terms

$$(39) \quad \int_{t^0}^{t^1} [G(t, x, P(t, x)) - G(t) + (p^k - P^k(t, x))G_{p^k}(t, x, P(t, x))]dt$$

where x, p are $\underline{x}(t), p(t)$ of a. Since G satisfies (10), then we can find a neighborhood $\bar{\bar{\mathscr{F}}}$ of a_0 in tx space such that

$$(40) \quad |G_{p^k}(t, x, P(t, x))| < \epsilon$$

for t, x in $\bar{\bar{\bar{\mathscr{F}}}}$. Then by (38) and (40) we have with $\mathscr{F}$ as the intersection of $\bar{\bar{\mathscr{F}}}$ and $\bar{\bar{\bar{\mathscr{F}}}}$ that if a is in $\mathscr{F}$

$$(41) \quad \left| \int_{t^0}^{t^1} (p^k - P^k(t, x))G_{p^k}(t, x, P(t, x))dt \right| < \epsilon c^{-1}[t^1 - t^0 + E_T(a)].$$

Also with $\mathscr{F}$ small enough we will have

$$(42) \quad |G(t, x, P(t, x) - G(t)| < \epsilon$$

for t, x in $\mathscr{F}$.

Thus by (41) and (42) we have that for a an admissible arc in $\mathscr{F}$, then

$$(43) \quad \int_{t^0}^{t^1} [G(t, x, P(t, x)) - G(t) + (p^k - P^k(t, x))G_{p^k}(t, x, P(t, x))]dt$$
$$< \epsilon[t^1 - t^0] + \epsilon c^{-1}[t^1 - t^0 + E_T(a)].$$

Now except for the quantities $\lambda_\gamma \Delta I_\gamma$ and $E_T(a)$, all other terms on the right hand side of (36) depend solely on txb values and vanish on a_0. Then by considering $\mathscr{F}$ as a neighborhood in txb space and making it small enough, we can make the sum of all of these quantities less than ϵ and then by (43) and (37) achieve for an admissible arc a in $\mathscr{F}$,

$$(44) \quad |\Delta I_0(a) + \lambda_\gamma \Delta I_\gamma(a) - E_T(a)| < \epsilon[t^1 - t^0 + 1] + \epsilon c^{-1}[t^1 - t^0 + E_T(a)].$$

Inequality (44) implies that

$$(45) \quad E_T(a) - \lambda_\gamma \Delta I_\gamma(a) - \epsilon[t^1 - t^0 + 1] - \epsilon c^{-1}[t^1 - t^0 + E_T(a)] < \Delta I_0(a).$$

Now by the admissibility of a and the nonnegativity of λ_γ (see (4)), we see that

$$(46) \quad -\lambda_\gamma \Delta I_\gamma(a) \geqq 0$$

then (46) implies that for a admissible and in $\mathscr{F}$

$$(47) \quad -\epsilon[(1 + c^{-1})(t^1 - t^0) + 1] + (1 - \epsilon c^{-1})E_T(a) < \Delta I_0(a).$$

Now select ϵ such that

$$(48) \quad \epsilon[(1 + c^{-1})(t^1 - t^0) + 1] < \epsilon' \quad \text{and} \quad \epsilon c^{-1} < \epsilon'$$

so that our lemma is proven.

Now with $V(a)$ as the functional defined below (29) we prove:

LEMMA 4.3. *If $V(t, x, p)$ satisfies*

$$(49) \quad V_{p^k}(t, x_0(t), p_0(t)) = 0 \qquad t^0 \leqq t \leqq t^1$$

and if V is E_T dominated near a_0 on R_0 with constant c, then for each $\epsilon' > 0$ there is a neighborhood $\mathscr{H}$ of a_0 in tx space such that

$$|V(a) - V(a_0)| < \epsilon' + (c^{-1} + \epsilon')E_T(a).$$

Proof. Set

$$(50) \quad V^*(a) = \int_{t^0}^{t^1} [V(t, x, P(t, x)) + (p^k - P^k(t, x))V_{p^k}(t, x, P(t, x))]dt$$

and

$$(51) \quad E_V^*(a) = \int_{t^0}^{t^1} E_V(t, x, P(t, x), p)dt$$

where all arguments are along a and E_V is the function of (30–2).   Then

(52) $$V(a) = V^*(a) + E^*_V(a)$$

and

(53) $$\Delta V(a) = \Delta V^*(a) + E^*_V(a).$$

Now by reasoning entirely analogous to that used in obtaining (43), then for any $\epsilon > 0$ we can find a neighborhood $\mathscr{K}$ in tx space about a_0 such that for any admissible arc a in $\mathscr{K}$ we have

(54) $$|\Delta V^*(a)| < \epsilon[t^1 - t^0] + \epsilon c^{-1}[t^1 - t^0 + E_T(a)].$$

Then by the E_T domination of V near a_0, we have by reducing $\mathscr{K}$ if necessary that

(55) $$\begin{aligned}
|\Delta V(a)| &\leqq |\Delta V^*(a)| + |E^*_V(a)| \\
&< \epsilon[t^1 - t^0] + \epsilon c^{-1}[t^1 - t^0 + E_T(a)] + c^{-1}E_T(a) \\
&= \epsilon(1 + c^{-1})[t^1 - t^0] + c^{-1}(1 + \epsilon)E_T(a).
\end{aligned}$$

Now select ϵ such that

(56) $$\epsilon(1 + c^{-1})[t^1 - t^0] < \epsilon' \quad \text{and} \quad \epsilon c^{-1} < \epsilon'$$

Then the lemma is proven.
      We next prove:

LEMMA 4.4.   *Let V satisfy the hypotheses of Lemma 4.3.   Then given $\epsilon < 0$, there exists $\eta > 0$ and a neighborhood $\mathscr{L}$ of a_0 in txb space such that if a is admissible and in $\mathscr{L}$ and satisfies*

(57) $$I_0(a) \leqq I_0(a_0) + \eta$$

then

(58) $$|V(a) - V(a_0)| < \epsilon.$$

Proof.   Assume the contrary, that is that there is a sequence $\{a_r\}$ of admissible arcs which converge to a_0 uniformly in txb space such that

(59) $$I_0(a_r) - I_0(a_0) \leqq r^{-1} \quad \text{and} \quad |V(a_r) - V(a_0)| > \epsilon.$$

By Lemma 4.2 given $\epsilon' > 0$, then if r is large enough so that a_r is in the neighborhood $\mathscr{F}$ of that lemma, then

(60) $\qquad -\epsilon' + (1-\epsilon')E_T(a_r) < I_0(a_r) - I_0(a_0) \leqq r^{-1}.$

Also, by Lemma 4.3, for r large enough so that a_r is in the neighborhood $\mathcal{K}$ of that lemma

(61) $|V(a_r) - V(a_0)| < \epsilon' + (c^{-1} + \epsilon')E_T(a_r) < \epsilon' + (c^{-1} + \epsilon')\dfrac{(r^{-1} + \epsilon')}{1 - \epsilon'}$.

Since this holds for all large r and since ϵ' can be made arbitrarily small, this gives a contradiction to the second part of (59) thus proving the Lemma.

With the help of these last three Lemmas, we can prove Theorem 4.1 as follows:

Proof. The integral of $K(a_r, a_0)$ defined in (25) involves the function $\bar{L}(t, x, p) \equiv L(p - p_0(t)) - 1$. This function is E_T dominated near a_0 on R_0 for the same reasons that (31) was true and furthermore satisfies

(62) $\qquad \dfrac{\partial \bar{L}(t, x, p)}{\partial p^i} = 0 \quad \text{along} \quad a_0$

so that $\bar{L}$ satisfies the hypothesis of Lemma 4.4. Also, we note that

(63) $\qquad \bar{L}(t, x, p_0(t)) = L(p_0(t) - p_0(t)) - 1 = 0.$

Then by Lemma 4.4 we have

(64) $\qquad \lim_{r \to \infty} K(a_r, a_0) = 0$

proving the first statement of Theorem 4.1.

Now by Hölder's Inequality

(65) $\left[\displaystyle\int_{t^0}^{t^1} |p_r - p_0|\, dt \right]^2 = \left[\displaystyle\int_{t^0}^{t^1} [L(p_r - p_0) - 1]^{\frac{1}{2}} [L(p_r - p_0) + 1]^{\frac{1}{2}}\, dt \right]^2$

$\qquad\qquad \leqq K(a_r, a_0) \displaystyle\int_{t^0}^{t^1} [L(p_r - p_0) + 1]\, dt$

$\qquad\qquad = K(a_r, a_0)[2(t^1 - t^0) + K(a_r, a_0)].$

where p_r, p_0 mean p evaluated respectively on a_r, a_0. Then by (64) we see that p_r converges to p_0 [mean] which by standard theorems implies the existence of a subsequence $\{a_{r_k}\}$ satisfying the second statement of Theorem 4.1 and proving the theorem.

5. Definition of the functions η_r, α_r, β_r. Now suppose that there is a sequence[10] $\{a_r\}$ of admissible arcs converging to a_0 uniformly in txb space with $I_0(a_r) \leqq I_0(a_0)$. Then by Theorem 4.1, we may replace this seqeunce by a subsequence which we again call $\{a_r\}$ such that with control p_r on a_r,

(66)
$$\lim_r p_r = p_0 \quad \text{[a. unif.]} \quad \text{on} \quad [t^0, t^1].$$

Also, the quantities

(67)
$$b_r - b_0|, \qquad \max_{t^0 \leqq t \leqq t^1} |\Delta x_r(t)|$$
$$\int_{t^0}^{t^1} \dot{\mu}_\alpha \bar{\psi}^\alpha(t, x_r) dt, \qquad \int_{t^0}^{t^1} |\dot{\mu}_\alpha \bar{\phi}^\alpha(t, x_r, p_r)| \, dt$$

all converge in bounded manner to zero as $r \to \infty$ where: (i) $\bar{\psi}^\alpha$, $\bar{\phi}^\alpha$ are the functions introduced in (3), (ii) the subscript r, as usual denotes values on the arc a_r, (iii) for the convergence of the integrals in (67) to zero we have used the fact that

(68)
$$\int_{t^0}^{t^1} \dot{\mu}_\alpha \bar{\phi}^\alpha(t) dt = 0 \quad \text{and} \quad \int_{t^0}^{t^1} \dot{\mu}_\alpha \bar{\psi}^\alpha(t) dt = 0$$

because of the properties of the functions $\dot{\mu}_\alpha(t)$.

Thus if we define:

(69)
$$k_r^2 = K(a_r, a_0) + \max \left[\int_{t^0}^{t^1} \dot{\mu}_\alpha \bar{\psi}^\alpha dt, \int_{t^0}^{t^1} |\dot{\mu}_\alpha \bar{\phi}^\alpha| \, dt \right]$$
$$+ |b_r - b_0|^2 + \max_{[t^0, t^1]} |\Delta x_r(t)|^2$$

where $\bar{\psi}^\alpha$, $\bar{\phi}^\alpha$ are evaluated along a_r, then by Theorem 4.1 and the above statements, we have that

(70)
$$\lim_{r \to \infty} k_r = 0.$$

Now define the vector functions

---

[10] Henceforth unless otherwise specified, all references to arcs a_r will mean members of subsequences (as defined in the following pages) of the sequence introduced here.

(71-1) $\eta_r^i(t) \equiv \Delta x_r^i(t)/k_r,$ $\alpha_r^k(t) \equiv \Delta p_r^k(t)/k_r,$ $\beta_r^\sigma \equiv \Delta b_r^\sigma/k_r,$

and

(71-2) $h_r(t) \equiv 1 + L(p_r(t) - p_0(t)) = k_r^2 |\alpha_r(t)|^2/(L(p_r(t) - p_0(t)) - 1)$

the second equality for h_r following from the above definitions. Then by (71-2)

(72) $\int_{t^0}^{t^1} |\alpha_r|^2/h_r dt = \int_{t^0}^{t^1} [L(p_r - p_0) - 1]/k_r^2 dt = K(a_r, a_0)/k_r^2$

which implies by the definition of k_r that

(73) $$\beta_r^2 + \max_{[t^0, t^1]} |\eta_r|^2 + \int_{t^0}^{t^1} \frac{|\alpha_r|^2}{h_r} \, dt \leq 1$$

and hence in particular that the integral in (73) is uniformly bounded with respect to r.

We next prove:

LEMMA 5.1.   *The integrals of $h_r(t)$ are absolutely continuous uniformly with respect to r.*

Proof.   This follows because $h_r(t)$ differs by 2 from the integrand of $K(a_r, a_0)$ and $\lim_{r \to \infty} K(a_r, a_0) = 0$.

LEMMA 5.2.   *The integrals $\int_M \alpha_r dt$ are absolutely continuous uniformly with respect to r.*

Proof.   By Hölder's Inequality and (73) we have that

(74) $$\left| \int_M \alpha_r dt \right|^2 \leq \int_M |\alpha_r|^2/h_r dt \cdot \int_M h_r dt \leq \int_M h_r dt$$

and by Lemma 5.1 this last integral is uniformly absolutely continuous.

6.   Existence of the variation η_0.   We next establish a number of results concerning the convergence of η_r, α_r, β_r. With $\{a_r\}$ always as the sequence of arcs considered above we prove:

THEOREM 6.1.   *With β_r defined in (71-1) then there exists a vector β_0 and a subsequence of arcs $\{a_{r_k}\}$ which we again call $\{a_r\}$ such that with $\{\beta_r\}$ as the associated values, then β_r converges to β_0.*

Proof. This follows from the definitions of k_r and β_r and the Bolzano Weierstrass Theorem, applied to the original sequence $\{\beta_r\}$.

THEOREM 6.2. *There exists a function α_0 in $\mathscr{L}_2[t^0, t^1]$ and a subsequence of arcs $\{a_{r_k}\}$ which we again call $\{a_r\}$ such that with $\{\alpha_r\}$ (defined in (71-1)) as the associated quantities, then for each bounded integrable function g*

$$
(75) \qquad \lim_{r \to \infty} \int_{t^0}^{t^1} g\alpha_r dt = \int_{t^0}^{t^1} g\alpha_0 dt
$$

and if s is a set on which p_r converges uniformly to p_0 and g is square integrable then

$$
(76) \qquad \lim_{r \to \infty} \int_s g\alpha_r dt = \int_s g\alpha_0 dt.
$$

Proof.   By the uniform boundedness of the integral in (73), there is a subsequence $\{\alpha_{r_k}\}$ which we again call $\{\alpha_r\}$ and a function $\bar\alpha_0$ in $\mathscr{L}_2[t^0, t^1]$ such that for every square integrable function g and any measurable set s.

$$
(77) \qquad \lim_{r \to \infty} \int \frac{g\alpha_r dt}{\sqrt{h_r}} = \int g\bar\alpha_0 dt.
$$

Now let s be $[t^0, t^1]$ if g is bounded and integrable while if g is only square integrable, let s be a set on which p_r converges uniformly to p_0. Then we can write

$$
(78) \qquad \int_s g\alpha_r dt = \sqrt{2} \int_s g \frac{\alpha_r}{\sqrt{h_r}} \, dt + \int_s g(\sqrt{h_r} - \sqrt{2}) \frac{\alpha_r}{\sqrt{h_r}} \, dt
$$

and by Hölder's Inequality together with (73)

$$
(79) \qquad \left[\int_s \left| g(\sqrt{h_r} - \sqrt{2}) \frac{\alpha_r}{\sqrt{h_r}} \right| dt \right]^2 \le \int_s g^2 (\sqrt{h_r} - \sqrt{2})^2 dt \cdot \int_s \frac{|\alpha_r|^2 dt}{h_r}
$$

$$
\le \int_s g^2 (\sqrt{h_r} - \sqrt{2})^2 dt.
$$

Since $h_r(t) \ge 2$, we have

$$(80) \quad 0 \leq \int_{t^0}^{t^1} (\sqrt{h_r} - \sqrt{2})dt \leq \int_{t^0}^{t^1} (\sqrt{h_r} - \sqrt{2})(\sqrt{h_r} + \sqrt{2})dt$$

$$= \int_{t^0}^{t^1} (h_r - 2)dt = \int_{t^0}^{t^1} [L(p_r - p_0) - 1]dt = K(a_r, a_0) \to 0 \quad \text{as} \quad r \to \infty$$

so that

$$(81) \qquad\qquad \lim_{r \to \infty} \int_{t^0}^{t^1} (\sqrt{h_r} - \sqrt{2})dt = 0.$$

Also since

$$(82) \quad \int_{t^0}^{t^1} (\sqrt{h_r} - \sqrt{2})^2 dt = \int_{t^0}^{t^1} (h_r - 2)dt - 2\sqrt{2} \cdot \int_{t^0}^{t^1} (\sqrt{h_r} - \sqrt{2})dt$$

then

$$(83) \qquad\qquad \lim_{r \to \infty} \int_{t^0}^{t^1} (\sqrt{h_r} - \sqrt{2})^2 dt = 0.$$

Now let g be a bounded integrable function and let s be $[t^0, t^1]$ in the statements, (78) and (79).   Then by (79), (83) and the boundedness of g we see that

$$(84) \qquad\qquad \lim_{r \to \infty} \int_{s} g(\sqrt{h_r} - \sqrt{2}) \frac{\alpha_r}{\sqrt{h_r}} \, dt = 0$$

so that by using (77), (78) and defining $\alpha_0 = \sqrt{2}\,\bar{\alpha}_0$ we have

$$(85) \quad \lim_{r \to \infty} \int_{s} g\alpha_r dt = \lim_{r \to \infty} \sqrt{2} \int_{s} g \frac{\alpha_r}{\sqrt{h_r}} \, dt = \sqrt{2} \int_{s} g\bar{\alpha}_0 dt = \int_{s} g\alpha_0 dt$$

proving the first statement of the Theorem.
    In order to prove the second statement let g be in $\mathscr{L}_2[t^0, t^1]$ and let s be a set on which p_r converges to p_0 uniformly.   Then our statements (78) and (79) hold also in this case, and on s we have

$$\lim_{r \to \infty} (h_r - 2) = 0 \quad \text{uniformly} \quad \text{on} \quad s$$

so that (84) and (85) hold also in this case, proving the Theorem.

Now with $\{a_r\}$ as the sequence of arcs yielded by the previous two theorems and with $\eta_r(t)$ defined in (71-1) as the associated quantities, we next prove:

THEOREM 6.3. *There exists a function $\eta_0(t)$ on $[t^0, t^1]$ with derivative $\dot\eta_0(t)$ in $\mathscr{L}_2[t^0, t^1]$ and a subsequence of arcs $\{a_{r_n}\}$ which we again call $\{a_r\}$ such that $\eta_r(t)$ converges to $\eta_0(t)$ uniformly on $[t^0, t^1]$. Furthermore, if g is a bounded integrable function on $[t^0, t^1]$, we have*

$$(86) \qquad \lim_{r\to\infty} \int_{t^0}^{t^1} g\dot\eta_r dt = \int_{t^0}^{t^1} g\dot\eta_0 dt$$

while if g is in $\mathscr{L}_2[t^0, t^1]$ and if s is a set on which p_r converges to p_0 uniformly, then

$$(87) \qquad \lim_{r\to\infty} \int_{s} g\dot\eta_r dt = \int_{s} g\dot\eta_0 dt$$

where $\dot\eta_r$ is the derivative of η_r.

Proof. We note that

$$(88) \qquad \dot\eta_r(t) = [\dot x_r(t) - \dot x_0(t)]/k_r = [f(t, x_r(t), p_r(t)) - f(t)]/k_r.$$

Now by assumption (vi), we have

$$(89) \qquad |\dot\eta_r(t)|^2 \leqq \rho^2[|\eta_r(t)|^2 + |\alpha_r(t)|^2]$$

so that

$$(90) \qquad |\eta_r(t)|^2 > \rho^{-2}|\dot\eta_r(t)|^2 - |\alpha_r(t)|^2.$$

Then by dividing (90) by h_r and integrating and then adding that inequality to (73) we get

$$(91) \qquad \beta_r^2 + \max_{[t^0, t^1]} |\eta_r|^2 + \frac{1}{\rho^2} \int_{t^0}^{t^1} |\dot\eta_r|^2/h_r dt < 1 + \int_{t^0}^{t^1} |\eta_r|^2/h_r dt.$$

Also by the definitions of k_r and h_r we see that

$$(92) \qquad \max_{[t^0, t^1]} |\eta_r|^2/h_r \leqq \tfrac{1}{2}$$

so that

(93)
$$\int_{t^0}^{t^1} |\eta_r|^2/h_r dt \leqq \tfrac{1}{2}[t^1 - t^0]$$

and then by (91), the first integral there is uniformly bounded with respect to r, that is

(94)
$$\int_{t^0}^{t^1} |\dot\eta_r|^2/h_r dt \leqq \Gamma \qquad \text{uniformly with respect to } r.$$

Then by using an argument similar to that used in the previous Theorem, there exists a function $\dot\eta_0(t)$ on $\mathscr{L}_2[t^0, t^1]$ such that the conclusions of the present theorem which concern the function $\dot\eta_0$ are true.

In order now to prove the remaining item of our Theorem, set

(95)
$$\eta_0^i(t) = X_{b^\sigma}^{i0} \beta_0^\sigma + \int_{t^0}^{t^1} \dot\eta_0^i dt \qquad t^0 \leqq t \leqq t^1 \qquad i = 1, \cdots, N$$

where X^{i0} are the functions of (1–4), β_0 is the vector of Theorem 6.1, and $X_{b^\sigma}^{i0}$ are evaluated on a_0.

By Hölder's Inequality together with (94) we have for any measurable set M

(96)
$$\left[\int_M |\dot\eta_r| dt\right]^2 \leqq \int_M |\dot\eta_r|^2/h_r dt \cdot \int_M h_r dt$$

$$\leqq \int_{t^0}^{t^1} |\dot\eta_r|^2/h_r dt \cdot \int_M h_r dt \leqq \Gamma \int_M h_r dt.$$

Then by the uniform absolute continuity of the integrals of $.h_r$ (Lemma 5.1) we see that the functions

(97)
$$\eta_r^i(t) = k_r^{-1} \Delta x_r^i(t^0) + \int_{t^0}^{t} \dot\eta_r^i dt \qquad t^0 \leqq t \leqq t^1$$

are absolutely continuous uniformly with respect to r.

Also by (86) which we've already proven,

(98)
$$\lim_{r\to\infty} \int_{t^0}^{t} \dot\eta_r dt = \int_{t^0}^{t} \dot\eta_0 dt \qquad t^0 \leqq t \leqq t^1.$$

Furthermore by Theorem 4.3 and the admissibility of our arcs we have that

(99)
$$\lim_{r\to\infty} \Delta x_r^i(t^0)/k_r = \lim_{r\to\infty} \frac{\Delta X^{i0}(b_r)}{\Delta b_r^\sigma} \beta_r^\sigma = X_{b^\sigma}^{i0} \beta_0^\sigma.$$

Thus by (97) through (99), the definition (95) and the uniform absolute continuity of $\eta_i'(t)$ we see that

$$(100) \qquad \lim_{r \to \infty} \eta_i^i(t) = \eta_0^i(t) \quad \text{uniformly on} \quad [t^0, t^1]$$

thus proving the theorem.

We have thus proven that if we define η_0 as the variation

$$(101) \qquad \eta_0\colon \eta_0(t) \qquad \alpha_0(t) \qquad \beta_0 \qquad t^0 \leqq t \leqq t^1$$

and η_r as the vector functions

$$(102) \qquad \eta_r\colon \eta_r(t) \qquad \alpha_r(t) \qquad \beta_r \qquad t^0 \leqq t \leqq t^1$$

then η_r converges to η_0 in the sense that

$$\left.\begin{array}{l} \beta_r \text{ converges to } \beta_0 \\[6pt] \eta_r(t) \text{ converges uniformly to } \eta_0(t) \\[6pt] \alpha_r(t) \text{ converges [a. unif.] to } \alpha_0(t) \end{array}\right\} \quad t^0 \leqq t \leqq t^1.$$

7. Additional properties of the variation η_0.

Now, using the functions $P(t, x)$ of Lemma 4.1, we recall that by our convention $P^k(t) \equiv P^k(t, x_0(t)) = p_0^k(t)$ the value of the control on a_0. For convenience, let us denote this as $P_0^k(t)$. Thus $P_0^k(t) = p_0^k(t)$. Next denote by $P_r(t)$ the values of P along the arc (a_r), i.e.,

$$(103) \qquad P_r^k(t) \equiv P^k(t, x_r(t)) \qquad k = 1, \cdots, K \qquad t^0 \leqq t \leqq t^1$$

and also define the vector functions

$$(104) \qquad \rho_r(t) \equiv \frac{P_r(t) - P_0(t)}{k_r} \qquad \rho_0(t) \equiv P_{x'}(t, x_0(t))\eta_0^i(t).$$

Then we state the following results which are proven in an analagous manner to Lemmas 8.1 through 8.4 of [1] except that derivatives of state functions are replaced by controls.

LEMMA 7.1. *The following relations are true:*

$$(105\text{-}1) \qquad \phi^\alpha(t, x_r, P_r) = \phi^\alpha(t, x_0, P_0) \qquad \alpha = 1, \cdots, m$$

(where ϕ^α are the functions introduced above (2))

(105–2) $\lim\limits_{r\to\infty} P_r(t) = P_0(t) = p_0(t)$ uniformly on $[t^0, t^1]$

(105–3) $\lim\limits_{r\to\infty} \rho_r(t) = \rho_0(t)$ uniformly on $[t^0, t^1]$.

If $N_r^k(t)$ are continuous functions which converge uniformly to $N_0^k(t)$ on $[t^0, t^1]$ and if g is square integrable then

(106–1) $\lim\limits_{r\to\infty} \int_s N_r^k(\alpha_r^k - \rho_r^k) dt = \int_s N_0^k(\alpha_0^k - \rho_0^k) dt$

for every measurable set s in $[t^0, t^1]$ and

(106–2) $\lim\limits_{r\to\infty} \int_s g(\alpha_r^k - \rho_r^k) dt = \int_s g(\alpha_0^k - \rho_0^k) dt$

for every measurable set s in $[t^0, t^1]$ upon which $p_r(t)$ converges uniformly to $p_0(t)$.   Furthermore if $\mathscr{W}(t, x, p)$ is any function of class C' near a_0, then

(107) $\lim\limits_{r\to\infty} k_r^{-1}[\mathscr{W}(t, x_r, P_r) - \mathscr{W}(t, x_0, P_0)] = \mathscr{W}_{x^i}\eta_0^i + \mathscr{W}_{p^k}\rho_0^k$

uniformly on $[t^0, t^1]$.

Finally with the functions ϕ^α of (105–1) we have

(108) $\phi_{x^i}^\alpha \eta_0^i + \phi_{p^k}^\alpha \rho_0^k = 0$ $\alpha = 1, \cdots, m$ $t^0 \leqq t \leqq t^1$.

Now using the functions $P(t, x)$ of Lemma 4.1 we extend to any measurable set s in $[t^0, t^1]$, the technique used to break up functionals in (50) as follows:

Let $\mathscr{V}(t, x, p)$ be a function of class C' near a_0 and a an arc and define

(109–1) $\mathscr{V}(a, s) = \int_s \mathscr{V}(t, x, p) dt$

(109–2) $\mathscr{V}^*(a, s) = \int_s (\mathscr{V}(t, x, P) + (p^k - P^k)\mathscr{V}_{p^k}(t, x, P)] dt$

(109–3) $E_\mathscr{V}^*(a, s) = \int_s E_\mathscr{V}(t, x, P, p) dt$.

Also, for a variation η of the arc a_0

$$\eta: \quad \eta(t) \qquad \alpha(t) \qquad \beta \qquad t^0 \leqq t \leqq t^1$$

define

$$(109\text{-}4) \qquad \mathcal{V}_1(\eta, s) = \int_s [\mathcal{V}_{x^i} \eta^i + \mathcal{V}_{p^k} \alpha^k] dt$$

where: (i) all values of x, p, P in (109–1) through (109–3) are along the arc a, (ii) the function $E_{\mathcal{V}}$ in (109–3) is the Weierstrass E function for $\mathcal{V}$, and (iii) the arguments of $\mathcal{V}_{x^i}$, $\mathcal{V}_{p^k}$ in (109–4) are from a_0.   Then as in (52) we see that

$$(110) \qquad \mathcal{V}(a, s) = \mathcal{V}^*(a, s) + E_{\mathcal{V}}^*(a, s).$$

and, in analagous manner to Lemma 9.1 of [1] we have

LEMMA 7.2.   *If $\mathcal{V}(t, x, p)$ is of class C' near a_0, then*

$$(111) \qquad \lim_{r \to \infty} k_r^{-1}[\mathcal{V}^*(a_r, s) - \mathcal{V}^*(a_0, s)] = \mathcal{V}_1(\eta_0, s).$$

8.   Evaluation of second order terms.   Our ultimate purpose now is to prove the admissibility of the variation η_0 defined in (101) and constructed in Theorems 4.3 through 5.1.   As a first step in this procedure, we evaluate certain second order terms.

Continuing with our sequence $\{a_r\}$, then dividing (36) by k_r^2, evaluating the expression on this sequence, moving most of the terms to the left side and taking superior limits, we obtain

$$(112) \quad \limsup_{r \to \infty} k_r^{-2}[-\Delta J_T^*(a_r) - \mu_\alpha(t^0)\Delta\psi^\alpha(t^0, x_r(t^0))]$$

$$+ \limsup_{r \to \infty} k_r^{-2}[-B_T^*(a_r)] + \limsup_{r \to \infty} k_r^{-2}\lambda_\gamma \Delta I_\gamma(a_r)$$

$$+ \limsup_{r \to \infty} k_r^{-2}\mu_\alpha(t^1)\Delta\psi^\alpha(t^1, x_r(t^1)) \geqq \limsup_{r \to \infty} k_r^{-2}E_T(a_r).$$

In order to establish the admissibility of η_0, we shall have to deal with the separate terms of (112).   The first term to come under consideration is the integral part of $\Delta J_T^*(a_r)$ which we denote by the symbol $\Delta_I J_T^*(a_r)$.   Thus

$$(113\text{-}1) \qquad \Delta_I J_T^*(a_r) \equiv \int_{t^0}^{t^1} [G(t, x_r, P_r) - G(t) + (p_r^k - P_r^k)G_{p^k}(t, x_r, P_r)] dt.$$

With $\eta_0(t)$, $\alpha_0(t)$ as quantities associated with the variation η_0 and ρ_r as the terms of (104) and finally with $\omega_G(\eta_0, \alpha_0)$ denoting the quadratic form

$$(113\text{-}2)\,\omega_G(\eta_0, \alpha_0) \equiv G_{x^ix^j}\eta_0^i\eta_0^j + 2G_{x^ip^k}\eta_0^i\alpha_0^k + G_{p^hp^k}\alpha_0^h\alpha_0^k \quad \begin{matrix} i, j = 1, \cdots, N \\ h, k = 1, \cdots, K \end{matrix}$$

where the partial derivatives are formed along a_0, we next prove:

LEMMA 8.1.

$$(114)\,\lim_{r\to\infty} k_r^{-2}\Delta_r J_T^*(a_r) = \frac{1}{2}\int_{t^0}^{t^1} [\omega_G(\eta_0, \alpha_0) - G_{p^hp^k}(\alpha_0^h - \rho_0^h)(\alpha_0^k - \rho_0^k)]dt$$

$$i, j = 1, \cdots, N \qquad h, k = 1, \cdots, K.$$

Proof. By Taylor's Theorem and the definitions of P_r, ρ_r

$$(115) \quad \frac{G(t, x_r, P_r) - G(t) + (p_r^k - P_r^k)G_{p^k}(t, x_r, P_r)}{k_r^2}$$

$$= \frac{k_r[G_{x^i}\eta_r^i + G_{p^k}\rho_r^k + (\alpha_r^k - \rho_r^k)G_{p^k}]}{k_r^2}$$

$$+ \frac{k_r^2(\alpha_r^k - \rho_r^k)[\tilde{\tilde{G}}_{p^kp^h}\rho_r^h + \tilde{\tilde{G}}_{p^kx^i}\eta_r^i]}{k_r^2}$$

$$+ \frac{\dfrac{k_r^2}{2}[\tilde{G}_{x^ix^j}\eta_r^i\eta_r^j + 2\tilde{G}_{x^ip^k}\eta_r^i\rho_r^k + \tilde{G}_{p^hp^k}\rho_r^h\rho_r^k]}{k_r^2}$$

where: (i) G_{x^i}, G_{p^h} are evaluated on the arc a_0 and (ii) $\tilde{G}_{x^ix^j}$, $\tilde{G}_{x^ip^k}$, $\tilde{G}_{p^hp^k}$, $\tilde{\tilde{G}}_{p^kp^h}$, $\tilde{\tilde{G}}_{p^hx^i}$ are all evaluated at intermediate points on the line segment

$$(t, x_0 + \theta\Delta x_r, P_0 + \theta\Delta P_r) \qquad 0 < \theta < 1.$$

Now using (10) together with Lemma 7.1 and the uniform convergence of x_r to x_0 and η_r to η_0 on $[t^0, t^1]$ we see by integrating (115) that

$$(116) \quad \lim_{r\to\infty} k_r^{-2}\Delta_r J_T^*(a_r) = \frac{1}{2}\int_{t^0}^{t^1} [G_{x^ix^j}\eta_0^i\eta_0^j + 2G_{x^ip^k}\eta_0^i\rho_0^k + G_{p^hp^k}\rho_0^h\rho_0^k$$

$$+ 2(\alpha_0^k - \rho_0^k)(G_{p^hp^k}\rho_0^h + G_{p^kx^i}\eta_0^i)]dt$$

where all partial derivatives are evaluated along a_0. Combining terms this becomes

$$(117) \quad \lim_{r \to \infty} k_r^{-2} \Delta_I J_T^*(a_r) = \frac{1}{2} \int_{t^0}^{t^1} [G_{x^i x^j} \eta_0^i \eta_0^j + 2 G_{x^i p^k} \eta_0^i \alpha_0^k - G_{p^h p^k} \rho_0^h \rho_0^k$$

$$+ 2 G_{p^h p^k} \rho_0^h \alpha_0^k] dt.$$

Now adding and subtracting the terms $G_{p^h p^k} \alpha_0^h \alpha_0^k$ in the above integrand produces (114) and the lemma is proven.

Next, by using the definition of $\Delta J_T^*(a_r)$ together with Lemma 8.1, the transversality relation (8) and the relation for λ_{p+i} in (4–1) we get

$$(118) \quad \limsup_{r \to \infty} k_r^{-2} [-\Delta J_T^*(a_r) - \mu_\alpha(t^0) \Delta \psi^\alpha(t^0, x_r(t^0))]$$

$$= \limsup_{r \to \infty} k_r^{-2} [K^\alpha \psi_{x^i}^\alpha(t^0) X_{b^\sigma}^{i0} \Delta b_r^\sigma - \mu_\alpha(t^0) \Delta \psi^\alpha(t^0, x_r(t^0))]$$

$$- \frac{1}{2} \int_{t^0}^{t^1} [\omega_G(\eta_0, \alpha_0) - G_{p^h p^k}(\alpha_0^h - \rho_0^h)(\alpha_0^k - \rho_0^k)] dt.$$

As a next step we prove

Lemma 8.2.

$$(119) \quad \limsup_{r \to \infty} k_r^{-2} [K^\alpha \psi_{x^i}^\alpha(t^0) X_{b^\sigma}^{i0} \Delta b_r^\sigma - \mu_\alpha(t^0) \Delta \psi^\alpha(t^0, x_r(t^0))]$$

$$\leq \limsup_{r \to \infty} k_r^{-2} [-(\mu_\alpha(t^0) - K^\alpha) \psi_{x^i}^\alpha(t^0) \Delta x_r^i(t^0)]$$

$$- \frac{1}{2} [K^\alpha \psi_{x^i}^\alpha(t^0) X_{b^\sigma b^\tau}^{i0} + K^\alpha \psi_{x^i x^j}^\alpha(t^0) X_{b^\sigma}^{i0} X_{b^\tau}^{j0}] \beta_0^\sigma \beta_0^\tau$$

$$- \frac{1}{2} (\mu_\alpha(t^0) - K^\alpha) \psi_{x^i x^j}^\alpha(t^0) \eta_0^i(t^0) \eta_0^j(t^0).$$

Proof. By Taylor's Theorem and the admissibility of our arcs, we have

$$(120\text{–}1) \qquad\qquad X_{b^\sigma}^{i0} \Delta b_r^\sigma = \Delta x_r^i(t^0) - \frac{1}{2} \tilde{X}_{b^\sigma b^\tau}^{i0} \Delta b_r^\sigma \Delta b_r^\tau$$

and

$$(120\text{–}2) \quad \Delta \psi^\alpha(t^0, x_r(t^0)) = \psi_{x^i}^\alpha(t^0) \Delta x_r^i(t^0) + \frac{1}{2} \tilde{\psi}_{x^i x^j}^\alpha \Delta x_r^i(t^0) \Delta x_r^j(t^0)$$

where $\tilde{X}_{b^\sigma b^\tau}^{i0}$, $\tilde{\psi}_{x^i x^j}^\alpha$ are intermediate values on the line segments

$$b_0 + \theta \Delta b_r \quad \text{and} \quad t^0, x_0(t^0) + \theta \Delta x_r(t^0) \qquad 0 < \theta < 1.$$

Then by using (120) and combining terms, we have that

$$K^\alpha \psi^\alpha_{x^i}(t^0) X^{i0}_{b^\sigma} \Delta b^\sigma_r - \mu_\alpha(t^0) \Delta \psi^\alpha(t^0, x_r(t^0)) = -(\mu_\alpha(t^0) - K^\alpha) \psi^\alpha_{x^i}(t^0) \Delta x^i_r(t^0)$$

(121)

$$- \frac{1}{2} K^\alpha \psi^\alpha_{x^i}(t^0) \tilde{X}^{i0}_{b^\sigma b^\tau} \Delta b^\sigma_r \Delta b^\tau_r - \frac{1}{2} \mu_\alpha(t^0) \tilde{\psi}^\alpha_{x^i x^j} \Delta x^i_r(t^0) \Delta x^j_r(t^0).$$

Now by the admissibility of our arcs, the terms,

$$(122) \qquad\qquad\qquad \Delta x^i_r(t^0) \qquad \Delta X^{i0}(b_r)$$

(where $\Delta X^{i0}(b_r)$ means $X^{i0}(b_r) - X^{i0}(b_0)$) are equal, so that we may add to and subtract from (121) the respective quantities

$$(123) \quad \frac{1}{2} K^\alpha \tilde{\psi}^\alpha_{x^i x^j} \Delta x^i_r(t^0) \Delta x^j_r(t^0) \quad \text{and} \quad \frac{1}{2} K^\alpha \tilde{\psi}^\alpha_{x^i x^j} \Delta X^{i0}(b_r) \Delta X^{j0}(b_r)$$

(where $\tilde{\psi}^\alpha_{x^i x^j}$ here means evaluation at the same point as for that term in (121)) to get

$$K^\alpha \psi^\alpha_{x^i}(t^0) X^{i0}_{b^\sigma} \Delta b^\sigma_r - \mu_\alpha(t^0) \Delta \psi^\alpha(t^0, x_r(t^0)) = -(\mu_\alpha(t^0) - K^\alpha) \psi^\alpha_{x^i}(t^0) \Delta x^i_r(t^0)$$

$$(124) \quad -\frac{1}{2} K^\alpha \psi^\alpha_{x^i}(t^0) \tilde{X}^{i0}_{b^\sigma b^\tau} \Delta b^\sigma_r \Delta b^\tau_r - \frac{1}{2} K^\alpha \tilde{\psi}^\alpha_{x^i x^j} \Delta X^{i0}(b_r) \Delta X^{j0}(b_r)$$

$$- \frac{1}{2} (\mu_\alpha(t^0) - K^\alpha) \tilde{\psi}^\alpha_{x^i x^j} \Delta x^i_r(t^0) \Delta x^j_r(t^0).$$

Now, dividing by k^2_r, taking superior limits and using the definition of β_0 yields (119) proving the lemma.

The next term of (112) which we consider is the one involving $B^*_T(a_r)$. We prove:

LEMMA 8.3.

$$(125) \quad \lim_{r \to \infty} k^{-2}_r B^*_T(a_r) = \frac{1}{2} [(z_i(t^s) X^{is}_{b^\sigma b^\tau})^{s=1}_{s=0} + g_{0b^\sigma b^\tau} + \lambda_\gamma g_{\gamma b^\sigma b^\tau}] \beta^\sigma_0 \beta^\tau_0.$$

Proof. By Taylor's Theorem and (27–3) we have

$$(126) \quad B^*_T(a_r) = \frac{1}{2} [(z'(t^s) \tilde{X}^{is}_{b^\sigma b^\tau})^{s=1}_{s=0} + \tilde{g}_{0b^\sigma b^\tau} + \lambda_\gamma \tilde{g}_{\gamma b^\sigma b^\tau}] \Delta b^\sigma_r \Delta b^\tau_r \quad \sigma, \tau = 1, \cdots, K$$

where $\tilde{X}^{is}_{b^\sigma b^\tau}$, $\tilde{g}_{0 b \sigma b \tau}$, $\tilde{g}_{\gamma b \sigma b \tau}$ indicate intermediate values on the line segment $b_0 + \theta \Delta b$, $0 < \theta < 1$. By dividing by k_r^2, taking limits and using the definition of B_0 in (101) we obtain (125).

Now by using (118) and (112) together with Lemmas 8.2 and 8.3 we obtain

$$(127) \qquad -\frac{1}{2}[(z^i(t^s)X^{is}_{b^\sigma b^\tau})^{s=1}_{s=0} + g_{0 b \sigma b \tau} + \lambda_\gamma g_{\gamma b \sigma b \tau} + K^\alpha \psi^\alpha_{x^i}(t^0)X^{i0}_{b^\sigma b^\tau}$$

$$+ K^\alpha \psi^\alpha_{x^i x^j}(t^0)X^{i0}_{b^\sigma}X^{j0}_{b^\tau}]\beta^\sigma_0 \beta^\tau_0$$

$$-\frac{1}{2}\int_{t^0}^{t^1}[\omega_G(\eta_0, \alpha_0) - G_{p^h p^k}(\alpha^h_0 - \rho^h_0)(\alpha^k_0 - \rho^k_0)]dt$$

$$-\frac{1}{2}(\mu_\alpha(t^0) - K^\alpha)\psi^\alpha_{x^i x^j}(t^0)\eta^i_0(t^0)\eta^j_0(t^0)$$

$$+ \limsup_{r \to \infty} k_r^{-2}[-(\mu_\alpha(t^0) - K^\alpha)\psi^\alpha_{x^i}(t^0)\Delta x^i_r(t^0)]$$

$$+ \limsup_{r \to \infty} k_r^{-2}\lambda_\gamma \Delta I_\gamma(a_r) + \limsup_{r \to \infty} k_r^{-2}\mu_\alpha(t^1)\Delta \psi^\alpha(t^1, x_r(t^1))$$

$$\geq \limsup_{r \to \infty} k_r^{-2}E_T(a_r).$$

Referring to (24) and (5) for the definition of $J_2(a_0, \eta_0)$ and $\mathcal{G}$ we see that (127) gives

$$(128) \qquad -\frac{1}{2}J_2(a_0, \eta_0) + \frac{1}{2}\int_{t^0}^{t^1} G_{p^h p^k}(\alpha^h_0 - \rho^h_0)(\alpha^k_0 - \rho^k_0)dt$$

$$-\frac{1}{2}(\mu_\alpha(t^0) - K^\alpha)\psi^\alpha_{x^i x^j}(t^0)\eta^i_0(t^0)\eta^j_0(t^0)$$

$$+ \limsup_{r \to \infty} k_r^{-2}[-(\mu_\alpha(t^0) - K^\alpha)\psi^\alpha_{x^i}(t^0)\Delta x^i_r(t^0)]$$

$$+ \limsup_{r \to \infty} k_r^{-2}\lambda_\gamma \Delta I_\gamma(a_r) + \limsup_{r \to \infty} k_r^{-2}\mu_\alpha(t^1)\Delta \psi^\alpha(t^1, x_r(t^1))$$

$$\geq \limsup_{r \to \infty} k_r^{-2}E_T(a_r).$$

Using the inequality (128) we now obtain an important relation which will aid us in proving the admissibility of the variation η_0.

LEMMA 8.4.

$$(129) \qquad \lim_{r \to \infty} k_r^{-1}E_T(a_r) = 0.$$

Proof. The first three terms in (128) are bounded quantities. Also by the last item in (4–1) we have that

(130) $\mu_\alpha(t^0) \leqq K^\alpha$ and $\mu_\alpha(t^0) = K^\alpha$ if $\psi^\alpha(t^0) < 0.$

Then by (130) together with the admissibility of our arcs which implies that for r large enough

(131) $\psi^\alpha_{x^i}(t^0)\Delta x^i_r(t^0) \leqq 0$ if $\psi^\alpha(t^0) = 0$

we have for each α that

$$- (\mu_\alpha(t^0) - K^\alpha)\psi^\alpha_{x^i}(t^0)\Delta x^i_r(t^0) \leqq 0 \qquad (\alpha \text{ not summed}).$$

Then by summing, multiplying by k_r^{-2} and taking the superior limit we get

(132) $\displaystyle\limsup_{r\to\infty} k_r^{-2}[-(\mu_\alpha(t^0) - K^\alpha)\psi^\alpha_{x^i}(t^0)\Delta x^i_r(t^0)] \leqq 0.$

Furthermore by the definition (4–1) of the terms λ_γ together with the admissibility of our arcs we see also that

(133) $\displaystyle\limsup_{r\to\infty} k_r^{-2}\lambda_\gamma \Delta I_\gamma(a_r) \leqq 0$

and by the properties of $\mu_\alpha(t)$ as listed in (4) and again the admissibility of our arcs, also that

(134) $\displaystyle\limsup_{r\to\infty} k_r^{-2}\mu_\alpha(t^1)\Delta\psi^\alpha(t^1, x_r(t^1)) \leqq 0.$

Thus all terms on the left hand side of (128) are either bounded or non positive. Putting this statement together with the nonnegativity of $E_T(a_r)$ (which follows for large enough r by (11)), then we get that

(135) $\displaystyle\limsup k_r^{-2}E_T(a_r)$ is finite

thus proving (129) and the lemma.

By using Lemma 8.4 together with the break-up of functionals as in (109), we are now able to prove the analogue of Lemma 9.4 of [1] which we just state, since the proof is directly analagous to that used in [1].

LEMMA 8.5. *If $\mathcal{V}(t, x, p)$ is of class C' near a_0 and is E_T dominated near a_0 then*

(136) $$\lim_{r \to \infty} k_r^{-1}[\mathcal{V}(a_r, s) - \mathcal{V}(a_0, s)] = \mathcal{V}_1(\eta_0, s).$$

by using (129) *together with other observations obtained from* (128), *we are now in position to establish most of the requirements for admissibility of the variation* η_0.

9. Admissibility of the variation η_0.

LEMMA 9.1. *The variation* η_0 *of* (101) *satisfies the conditions* (23-3), (23-5), (23-6), (23-7), (23-8), *which are respectively*

(137-1) $\quad \psi_{x^i}^{\alpha}(t^1)\eta_0^i(t^1) = 0 \quad if \quad \mu_\alpha(t^1) \neq 0$

(137-2) $\quad \psi_{x^i}^{\alpha}(t^0)\eta_0^i(t^0) = 0 \quad if \quad \mu_\alpha(t^0) \neq K^\alpha$

(137-3) $\quad J_\gamma'(a_0, \eta_0) = 0 \quad if \quad \lambda_\gamma \neq 0 \qquad 1 \leqq \gamma \leqq p'$

(137-4) $\quad J_\gamma'(a_0, \eta_0) \leqq 0 \quad if \quad \lambda_\gamma = 0 \qquad \gamma \neq \gamma_k \qquad 1 \leqq \gamma \leqq p'$

(137-5) $\quad J_\rho'(a_0, \eta_0) = 0 \qquad p' < \rho \leqq p + 2N$

where γ_k *are the indices of* (22).

Proof. According to the statements used in proving Lemma 8.4 concerning the boundedness or the signs of terms in (128) we see in particular that

(138-1) $\quad \limsup\limits_{r \to \infty} k_r^{-2}\mu_\alpha(t^1)\Delta\psi^\alpha(t^1, x_r(t^1)) \qquad$ is finite

(138-2) $\quad \limsup\limits_{r \to \infty} k_r^{-2}[-(\mu_\alpha(t^0) - K^\alpha)\psi_{x^i}^{\alpha}(t^0)\Delta x_r^i(t^0)] \quad$ is finite

and

(138-3) $\quad \limsup\limits_{r \to \infty} k_r^{-2}\lambda_\gamma\Delta I_\gamma(a_r) \qquad\qquad$ is finite.

Now by an application of Taylor's Theorem together with the convergence of η_r to η_0 we see that for each α

(139-1)$\lim\limits_{r \to \infty} k_r^{-1}\mu_\alpha(t^1)\Delta\psi^\alpha(t^1, x_r(t^1)) = \mu_\alpha(t^1)\psi_{x^i}^{\alpha}(t^1)\eta_0^i(t^1)$ (α not summed).

Thus this limit exists for each α. By summing on α, we have that the sum of the limits exists and that by (138-1) this sum must vanish, that is

(139-2) $\quad 0 = \lim\limits_{r \to \infty} k_r^{-1}\mu_\alpha(t^1)\Delta\psi^\alpha(t^1, x_r(t^1)) = \mu_\alpha(t^1)\psi_{x^i}^{\alpha}(t^1)\eta_0^i(t^1).$

However by reasoning as used in obtaining (134), for each α index the product of the terms on the left side of (139–1) and hence also on the right side, is nonpositive. Putting this statement together with (139–2) establishes (137–1).

Next, we see by (15) that, L_γ is E_T dominated near a_0 so that by using Lemma 8.5 with $\mathscr{V} = L_\gamma$ we see that for each γ

$$(140) \quad \lim_{r \to \infty} k_r^{-1} \lambda_\gamma \Delta I_\gamma(a_r) = \lambda_\gamma \left[g_{\gamma b \sigma} \beta_0^\sigma + \int_{t^0}^{t^1} [L_{\gamma x^i} \eta_0^i + L_{\gamma_p h} \alpha_0^h] dt \right]$$

$$= \lambda_\gamma J_\gamma'(a_0, \eta_0) \qquad (\gamma \text{ not summed})$$

where the last equality follows from (20). Thus this limit exists, for each γ. By summing on γ and using (139–3) we see that the sum of the limits exists and must vanish, that is

$$(141) \qquad 0 = \lim_{r \to \infty} k_r^{-1} \lambda_\gamma \Delta I_\gamma(a_r) = \lambda_\gamma J_\gamma'(a_0, \eta_0).$$

By similar statements as below (139–2), we see that for each γ

$$(142) \qquad \lambda_\gamma J_\gamma'(a_0, \eta_0) = 0 \qquad (\gamma \text{ not summed}).$$

Then by using the properties of the terms λ_γ together with (140) and the admissibility of our arcs we get that (137–3) and (137–4) and (137–5) hold for $p' < \rho \leqq p$. The remainder of (137–5) follows from the admissibility of our arcs and the definition (20).

Finally we note that the limit

$$(143) \lim_{r \to \infty} k_r^{-1} [-(\mu_\alpha(t^0) - K^\alpha) \psi_{x^i}^\alpha(t^0) \Delta x_r^i(t^0)] = -(\mu_\alpha(t^0) - K^\alpha) \psi_{x^i}^\alpha(t^0) \eta_0^i(t^0)$$

certainly exists and then by steps similar to the above, but using (138–2), we get that (137–2) and hence the lemma is proven.

In order to establish the admissibility of the variation η_0, it remains only to prove that properties (23–1), (23–2) and (23–4) are satisfied. The property (23–2) follows from Taylor's Theorem together with the admissibility of our arcs so that

$$(144) \qquad \psi_{x^i}^\alpha(t) \eta_0^i(t) \leqq 0 \quad \text{on} \quad S^\alpha \qquad \alpha = 1, \cdots, m.$$

The property (23–4) is proven in the following lemma:

LEMMA 9.2. *The variation η_0 satisfies condition (23–4).*

Proof. By the properties[11] of $\dot{\mu}_\alpha(t)$ and by (144) we see that for each α

$$(145) \qquad \dot{\mu}_\alpha(t)\psi_{x^i}^\alpha(t)\eta_0^i(t) \geqq 0 \qquad t^0 \leqq t \leqq t^1 \qquad (\alpha \text{ not summed})$$

so that we will prove the desired result if we prove that

$$(146) \qquad \int_{t^0}^{t^1} \dot{\mu}_\alpha(t)\psi_{x^i}^\alpha(t)\eta_0^i(t)dt = 0.$$

Now, using the functions $\bar{\psi}^\alpha$ of (3), the definition of k_r in (69) and the admissibility of our arcs we see that

$$(147) \qquad 0 \leqq k_r^{-1}\int_{t^0}^{t^1} \dot{\mu}_\alpha\psi^\alpha(t,x_r)dt = k_r^{-1}\int_{S^\alpha} \dot{\mu}_\alpha\psi^\alpha(t,x_r)dt$$

$$= k_r^{-1}\int_{S^\alpha} \dot{\mu}\bar{\psi}_\alpha^\alpha(t,x_r)dt = k_r^{-1}\int_{t^0}^{t^1} \dot{\mu}_\alpha\bar{\psi}^\alpha(t,x_r)dt \leqq k_r^{-1}k_r^2 = k_r.$$

Thus

$$(148) \qquad \lim_{r\to\infty} k_r^{-1}\int_{t^0}^{t^1} \dot{\mu}_\alpha\psi^\alpha(t,x_r)dt = 0.$$

Now by the properties of $\dot{\mu}_\alpha$, we also have for each α

$$(149) \qquad \dot{\mu}_\alpha(t)\psi^\alpha(t) = 0 \qquad t^0 \leqq t \leqq t^1 \qquad (\alpha \text{ not summed})$$

so that we may add this to (148) and then get by Taylor's Theorem

$$(150) \qquad 0 = \lim_{r\to\infty} k_r^{-1}\int_{t^0}^{t^1} \dot{\mu}_\alpha\psi^\alpha(t,x_r)dt = \lim_{r\to\infty} k_r^{-1}\int_{t^0}^{t^1} \dot{\mu}_\alpha[\psi^\alpha(t,x_r) - \psi^\alpha(t)]dt$$

$$= \lim_{r\to\infty} \int_{t^0}^{t^1} \dot{\mu}_\alpha\tilde{\psi}_{x^i}^\alpha\eta_r^i(t)dt = \int_{t^0}^{t^1} \dot{\mu}_\alpha\psi_{x^i}^\alpha(t)\eta_0^i(t)dt$$

where $\tilde{\psi}_{x^i}^\alpha$ indicates evaluation on the line segment t, $x_0(t) + \theta\Delta x_r(t)$ $0 < \theta < 1$ and where the last equality follows from the uniform convergence of x_r to x_0 and η_r to η_0. Thus the lemma is proven.

The last required property for the admissibility of η_0 is established in the following lemma:

---

[11] See the remarks below (4).

LEMMA 9.3. *With f as the functions of (1–1), then the variation η_0 with quantities $\eta_0(t)$, $\alpha_0(t)$ satisfies condition (23–1), that is*

$$(151) \qquad \dot{\eta}_0^i(t) = f_{x^j}^i(t)\eta_0^j(t) + f_{p^k}^i(t)\alpha_0^k(t) \quad \text{a.e. on} \quad [t^0, t^1]$$

Proof. By Taylor's Theorem together with our Δ convention

$$(152) \quad \dot{x}_r(t) - \dot{x}_0(t) = f_{x^i}(t)\Delta x_r^i(t) + f_{p^k}(t)\Delta p_r^k(t) + R_r(t) \quad \text{on} \quad [t^0, t^1]$$

where

$$(153) \qquad R_r(t) \equiv \int_0^1 (1-\theta)[f_{x^i x^j}\Delta x_r^i \Delta x_r^j + 2f_{x^i p^k}\Delta x_r^i \Delta p_r^k + f_{p^h p^k}\Delta p_r^h \Delta p_r^k]d\theta$$

$$i, j = 1, \cdots, N, \qquad h, k = 1, \cdots, K$$

with the arguments of the f partials being at

$$(154) \qquad\qquad (t, x_0 + \theta\Delta x_r, p_0 + \theta\Delta p_r).$$

Now let s be a set on which p_r converges uniformly to p_0, then

$$(155) \qquad \int_s \left| \frac{R_r(t)}{k_r} \right| dt \leqq \epsilon_r \int_s [|\Delta x_r^i \eta_r^j| + |\Delta x_r^i \alpha_r^k| + |\Delta p_r^h \alpha_r^k|]dt$$

where ϵ_r is a constant which bounds the mixed partials of f in the integrand of (153) and exists since x_r converges uniformly to x_0 and p_r converges to p_0 uniformly on s.

By Lemma 5.2 the integrals $\int_s \alpha_r dt$ are uniformly (with respect to r) bounded. Then by using this fact together with: (i) the uniform convergence of x_r to x_0, and η_r to η_0, (ii) the uniform convergence of p_r to p_0 on s and (iii) the fact that $\lim_{r\to\infty}\epsilon_r = 0$, we see that

$$(156) \qquad\qquad \lim_{r\to\infty} \int_s \frac{R_r(t)}{k_r} dt = 0.$$

Thus by (139) and (152) we obtain

$$(157) \qquad\qquad \lim_{r\to\infty} \int_s \dot{\eta}_r^i dt = \lim_{r\to\infty} \int_s [f_{x^j}^i \eta_r^j + f_{p^k}^i \alpha_r^k]dt.$$

Now by Theorem 6.2 and 6.3 together with (157) we have

(158)
$$\int_s \dot{\eta}_0^i dt = \lim_{r \to \infty} \int_s \dot{\eta}_r^i dt = \lim_{r \to \infty} \int_s [f_{x^j}^i \eta_r^j + f_{p^k}^i \alpha_r^k] dt$$

$$= \int_s [f_{x^j}^i \eta_0^j + f_{p^k}^i \alpha_0^k] dt$$

so that on sets s upon which p_r converges uniformly to p_0, we have

(159)
$$\int_s \dot{\eta}_0^i dt = \int_s [f_{x^j}^i \eta_0^j + f_{p^k}^i \alpha_0^k] dt.$$

Since p_r converges [a.unif.] to p_0 and (159) holds on each such set where this convergence is uniform then for any set M

(160)
$$\int_M \dot{\eta}_0^i dt = \int_M [f_{x^j}^i \eta_0^j + f_{p^k}^i \alpha_0^k] dt$$

and the Theorem is proven.

Thus by Lemmas 9.1 through 9.3 together with (144) we see that the variation η_0 is admissible.

10. Proof of Theorem 3.1.

We are now in position to prove Theorem 3.1 as follows:

THEOREM 10.1. *If the following two inequalities are true, then Theorem 3.1 is true.*

$$\limsup_{r \to \infty} k_r^{-2} E_T(a_r) \geqq \frac{1}{2} \int_{t^0}^{t^1} G_{p^k p^h} (\alpha_0^k - \rho_0^k)(\alpha_0^h - \rho_0^h) dt \quad h, k = 1, \cdots, K.$$

$$0 \geqq \limsup_{r \to \infty} k_r^{-2} [- (\mu_\alpha(t^0) - K^\alpha)\psi_{x^i}^\alpha(t^0) \Delta x_r^i(t^0)]$$

$$- \frac{1}{2} (\mu_\alpha(t^0) - K^\alpha)\psi_{x^i x^j}^\alpha(t^0)\eta_0^i(t^0)\eta_0^j(t^0) \qquad i, j = 1, \cdots, N$$

(where the arguments of $G_{p^k p^h}$ are those of a_0).

Proof.   Referring to inequality (128) and using arguments as used in the proof of Lemma 8.4, we see that the fifth and sixth terms of (128) are non positive.   Now assuming the truth of the above listed inequalities, we see that (128) implies that

(161)
$$J_2(a_0, \eta_0) \leqq 0$$

which according to the hypotheses of Theorem 3.1, implies that the variation η_0 of (101) is null. According to the formula for ρ_0 as listed in (104), then this quantity is also null so that

$$(162) \qquad \int_{t^0}^{t^1} G_{p^k p^h}(\alpha_0^h - \rho_0^h)(\alpha_0^k - \rho_0^k)dt = 0$$

but then by (128) again, this implies that

$$(163) \qquad 0 \geqq \limsup_{r\to\infty} k_r^{-2}E_T(a_r),$$

which by the nonnegativity of this quantity for large r, implies that

$$(164) \qquad 0 = \limsup_{r\to\infty} k_r^{-2}E_T(a_r).$$

Next by reasoning similar to that used in [1] p. 47, we see that there is a positive number b^* such that for r large enough

$$(165) \qquad E_L(P_r, p_r) \geqq b^* E_L(P_r - p_0, p_r - p_0).$$

Then by (11), we get that for large r,

$$
\begin{aligned}
(166) \quad k_r^{-2}E_T(a_r) \geqq{} & bk_r^{-2}\left[\int_{t^0}^{t^1} E_L(P_r, p_r)dt \right. \\
& \left. + \max\left(\int_{t^0}^{t^1} \mu_\alpha \bar\psi^\alpha(t, x_r)dt, \int_{t^0}^{t^1} |\mu_\alpha \bar\phi^\alpha(t, x_r, p_r)|dt\right)\right] \\
\geqq{} & bb^* k_r^{-2}\left[\int_{t^0}^{t^1} E_L(P_r - p_0, p_r - p_0)dt \right. \\
& \left. + \max\left(\int_{t^0}^{t^1} \mu_\alpha \bar\psi^\alpha(t, x_r)dt, \int_{t^0}^{t^1} |\mu_\alpha \bar\phi^\alpha(t, x_r, p_r)|dt\right)\right] \\
={} & bb^* k_r^{-2}\left[\int_{t^0}^{t^1} \left[L(p_r - p_0) - \frac{1 + k_r^2 \rho_r^k \alpha_r^k}{L(P_r - p_0)}\right]dt \right. \\
& \left. + \max\left(\int_{t^0}^{t^1} \mu_\alpha \bar\psi^\alpha(t, x_r)dt, \int_{t^0}^{t^1} |\mu_\alpha \bar\phi^\alpha(t, x_r, p_r)|dt\right)\right] \\
\geqq{} & bb^* k_r^{-2}\left[\int_{t^0}^{t^1} \left[L(p_r - p_0) - 1 - \frac{k_r^2 \rho_r^k \alpha_r^k}{L(P_r - p_0)}\right]dt \right. \\
& \left. + \max\left(\int_{t^0}^{t^1} \mu_\alpha \bar\psi^\alpha(t, x_r)dt, \int_{t^0}^{t^1} |\mu_\alpha \bar\phi^\alpha(t, x_r, p_r)|dt\right)\right].
\end{aligned}
$$

By Lemma 5.2 together with the fact that the variation η_0 is null so that ρ_r converges uniformly to zero, $\beta_0^2 = 0$, $\max_{[t^0, t^1]} |\eta_0(t)|^2 = 0$, and by taking superior limits of (166), we obtain:

$$(167)\ \limsup_{r\to\infty} k_r^{-2} E_T(a_r) \geq \limsup_{r\to\infty} bb^* k_r^{-2} \left[K(a_r, a_0) \right.$$

$$+ \max\left(\int_{t^0}^{t^1} \dot\mu_\alpha \bar\psi^\alpha(t, x_r)dt, \int_{t^0}^{t^1} |\dot\mu_\alpha \bar\phi^\alpha(t, x_r, p_r)|\, dt \right)$$

$$\left. + |\beta_r|^2 + \max_{[t^0, t^1]} |\eta_r(t)|^2 \right] = bb^* > 0$$

which is a contradiction, thus proving the theorem and hence also Theorem 3.1.

We now prove the second of the inequalities listed in Theorem 10.1.

LEMMA 10.1.   *The following inequality is true*

$$(168)\qquad 0 \geq \limsup_{r\to\infty} k_r^{-2} [-(\mu_\alpha(t^0) - K^\alpha)\psi_{x^i}^\alpha(t^0)\Delta x_r^i(t^0)]$$

$$-\frac{1}{2} (\mu_\alpha(t^0) - K^\alpha)\psi_{x^i x^j}^\alpha(t^0)\eta_0^i(t^0)\eta_0^j(t^0).$$

$$i, j = 1, \cdots, N \qquad \alpha = 1, \cdots, m.$$

Proof.   According to the last property in (4–1), we see that we need only consider those indices $\bar\alpha$ such that

$$(169)\qquad\qquad\qquad \psi^{\bar\alpha}(t^0) = 0.$$

Then by Taylor's theorem together with the admissibility of our arcs, we have that for each such $\bar\alpha$

$$(170)\qquad 0 \geq \Delta\psi^{\bar\alpha}(t^0, x_r(t^0)) = \psi_x^{\bar\alpha}(t^0)\Delta x_r^i(t^0)$$

$$+\frac{1}{2} \tilde\psi_{x^i x^j}^{\bar\alpha}\Delta x_r^i(t^0)\Delta x_r^j(t^0) \qquad i, j = 1, \cdots, N$$

where $\tilde\psi_{x^i x^j}^{\bar\alpha}$ indicates evaluation at an intermediate point on the line segment $[x_0(t^0) + \theta\Delta x_r(t^0)]$ $0 < \theta < 1$. Now by multiplying by $- k_r^{-2}(\mu_{\bar\alpha}(t^0) - K^{\bar\alpha})$, taking superior limits, using the convergence of $\eta_r(t^0)$ to $\eta_0(t^0)$ and the last property in (4–1), we get (168) for each such index $\bar\alpha$ and hence for the sum of those indices, proving the lemma.

It remains only to prove the first inequality listed in the hypothesis of Theorem 10.1.   By using arguments directly analagous to those used in

Lemma 11.3 of [1], but with $\dot{\mu}_\alpha$ replacing λ^β, one proves the required inequality with lim inf replacing lim sup. Since lim sup $\geqq$ lim inf, the required inequality is certainly true. We state this result

LEMMA 10.2. *The following inequality is true*

$$(171) \qquad \limsup k_r^{-2} E_T(a_r) \geqq \frac{1}{2} \int_{t^0}^{t^1} G_{p^h p^k}(\alpha_0^h - \rho_0^h)(\alpha_0^k - \rho_0^k)\,dt$$

$$h, k = 1, \cdots, K.$$

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Pacific Journal of Mathematics
Vol. 62, No. 1   January, 1976