Pacific Journal of Mathematics

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Vol. 62, No. 1 January 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 1, 1976

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Three notions of a basis for an ultrafilter in a Boolean algebra are investigated in this paper, namely having an independent set of generators, a weakly independent set of generators and a weakly independent set of generators over a proper subfilter. In general these three notions are distinct, but for a Boolean algebra with an ordered base the latter two are equivalent. This paper shows that a large class of Boolean algebras do not possess ultrafilters with a basis, in particular no infinite homomorphic image of a σ-complete Boolean algebra has a nonprincipal ultrafilter with a basis. For Boolean algebras with an ordered base necessary and sufficient conditions on the order type of the base are given for the Boolean algebra to have the basis property.

Introduction. The notion of an independent family of sets was first introduced in **[1]** by Fichtenholz–Kantorovitch. Their results were generalized in **[3]** by Hausdorff where it was shown that, if $|I| = m$, there exists an independent family of subsets of I of power 2^m. It is well known that the free Boolean algebra on m generators is generated by an independent family of elements of power m and that every ultrafilter in this algebra has an independent set of generators. A weaker notion, that of an irredundant set of generators, or a weakly independent set of generators, has been considered by both A. Tarski **[11]** and I. Reznikoff **[7]** in the setting of mathematical logic, and it is the algebraic version of this notion which we call a basis for an ultrafilter. Boolean algebras in which every ultrafilter has a basis are said to have the basis property. The idea of an independent set modulo a filter has been used by K. Kunen in **[4]** and this leads to the property considered here, that of a basis over a filter.

The first section consists of the definitions and basic lemmas concerning the above mentioned three notions and a theorem showing that a large class of Boolean algebras do not have the basis property. In §2 Boolean algebras with an ordered base are considered and for this class of Boolean algebras necessary and sufficient conditions on the order type of the base are given for the Boolean algebra to have the basis property. For these Boolean algebras, the latter two notions of a basis are shown to be equivalent and further, any such Boolean algebra with the basis property must have cardinality less than or equal to $2^{\aleph_0}$. Finally a summary of the relationships between these three concepts of basis is given.

Preliminaries. If $\mathfrak{A}$ is a Boolean algebra we assume $\mathfrak{A} = \langle A, \vee, \wedge, -, 0, 1\rangle$ and if $\mathcal{F}$ is a filter in $\mathfrak{A}$, and $a \in A$, we write $a/\mathcal{F} = \bar{a}$. The basic results concerning Boolean algebras may be found in **[2]** or **[10]**. We recall that a filter in a Boolean algebra is generated by $\{b_\nu\}_{\nu<\alpha} \subset \mathcal{F}$ if for each $a \in \mathcal{F}$ there exists $\nu_1, \cdots, \nu_k < \alpha$ with $b_{\nu_1} \wedge \cdots \wedge b_{\nu_k} \leqq a$. A family of elements $\{a_\nu\}_{\nu<\alpha} \subset A$ is independent if, for all $\nu_1, \cdots, \nu_n < \alpha$ which are distinct, $b_{\nu_1} \wedge \cdots \wedge b_{\nu_n} \neq 0$ where for each i, $b_{\nu_i} = a_{\nu_i}$ or $-a_{\nu_i}$.

1. DEFINITION 1.1. A filter $\mathcal{F}$ in a Boolean algebra $\mathfrak{A}$ is said to have a *basis* $\{a_\nu\}_{\nu<\alpha}$ if

(i) $\{a_\nu\}_{\nu<\alpha}$ generates $\mathcal{F}$, and

(ii) if $\nu_0, \cdots, \nu_{n+1} < \alpha$ are distinct, then $a_{\nu_0} \wedge \cdots \wedge a_{\nu_n} \not\leqq a_{\nu_{n+1}}$.

A Boolean algebra is said to have the *basis property* if each ultrafilter has a basis. The definition of basis is weaker than that of independent set of generators. For example, in the Boolean algebra of finite and cofinite subsets of ω, there is only one nonprincipal ultrafilter and it has a basis but does not have an independent set of generators:

If $\{A_n\}_{n\in\omega} \subset \mathcal{F}$ is an independent set of generators then each A_n is cofinite. Let $B_0 = \{n_1, \cdots, n_k\} = \sim A_0$ and, for $i = 1, \cdots, k$, let

$$B_i = \begin{cases} A_i & \text{if } n_i \in A_i \\ \sim A_i & \text{otherwise} \end{cases}$$

Then $B_0 \subset B_1 \cup \cdots \cup B_k$. We may assume without loss of generality that $B_0 \subset (\sim A_1 \cup \cdots \cup \sim A_j) \cup (A_{j+1} \cup \cdots \cup A_k)$. Thus, since $B_0 = \sim A_0$, we have $(A_1 \cap \cdots \cap A_j) \cap \sim A_0 \subset (A_{j+1} \cup \cdots \cup A_k)$ and hence $A_1 \cap \cdots \cap A_j \subset A_0 \cup A_{j+1} \cup \cdots \cup A_k$ which contradicts the independence of the A_i's. On the other hand, the complements of singletons form a basis for $\mathcal{F}$.

Condition (ii) is an irredundancy condition and is the algebraic translation of the logical notion of an independent set of formulas, apparently first introduced by Tarski **[11]**. This algebraic version will be referred to as weak independence. In this connection, it is interesting to note that Reznikoff in **[7]** showed that every filter in a free Boolean algebra has a basis. The following notion of a basis for one filter over another filter is a modification of the definition of being independent modulo a filter (See **[4]**):

DEFINITION 1.2. Let $\mathcal{G}$ and $\mathcal{F}$ be filters in the Boolean algebra $\mathfrak{A}$ with $\mathcal{F} \supsetneqq \mathcal{G}$. $\{a_\nu\}_{\nu<\alpha} \subset \mathcal{F}$ is a basis for $\mathcal{F}$ over $\mathcal{G}$ if

(i) $\mathcal{G} \cup \{a_\nu\}_{\nu<\alpha}$ generates $\mathcal{F}$ and

(ii) if $\nu_0, \cdots, \nu_{n+1} < \alpha$ are distinct, then $-a_{\nu_0} \vee \cdots \vee -a_{\nu_n} \vee a_{\nu_{n+1}} \notin \mathcal{G}$.

In particular, condition (ii) allows one to extend $\mathcal{G}$ to a proper filter containing $a_{\nu_0} \wedge \cdots \wedge a_{\nu_n} \wedge -a_{\nu_{n+1}}$.

If $\mathcal{F}$ is a filter in a Boolean algebra $\mathfrak{A}$, and $\mathcal{I} = \{-x : x \in \mathcal{F}\}$ then $\mathcal{I}$ is an ideal and, by $\mathfrak{A}/\mathcal{F}$ we mean the quotient algebra $\mathfrak{A}/\mathcal{I}$.

LEMMA 1.3. *Let $\mathfrak{A}$ be a Boolean algebra, $\mathcal{F}$ an ultrafilter in $\mathfrak{A}$, $\mathcal{G}$ a filter in $\mathfrak{A}$, $\bar{\mathfrak{A}} = \mathfrak{A}/\mathcal{G}$ and $\bar{\mathcal{F}} = \{\bar{a} : a \in \mathcal{F}\}$. Then $\mathcal{F}$ has a basis over $\mathcal{F} \cap \mathcal{G}$ if and only if $\bar{\mathcal{F}}$ has a basis in $\bar{\mathfrak{A}}$.*

Proof. It is straightforward to verify that $\{a_\nu\}_{\nu<\alpha}$ is a basis for $\mathcal{F}$ over $\mathcal{F} \cap \mathcal{G}$ if and only if $\{\bar{a}_\nu\}_{\nu<\alpha}$ is a basis for $\bar{\mathcal{F}}$ in $\bar{\mathfrak{A}}$.

LEMMA 1.4. *Let $\mathfrak{A}$ be an infinite Boolean algebra and $\{x_n\}_{n\in\omega}$ an infinite set of distinct ultrafilters in $\mathfrak{A}$. Then there exists an infinite subsequence $\{x_{n_k}\}_{k\in\omega} \subset \{x_n\}_{n\in\omega}$ and $\{a_k\}_{k\in\omega} \subset A$ of pairwise disjoint elements with $a_k \in x_{n_k}$.*

Proof. Let $b_0 \in x_0$ with $-b_0 \in x_1$ and let $B_0^+ = \{x_n : b_0 \in x_n\}$ $B_0^- = \{x_n : -b_0 \in x_n\}$. If $|B_0^+| = \aleph_0$, set $B_0 = B_0^+$, $a_0 = -b_0$, $c_0 = b_0$ and $n_0 = 1$. If $|B_0^+| \neq \aleph_0$, set $B_0 = B_0^-$, $a_0 = b_0$, $c_0 = -b_0$ and $n_0 = 0$. Suppose $\{a_k\}_{k\leqq m}$, $\{c_k\}_{k\leqq m}$ $\{x_{n_k}\}_{k\leqq m}$ and $\{B_k\}_{k\leqq m}$ have been defined with B_k infinite for $k \leqq m$ and $i < j \leqq m$ implies $B_i \supsetneqq B_j$ and for all $k \leqq m$ we have

(i) $a_k \in x_{n_k}$, $c_k \in x$ for all $x \in B_k$, $a_k \wedge a_i = 0$ if $k \neq i$ and $a_k \wedge c_i = 0$ for all $k \leqq i$.

Let x_0^m and x_1^m be distinct elements of B_m of the form x_n, where $n > n_m$, and let $b_m \in x_0^m$ with $-b_m \in x_1^m$. Then $c_m \wedge b_m \in x_0^m$ and $c_m \wedge -b_m \in x_1^m$. Let $B_{m+1}^+ = \{y \in B_m : c_m \wedge b_m \in y\}$, $B_{m+1}^- = \{y \in B_n : c_m \wedge -b_m \in y\}$. If $|B_{m+1}^+| = \aleph_0$, let $B_{m+1} = B_{m+1}^-$, $a_{m+1} = c_m \wedge -b_m$, $c_{m+1} = c_m \wedge b_m$ and $x_{n_{m+1}} = x_1^m$. If $|B_{m+1}^+| \neq \aleph_0$, let $B_{m+1} = B_{m+1}^-$, $a_{m+1} = c_m \wedge b_m$, $c_{m+1} = c_m \wedge -b_m$ and $x_{n_{m+1}} = x_0^m$. Clearly $\{x_{n_k}\}_{k\in\omega}$ and $\{a_k\}_{k\in\omega}$ so defined have the desired properties.

THEOREM 1.5. *Let $\mathfrak{A}$ be a σ-complete Boolean algebra, $\mathcal{G}$ a filter in $\mathfrak{A}$ and $\mathcal{F}$ an ultrafilter in $\mathfrak{A}$ with $\mathcal{F} \supsetneqq \mathcal{G}$. Then $\mathcal{F}$ has a basis over $\mathcal{G}$ if and only if there exists a $b \in A$ such that $\mathcal{F}$ is generated by $\mathcal{G} \cup \{b\}$.*

Proof. Obviously if $\mathcal{F}$ is generated by $\mathcal{G} \cup \{b\}$ then $\mathcal{F}$ has a basis over $\mathcal{G}$. Now suppose $\mathcal{F}$ has a basis $\{a_\nu\}_{\nu<\alpha}$ over $\mathcal{G}$ and that α is infinite (if α is finite, then clearly $\mathcal{G} \cup \{b\}$ generates $\mathcal{F}$ where $b =$

$a_0 \wedge a_1 \wedge \cdots \wedge a_{\alpha-1}$). Let $\mathcal{F}_\nu$ be an ultrafilter in $\mathfrak{A}$ such that $\mathcal{F}_\nu \supset \mathcal{G} \cup \{a_\mu\}_{\mu<\alpha,\, \mu\neq\nu} \cup \{-a_\nu\}$. This is possible by the definition of a basis over $\mathcal{G}$.

(1) If $a \in \mathcal{F}$ then $\{\nu < \alpha : a \notin \mathcal{F}_\nu\}$ is finite. Since $a \in \mathcal{F}$ there exists $b \in \mathcal{G}$ and $\nu_0, \cdots, \nu_n < \alpha$ such that $b \wedge a_{\nu_0} \wedge \cdots \wedge a_{\nu_n} \leqq a$ and $a \in \mathcal{F}_\nu$ for all $\nu \neq \nu_0, \cdots, \nu_n$.

By Lemma 1.4 there exists a subsequence $\{\mathcal{F}_{\nu_k}\}$ of $\{\mathcal{F}_\nu\}_{\nu<\alpha}$ and $b_k \in \mathcal{F}_{\nu_k}$ such that $b_k \wedge b_j = 0$ for $k \neq j$. Since $\mathfrak{A}$ is σ-complete there exist $b = \bigvee_{k\in\omega} b_{2k}$ and $c = \bigvee_{k\in\omega} b_{2k+1}$. Since $b \in \mathcal{F}_{\nu_{2k}}$ for all k we have $-b \notin \mathcal{F}$ by (1) and therefore $b \in \mathcal{F}$. Similarly $c \in \mathcal{F}$. But $b \wedge c = 0$ which contradicts $0 \notin \mathcal{F}$. Thus if $\mathcal{F}$ has a basis in $\mathfrak{A}$, the basis is finite.

COROLLARY 1.6. *Let $\mathfrak{A}$ be a σ-complete Boolean algebra and $\mathfrak{B}$ a homomorphic image of $\mathfrak{A}$. Then no nonprincipal ultrafilter in $\mathfrak{B}$ has a basis.*

Proof. Immediate by the previous theorem and Lemma 1.3.

One easily sees that if $\mathfrak{A}$ has the basis property, this does not imply that, given an ultrafilter $\mathcal{F}$ extending a filter $\mathcal{G}$, $\mathcal{F}$ has a basis over $\mathcal{G}$. For example let $\mathfrak{A}_{\boldsymbol{m}}$ be the free Boolean algebra on $\boldsymbol{m}$ generators and let $\mathfrak{B}$ be an atomless σ-complete Boolean algebra with $|\mathfrak{B}| < \boldsymbol{m}$. Then there exists a filter $\mathcal{G}$ in $\mathfrak{A}_{\boldsymbol{m}}$ such that $\mathfrak{A}_{\boldsymbol{m}}/\mathcal{G} \cong \mathfrak{B}$, but if $\mathcal{F}$ is an ultrafilter extending $\mathcal{G}$, then $\mathcal{F}$ has no basis over $\mathcal{G}$ by Lemma 1.3 and Corollary 1.6. It is interesting to note that for Boolean algebras with an ordered base, if an ultrafilter has a basis then it has a basis over every filter which it extends (see 2.5 and the remark preceding it). If an ultrafilter has a basis over every proper filter which it extends then it does have a basis since it has a basis over $\{1\}$. In fact if $\mathcal{G} \subset \mathcal{F}$, such that $\mathcal{F}$ has a basis over $\mathcal{G}$ and there exists $a \in \mathcal{F}$ with $a \leqq b$ for all $b \in \mathcal{G}$, then $\mathcal{F}$ has a basis in $\mathfrak{A}$.

COROLLARY 1.7. *Let $\mathfrak{A}$ be Boolean algebra and $\mathcal{F}$ an ultrafilter in $\mathfrak{A}$. If $\mathcal{F}$ has a basis over every proper subfilter, then $\mathcal{F}$ has a basis in $\mathfrak{A}$.*

In addition to free Boolean algebras, it is well known that every countable Boolean algebra has the property that all nonprinciple ultrafilters have a basis.

The following lemma, probably first proved by Tarski **[9]** establishes this result:

LEMMA 1.8. *Let $\mathcal{F}$ be an ultrafilter in a Boolean algebra $\mathfrak{A}$ such that $\mathcal{F}$ has a countable set $\{a_n\}_{n\in\omega}$ of generators. Then $\mathcal{F}$ has a basis.*

Proof. Assume without loss of generality that $a_0 \neq 1$ and for $n < m$, $a_n > a_m$. Let $b_n = a_n \vee (\bigvee_{k<n} - a_k)$.

(1) $a_n \leqq b_n$ for all $n \in \omega$,

(2) $- a_n \leqq b_m$ for all $n < m$,

(3) $b_n \vee b_m = 1$ for all $n \neq m$.

Now

(4) $\{b_n\}_{n\in\omega}$ is weakly independent if $\{b_{k_1} \wedge \cdots \wedge b_{k_n}\} \leqq b_{k_{n+1}}$ then $-(b_{k_1} \wedge \cdots \wedge b_{k_n}) \wedge b_{k_{n+1}} = 1$.

But, since $b_{k_i} \vee b_{k_{n+1}} = 1$ for $1 \leqq i \leqq k$ by (3), we have $(b_{k_1} \wedge \cdots \wedge b_{k_n}) \vee b_{k_{n+1}} = 1$. Since $b_{k_{n+1}} \neq 1$, this is a contradiction.

(5) $\{b_n\}_{n\in\omega}$ generates $\mathcal{F}$.

A simple inductive proof shows $a_n = \bigwedge_{i \leqq n} b_i$.

COROLLARY 1.9. *Let $\mathfrak{A}$ be a σ-complete Boolean algebra and $\mathfrak{B}$ a homomorphic image of $\mathfrak{A}$. Then no nonprincipal ultrafilter in $\mathfrak{B}$ has a countable set of generators.*

Proof. By 1.6 and 1.8.

From 1.9, the well-known result that no infinite homomorphic image of a σ-complete Boolean algebra is countable is immediate.

The question of whether every projective Boolean algebra has the basis property is open. Since little is known about projective Boolean algebras a positive answer to this question would be most interesting. A characterization of those Boolean algebras with the basis property, or one for those ultrafilters with a basis — perhaps in terms of chains in the filter — are additional areas of investigation. These latter two problems are answered completely in the case of Boolean algebras with an ordered base in the next section.

2. In this section we restrict the discussion to Boolean algebras with an ordered base. These Boolean algebras were first introduced by Mostowski and Tarski in **[6]** and have been studied more recently by Mayer and Pierce **[5]** and Rotman **[8]** where additional references may be found. Rotman shows that in a Boolean algebra with an ordered base there are at most countably many independent elements. The question for weakly independent elements appears to be open.

DEFINITION 2.1. A Boolean algebra $\mathfrak{A}$ has an ordered base X if X is linearly ordered by $<$ (the order in $\mathfrak{A}$), X generates $\mathfrak{A}$, $0 \in X$ and $1 \notin X$.

If $(A, \leqq)$ is a linearly ordered set, then the cofinality of A (cf(A)) is

$\inf\{|B|$: for all $a \in A$ there exists $b \in B \subset A$ with $a \leqq b\}$. The coinitiality of A ($\mathrm{ci}(A)$) is the $\inf\{|B|$: for all $a \in A$ there exists a $b \in B \subset A$ with $b \leqq a\}$. An initial segment of A is a set $B \subset A$ such that if $b \in B$ and $a < b$ then $a \in B$. A tail of A is the complement of an initial segment.

LEMMA 2.2. *Let $\mathfrak{A}$ be a Boolean algebra with ordered base X, Y an initial segment of X and $\mathscr{F}$ an ultrafilter in $\mathfrak{A}$ containing $\{-y; y \in Y\} \cup (X \sim Y)$. If $x \in \mathscr{F}$, then there exists $y \in Y$ and $z \in X \sim Y$ such that $x \geqq -y \wedge z$.*

Proof. Since X is a set of generators for the Boolean algebra and $\mathscr{F}$ is an ultrafilter, the conclusion is obvious.

THEOREM 2.3. *If $\mathfrak{A}$ is a Boolean algebra with ordered base X, and there exists an initial segment $Y \subset X$ with $\mathrm{cf}(Y) > \aleph_0$ or there exists a tail $Z \subset X$ with $\mathrm{ci}(Z) > \aleph_0$, then $\mathfrak{A}$ does not have the basis property of ultrafilter.*

Proof. We may assume there exists an initial segment $Y \subset X$ with $\mathrm{cf}(Y) > \aleph_0$, for otherwise $\{0\} \cup \{-x: x \in X \sim \{0\}\}$ is an ordered basis with an initial segment Z having cofinality greater than $\aleph_0$. Let $Y = \{a_i\}_{i \in I}$ and let $\mathscr{F}$ be an ultrafilter such that $\mathscr{F} \supset \{-a_i\}_{i \in I}$ and $\mathscr{F} \supset X - Y$. Suppose $\mathscr{F}$ has a basis $\{c_\nu\}_{\nu < \lambda}$.

(1) $|\lambda| > \aleph_0$ — we first note that no finite meet d of basis elements is less than or equal to all $-a_i$, for otherwise by Lemma 2.2 there exists $-a_j$ and $x \in X - Y$ with $-a_j \wedge x \leqq d \leqq -a_i$ for all $i \in I$. Hence $-a_j \wedge x \leqq -a_i \wedge x$ for all $i \in I$. Choosing $a_i > a_j$ we have $a_i < x$ since Y is an initial segment of X. Thus $a_i \wedge -a_j \wedge x = a_i \wedge -a_j = 0$ so $a_i \leqq a_j$, a contradiction.

By the above argument, $|\lambda| \geqq \aleph_0$, so assume $|\lambda| = \aleph_0$. Let $d_n = c_0 \wedge \cdots \wedge c_n$. Again by above argument, for each $n \in \omega$ there exists $-a_{i_n}$ such that $d_n \nleqq -a_{i_n}$. Since $\mathrm{cf}(Y) > \aleph_0$, we arrive at an obvious contradiction — hence (1) is established.

(2) $\mathscr{F}$ has no basis.

Case 1. $\mathrm{ci}(X \sim Y) > \aleph_0$.

Let $\{b_j\}_{j \in J} = X \sim Y$. By Lemma 2.2 there exists $i_0 \in I$ and $j_0 \in J$ with $c_0 \geqq -a_{i_0} \wedge b_{j_0} \geqq d_0$ where d_0 is a finite meet of the basis elements $\{c_\nu\}_{\nu < \lambda}$. By Lemma 2.2, choose $i_1 \in I$ and $j_1 \in J$ such that $-a_{i_1} \wedge b_{j_1} < d_0$. Proceeding in this manner we construct $c_0 \geqq -a_{i_0} \wedge b_{i_0} \geqq d_0 \geqq \cdots \geqq$

$-a_{i_n} \wedge b_{i_n} \gneqq d_n \geqq \cdots$ where $-a_{i_n} \wedge b_{i_n} > -a_{i_{n+1}} \wedge b_{i_{n+1}}$ and d_n is a finite meet of basis elements. Since $\mathrm{cf}(Y) > \aleph_0$, there exists $a \in Y$ with $-a \leqq -a_{i_n}$ for all $n \in \omega$ and $b \in X \sim Y$ with $b \leqq b_{i_n}$ for all $n \in \omega$. Hence $-a \cdot b \leqq d_n$ for all $n \in \omega$. Now there exist $\nu_1, \cdots, \nu_k$ with $c_{\nu_1} \wedge \cdots \wedge c_{\nu_k} \leqq -a \cdot b \leqq d_n$ for all $n \in \omega$. Since the d_n's are strictly decreasing, there exists c_{ν_0} occurring in some d_{n_0} with $c_{\nu_0} \neq c_{\nu_1}, \cdots, c_{\nu_k}$. Hence $c_{\nu_1} \wedge \cdots \wedge c_{\nu_k} \leqq d_{n_0} < c_{\nu_0}$ contradicting the weak independence of the c_ν's.

Case 2. $\mathrm{ci}(X \sim Y) \leqq \aleph_0$.

We observe that for each $a \in Y$ and $b \in X \sim Y$, $|\{c_\nu : -a \wedge b \leqq c_\nu\}| < \aleph_0$ — for otherwise there exist $c_{\nu_1}, \cdots, c_{\nu_k}$ with $c_{\nu_1} \wedge \cdots \wedge c_{\nu_k} \leqq -a \wedge b$ with $-a \wedge b$ less than infinitely many c_ν's which contradicts the weak independence of the c_ν's. Let $\{b_n\}_{n \in \omega}$ be coinitial with $X \sim Y$. Let $B_n = \{c_\nu : -a_i \wedge b_n \leqq c_\nu$ for some $i \in I\}$. Since $\bigcup_{n \in \omega} B_n = \{c_\nu\}_{\nu < \lambda}$ and λ is uncountable there is an n_0 with B_{n_0} infinite. This implies there is a countable $I_0 \subset I$ such that $|\{c_\nu : -a_i \wedge b_{n_0} \leqq c_\nu$ for some $i \in I_0\}| \geqq \aleph_0$. Since $\mathrm{cf}(Y) > \aleph_0$ there exists $-a \in Y$ with $-a \leqq -a_i$ for all $i \in I_0$. Hence $-a \wedge b_{n_0}$ is less than or equal to infinitely many basis elements — a contradiction.

THEOREM 2.4. *Let $\mathfrak{A}$ be a Boolean algebra with ordered base X. If* $\mathrm{cf}(Y) \leqq \aleph_0$ *for every initial segment Y of X and* $c_i(Z) \leqq \aleph_0$ *for every tail Z of X, then $\mathfrak{A}$ has the basis property for ultrafilters.*

Proof. Let $\mathcal{F}$ be an ultrafilter in $\mathfrak{A}$. Let $Y = \{y \in X : y \notin \mathcal{F}\}$. Then Y is an initial segment of X and, by Lemma 2.2, $\mathcal{F}$ is generated by $\{-y : y \in Y\} \vee (X \sim Y)$. By hypothesis there is a countable sequence $\{x_n\}$ which is coinitial in $X \sim Y$ and a countable sequence $\{y_n\}$ which is cofinal in Y. Clearly $\mathcal{F}$ is generated by $\{-y_n\} \cup \{x_n\}$. Therefore, by Lemma 1.8 $\mathcal{F}$ has a basis.

Theorems 2.3 and 2.4 completely characterize the order types of ordered bases which give rise to Boolean algebras with the basis property for ultrafilters.

Boolean algebras with an ordered base which have the basis property for ultrafilters have in fact a stronger property — namely, every ultrafilter has a basis over every filter which it contains (see remark following 1.6). This is established by Lemma 1.3 and the following:

COROLLARY 2.5. *Let $\mathfrak{A}$ be a Boolean algebra with ordered base X and suppose $\mathfrak{A}$ has the basis property for ultrafilters. If $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$ then $\mathfrak{B}$ has the basis property for ultrafilters.*

Proof. Let $h: \mathfrak{A} \to \mathfrak{B}$ be a homomorphism and let $X' = h(X) \sim \{1\}$. Then one easily checks that X' is an ordered base for $\mathfrak{B}$. By 2.4 it suffices to check that if Y' is any initial segment of X' and Z' is any tail of X', then $\mathrm{cf}(Y') \leqq \aleph_0$ and $\mathrm{ci}(Z') \leqq \aleph_0$. Since $Y = \{y \in X: h(y) \in Y'\}$ has cofinality $\leqq \aleph_0$ by 2.3, it is easy to verify that $\mathrm{cf}(Y') \leqq \aleph_0$. Similarly, one sees that $\mathrm{ci}(Z') \leqq \aleph_0$.

By the proof of 2.3 and 2.4 it is clear that if $\mathfrak{A}$ is a Boolean algebra with an ordered basis X and $\mathscr{F}$ is an ultrafilter in $\mathfrak{A}$ then $\mathscr{F}$ has a basis iff $\mathrm{cf}(Y) \leqq \aleph_0$ and $\mathrm{ci}(Z) \leqq \aleph_0$ where $Y = \{x \in \mathscr{F}: -x \in X\}$ and $Z = \{x \in \mathscr{F}: x \in X\}$. Similarly, as in 2.5, if $\mathscr{F}$ has a basis and $\mathscr{G} \subset \mathscr{F}$, then $\bar{\mathscr{F}}$ has a basis in $\mathfrak{A}/\mathscr{G}$. Combined with 1.8 this gives us

COROLLARY 2.6. *If $\mathfrak{A}$ is a Boolean algebra with an ordered base and $\mathscr{F}$ is an ultrafilter in $\mathfrak{A}$ then the following are equivalent:*

(i) *$\mathscr{F}$ has a basis*

(ii) *$\mathscr{F}$ has a basis over every filter $\mathscr{G} \subset \mathscr{F}$.*

THEOREM 2.7. *Let $\mathfrak{A}$ be a Boolean algebra with an ordered base X. If $\mathfrak{A}$ has the basis property for ultrafilters, then $|\mathfrak{A}| \leqq 2^{\aleph_0}$.*

Proof. Let X^* be the completion by cuts of X. Then X^* is a compact, first countable Hausdorff space under the order topology and hence has cardinality $\leqq 2^{\aleph_0}$ (see **[9]**). As a consequence $|\mathfrak{A} \leqq 2^{\aleph_0}$.

We summarize results concerning our three notions of basis in the following table where we use the notation:

(I) $\mathscr{F}$ has an independent set of generators

(II) $\mathscr{F}$ has a basis

(III) $\mathscr{F}$ has a basis over every proper subfilter.

	Boolean algebras	Boolean algebras with an ordered base
I → II	Yes	Yes
II → III	No	Yes
III → I	No	No
I → III	No	Yes
II → I	No	No
III → II	Yes	Yes

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Received June 26, 1975, and in revised form November 3, 1975.

EMORY UNIVERSITY

Pacific Journal of Mathematics

Vol. 62, No. 1 January, 1976