IN Variant Manifolds on Non-Linear Operators

John Campbell Wells
INVARIANT MANIFOLDS OF NON-LINEAR OPERATORS

JOHN C. WELLS

In this paper we generalize the classical stable manifold theorem at a point as well as a recent result of M. Hirsch, C. Pugh and M. Shub. We deduce the existence of the invariant manifolds, their smoothness and their continuity under small perturbations of the underlying endomorphism entirely from the inverse function theorem and an easy proposition about smoothness of maps on $c_0(E)$. The constructive nature of our proof has the advantage of ready adaptation to numerical methods.

Introduction. The following classes of maps between Banach spaces will be used (all derivatives are Frechet derivatives). Let $\text{Lip}(f)$ denote the Lipschitz constant of $f$ and let $\mathcal{H}(E, F) = \{f | \text{Lip}(f) < \infty\}$. For $p \geq 1$ and $0 < \alpha \leq 1$ let $\mathcal{H}(E, E)$ denote the classes

$$
C^p = \{f | f \text{ has } p \text{ continuous derivatives}\} \\
B^p_\alpha = \{f | f \in C^p, \|D^p f(x + h) - D^p f(x)\| \leq M \|h\|^\alpha \text{ for some } M\} \\
C^\alpha_p = \{f | f \in C^p \text{ and } D^p \text{ is uniformly continuous}\} \\
C^\beta_p = \{f | f \in C^p \text{ and } D^p \text{ is bounded}\} \\
C^\infty_p = \{f | f \in C^p \text{ for all } p\} \\
B^\infty_p = \{f | f \in B^p_\alpha \text{ for all } p\}
$$

We will also use the following norms (or pseudo-norms)

$$
\|f\|_0 = \sup_x \|f(x)\| \\
\|f\|_p = \max(\|f\|_0, \text{Lip } f, \cdots, \text{Lip } D^{p-1} f)
$$

If $f \in C^p$ then $\|f\|_p = \max_{0 \leq i \leq p} \sup_x \|D^i f(x)\|$.

For maps in these classes we have an inverse function theorem. We start with a Lipschitz inverse function theorem. A stronger version of this theorem is given in Hirsch-Pugh [3]. We provide a proof along their lines for completeness.

Lipschitz Inverse Function Theorem. Let $T$ be a linear invertible map from $E$ to $E$. Suppose $f: U \to E$, $U$ an open nbhd of $0$ in $E$, $f(0) = 0$ and $\text{Lip}(f) \cdot \|T^{-1}\| = \lambda < 1$. Then $T + f$ is a homeomorphism of $U$ onto an open subset $V$ of $E$, $(T + f)^{-1}$ is Lipschitz and $\text{Lip}(T + f)^{-1} \leq \frac{\lambda}{1 - \lambda}$.
If $U$ contains the ball $B_r$ of radius $r$ and center $0$, then $V$ contains the ball $B_{r'}$ of radius $r' = r(1 - \lambda)/\|T^{-1}\|$ and center $0$. The map $f \to f^{-1}$ from Lip to Lip is continuous in the $\|\|_0$ topology on the range and domain of $\to$.

**Proof.** Consider the set $\mathcal{L}$ of maps $g : B_r \to E$ for which $\text{Lip} g \leq \lambda \cdot \|T^{-1}\|/(1 - \lambda)$ and $g(0) = 0$. If $g \in \mathcal{L}$ then $(T^{-1} + g)(B_r) \subset B_r$, so the map $g' = -T^{-1} \circ f \circ (T^{-1} + g)$ is defined. Furthermore $\text{Lip} g' \leq \lambda \cdot \|T^{-1}\|/(1 - \lambda)$, so $g' \in \mathcal{L}$. If $h' = -T^{-1} \circ f \circ (T^{-1} + h)$ with $h \in \mathcal{L}$ then $\|h' - g'\|_0 \leq \lambda \cdot \|h - g\|_0$. Thus the map $g \to -T^{-1} \circ f \circ (T^{-1} + g)$ is a contraction of the complete space $\mathcal{L}$ in the topology $\|\|_0$. Thus there is a unique fixed point $g \in \mathcal{L}$ satisfying

$$g = -T^{-1} \circ f \circ (T^{-1} + g)$$

This last equation implies $\text{Id} + T \cdot g + f \circ (T^{-1} + g) = \text{Id}$ so that $(T + f) \circ (T^{-1} + g) = \text{Id}$. Observe that

$$\|(T + f)(x) - (T + f)(y)\| \geq \|T\| \cdot \|x + T^{-1} \circ f(x) - (y + T^{-1} \circ f(y))\| \geq \|T\| \cdot \{\|x - y\| - ||T^{-1}\| \cdot \text{Lip}(f) \cdot \|x - y\|\} \geq \|T\| \cdot (1 - \lambda) \cdot \|x - y\|.$$  

Thus $T + f$ is 1–1 on all of $U$ and $T^{-1} + g$ is the inverse of $T + f$ on $B_r$. Also $\text{Lip}(T^{-1} + g) \leq \|T^{-1}\| + \lambda \cdot \|T^{-1}\|/(1 - \lambda) \leq \|T^{-1}\|/(1 - \lambda)$. By translating coordinates so that $x_0 \to 0$ and $(T + f)(x_0) \to 0$, the above reasoning show that $\text{Lip}(T + f)^{-1} \leq \|T^{-1}\|/(1 - \lambda)$ on all of $V$. The openness of $V$ is also obtained. Finally if $f$ and $f'$ are invertible maps with $f^{-1}$ Lipschitz then

$$\|f^{-1} - f'^{-1}\|_0 \leq |f^{-1} \circ f \circ f'^{-1} - f^{-1} \circ f' \circ f'^{-1}| \leq \text{Lip}(f^{-1}) \cdot \|f - f'|_0.$$  

This proves the last statement.

**Inverse Function Theorem.** Suppose $U$ is an open subset of $E$, $f : U \to E$ is Lip or is one of the Classes $\mathcal{K}(E, E)$. Suppose $T : E \to E$ is a linear invertible map from $E$ to $E$ such that $\text{Lip}(f - T) \cdot \|T^{-1}\| \leq \lambda$ for some $\lambda < 1$. Then $f$ is a homeomorphism of $U$ onto an open subset of $E$ and $f^{-1}$ is in the same class as $f$. The map $f \to f^{-1}$ from $\{f \mid \text{Lip}(f - T) \cdot \|T^{-1}\| \leq \lambda\}$ to $f^{-1}$ is continuous in the following way (the indicated topologies apply to both $f$ and $f^{-1}$).

**Table 1**

<table>
<thead>
<tr>
<th>Classes</th>
<th>Pseudo-norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lip or $C_p^r$, $1 \leq p \leq \infty$</td>
<td>$||<em>0$ or $||</em>{p-1}$</td>
</tr>
<tr>
<td>$C_b^r$</td>
<td>$||_0$, $||_b$, $||<em>r$, $||</em>{p-1}$</td>
</tr>
<tr>
<td>$B^r$, $C_b^r$</td>
<td>$||_0$, $||_1$, $||_r$ or $||_p$</td>
</tr>
<tr>
<td>$B^*$</td>
<td>$||_0$, $||_p$ or $||_r$</td>
</tr>
</tbody>
</table>

**Remark.** If $E$ is finite dimensional then $C_p = C_b^\infty$. 
Proof. The case $f \in \text{Lip}$ is a restatement of the Lipschitz Inverse Function Theorem. In all other cases we can conclude from the LIFT that $f$ is a homeomorphism of $U$ onto an open subset and that $f^{-1}$ is Lipschitz.

Observe that $f, g$ in any of the classes with domain $(g)$ bounded implies $f \circ g$ is of the same class. Now if $f$ is at least $C^1$ then $f^{-1}$ is differentiable by the usual argument which we give for completeness: It suffices to assume $f(0) = 0$ and show $f^{-1}$ differentiable at $0$.

Observe $f(0 + h) - f(0) - Df(0)[h] = f(h) - Df(0)[h] = o(\|h\|)$. So

$$f(f^{-1}(h')) - Df(0)[f^{-1}(h')] = o(\|f^{-1}(h')\|)$$

which gives $Df^{-1}(0) \cdot [f(f^{-1}(h')) - f^{-1}(h')] = Df^{-1}(0) \cdot o(\|f^{-1}(h)\|)$ so $f^{-1}(h') - Df^{-1}(0)[h'] = o(\|h\|)$. Thus $f^{-1}$ is differentiable and $Df^{-1}(y) = (Df^{-1}(y))^{-1}$. Now suppose $f$ is in one of $C^p, B^p_\alpha, C^p_\alpha$ or $C^p_\beta$, $p \geq 1$ and that $f^{-1}$ has been shown to be of class $C^{k-1}, B^k_\alpha, C^k_\alpha$ or $C^k_\beta$, respectively, where $k \leq p$. The map

$$L \to L^{-1}: \{L \mid L \in L(E, E), \text{Lip}(L - T) \cdot \|T^{-1}\| \leq \lambda < 1\} \to L(E, E)$$

is of class $B^\alpha$. Thus $Df^{-1}$ which is the composition: Inverse $\circ Df \circ f^{-1}$ is of class $C^{k-1}, B^{k-1}_\alpha, C^{k-1}_\alpha$ or $C^{k-1}_\beta$ respectively and so $f^{-1}$ is of class $C^k, B^k_\alpha, C^k_\alpha$ or $C^k_\beta$ respectively. Repeating the argument gives $f^{-1} \in C^p, B^p_\alpha, C^p_\alpha$ or $C^p_\beta$ respectively.

Now we prove the continuity table for $f \to f^{-1}$. The continuity in $\| \|_0$ for $C^p$ functions, $1 \leq p \leq \infty$ is implied by the continuity in $\| \|_0$ for Lipschitz functions. Since $B^p_\alpha \subset C^p_\alpha$, $p \geq 1$ and $C^p_\alpha \subset C^{k-1}_\alpha$, to prove the continuity results of the table, it suffices to show that $f \to f^{-1}$ is continuous in $\| \|_0$ for $f$ in $C^p_\alpha$. Suppose it has been shown for the pseudonorm $\| \|_{k-1}, k \leq p$. We have

$$D^k f^{-1}(y) = D^{k-1}(Df(f^{-1}(y)))^{-1} = [(Df(f^{-1}(y)))^{-1}] \cdot P_k(Df(f^{-1}(y))) \cdot \cdots \cdot D^k f^{-1}(y),$$

where $P_k$ is a polynomial. Now $Df^{-1}, \cdots, D^k f^{-1}$ vary in $\| \|_0$ continuously as $f$ varies in $\| \|_0$ by assumption. Also $Df, \cdots, D^k f$ are uniformly continuous by assumption so $Df^{-1} \cdot \cdots \cdot D^k f^{-1}$ vary in $\| \|_0$ as $f$ varies in $\| \|_0$. Finally $[Df(f^{-1})]^{-1}$ varies in $\| \|_0$ as $f$ varies in $\| \|_0$ by an earlier statement. Thus $D^k f^{-1}$ varies in $\| \|_0$ as $f$ varies in $\| \|_p$. A repetition of this argument implies that $f^{-1}$ varies in $\| \|_p$ as $f$ varies in $\| \|_p$.

**Definition.** $c_0(E) = \{(x_0, x_1, \cdots) \mid x_i \in E \text{ for } i \geq 0 \text{ and } \text{Lim}_i \|x_i\| = 0\}$

$c_0(E)$ is a Banach space with norm $\|x\| = \sup_i \|x_i\|.$

**Definition.** If $f$ is at least in Lip$(E, E)$ and $f(0) = 0$ and $r \geq 1$ let $C_f$ be the map from $c_0(E)$ to $c_0(E)$ defined by $[C_f(x)]_i = r^i f(x_i/r^i)$, $i = 0, 1, \cdots$. 


PROPOSITION. If \( f(0) = 0 \) and \( f \in \text{Lip} \) or \( f \) is in one of the classes \( \mathcal{K} \) and \( r \geq 1 \) then \( Cf \) is of the same class as \( f \). \( \text{Lip} \mathcal{C}f = \text{Lip}f \) and the map \( f \to Cf \) is continuous according to the following table.

<table>
<thead>
<tr>
<th>Classes</th>
<th>Pseudo-norms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Lip} )</td>
<td>( | |_0 \text{ or } | |_1 )</td>
</tr>
<tr>
<td>( C^p, C^p_k, B^p, C^p_k )</td>
<td>( | |<em>0, \ldots, | |</em>{p+1} )</td>
</tr>
<tr>
<td>( C^<em>, B^</em> )</td>
<td>( | |, \ p \geq 0 )</td>
</tr>
</tbody>
</table>

Proof. The conclusion of the theorem for Lipschitz functions is obvious. The validity of the general case results from showing that \( f \in \text{Lip} \) and \( C^k \) implies \( Cf \in C^k \) and

\[
D^k Cf(x)[h] = (D^k f(x)[0], \ldots, r^{(1-k)n} D^k f(x/r^n)[h], \ldots)
\]

For \( k = 0 \), (1) is the definition of \( Cf \). Suppose it has been proved for \( k = 0, \ldots, k - 1 \). Then

\[
\sup_n \| r^n f((x_n + h_n)/r^n) - r^n f(x_n/r^n) - Df(x_n/r^n)[h_n] - \cdots - r^{(1-k)n} D^k f(x_n/r^n)[h_n] \| \\
\leq \sup_{n, 0 < \theta_n < 1} \| (1 - \theta_n) (D^k f((x_n + \theta_n h_n)/r^n) - D^k f(x_n/r^n))[h_n] \|
\]

by the mean value theorem.

Given \( \varepsilon > 0 \) choose \( \delta \) s.t. \( \|x\| < 2\delta \) implies \( \|D^k f(x) - D^k f(0)\| < \varepsilon /2 \). Choose \( N \) so large that \( \|x_i/r^i\| < \delta \) when \( i > n \) and then choose \( \delta' < \delta \) such that when \( \|h\| < \delta' \)

\[
\| r^{(1-k)n} (D^k f((x_i + h_i)/r^i)) - D^k f(x_i/r_i)[h_i] \| < \varepsilon /2 \text{ for } 0 \leq i \leq N.
\]

Then whenever \( \|h\| < \delta' \), \( 0 < \theta_i < 1 \),

\[
\sup_i \| r^{(1-k)n} (D^k f((x_i + \theta_i h_i)/r^i)) - D^k f(x_i/r_i)[h_i] \| \leq \varepsilon \cdot \|h\|^k.
\]

By the inverse mean value theorem (see Abraham and Robbin [1]) \( Cf \in C^k \), equation (1) is true and hence the proposition is true.

Theorem 1. Suppose \( E = E_1 \oplus E_2 \) is a direct sum decomposition of a Banach space \( E \) into two Banach spaces. Suppose that \( E \) has been renormalized (isomorphically) if necessary so that \( \|(x, y)\|_E = \)
max(∥x∥_E^1, ∥y∥_E^2). Suppose L^1: E^1 → E^1 and L^2: E^2 → E^2 are bounded linear maps such that L^1 has a right inverse L^1' and ∥L^2∥ < ∥L^1∥^{-1}. Suppose that U^1 and U^2 are open spheres with center zero in E^1, E^2 and U = U^1 × U^2. Suppose f: U → E is Lip or is in one of the classes 𝒫. In addition suppose f(0) = 0, and that Lip f ≤ σ for some ∥L^2∥ < r < ∥L^1∥^{-1} and σ < min(r^{-1} · (1 - r · ∥L^1∥), r - ∥L^2∥) where \( f = f - \left( \begin{array}{c} L^1 \\ 0 \\ L^2 \end{array} \right) \). Then

(Case 1) \( r \geq 1 \). The set \( W_{E^1} = \{ x^0 \in E^1 | \exists x^1, x^2, x^3 \cdots \text{ with } f(x^{n+1}) = x^n \text{ for } n \geq 0 \text{ and } x^n = o(r^{-n}) \} \) is invariant under f and there exists \( g_1: U^1 → E^2 \) of the same class as f such that \( W_{E^1} = \{(x, g_1(x)) | x \in U^1 \} \).

(Case 2) \( r \leq 1 \). The set \( W_{E^2} = \{ x | f^n(x) = o(r^n) \} \) is invariant under f and there exists \( g_2: U^2 → E^1 \) of the same class as f such that \( W_{E^2} = \{(y, g_2(y)) | y \in U^2 \} \). \( W_{E^1}, W_{E^2} \) and \( g_1 \) and \( g_2 \) are independent of \( r \) satisfying the above conditions. In both cases \( g_1 \) and \( g_2 \) vary topologically with \( f \in \{ f | \text{Lip} f \leq \sigma \} \) according to the following table.

<table>
<thead>
<tr>
<th>Class</th>
<th>Pseudo-norm on f and ( g_1 ) or ( g_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lip, ( C^r, p \geq 1 ) and ( C^\infty )</td>
<td>( | \cdot |_p )</td>
</tr>
<tr>
<td>( C^p_0 )</td>
<td>( | \cdot |<em>0, \cdots \text{ or } | \cdot |</em>{p-1} )</td>
</tr>
<tr>
<td>( C^p_k )</td>
<td>( | \cdot |_0, \cdots \text{ or } | \cdot |_p )</td>
</tr>
<tr>
<td>( B^p )</td>
<td>( | \cdot |_p ) for all ( p )</td>
</tr>
</tbody>
</table>

**Remarks.** The case \( r = 1 \) with \( L^1 \) and \( L^2 \) invertible is the classic stable manifold and unstable manifold theorem (see A. Kelley [6] or Hirsch-Pugh [3]). The case of arbitrary \( r \), \( L^1 \) invertible and \( f \in C^r, p \geq 1 \) has been proved in Hirsch-Pugh-Shub [4] using other methods. Refer to M. Irwin [5] for a proof similar to ours in the \( r = 1 \), \( L^1 \) and \( L^2 \) invertible case.

**Proof.** In either case the sets \( W_{E^1} \) and \( W_{E^2} \) are clearly invariant under f. Let \( \Pi_i: E^1 \oplus E^2 → E_i \) be the projections. We consider first case 1.

Let \( \mathcal{X} = ((x_0, y_0), (x_1, y_1), \cdots) \) be an element of \( c_0(E) \), Define \( g: \mathcal{X}_0 → c_0(E) \) by \( g = \mathcal{L} + h \) on \( \mathcal{U}_0 = \{ \mathcal{X} | (x_i, y_i) \in U \text{ for all } i \} \) where

\[
\mathcal{L}(\mathcal{X})_i = (r \cdot L_1'(x_{i-1}), L_2(y_{i+1})/r), \quad i \geq 1
\]

\[
\mathcal{L}(\mathcal{X})_0 = (0, L_2(y_1)/r)
\]

and
By the proposition, \( \mathcal{L}, h \) and hence \( g \) is of the same class as \( f \) and furthermore 
\[
\text{Lip}(g) \leq \max (r \cdot \| L_1 \| + \| L_2 \| \cdot \text{Lip} f, \ (\| L_2 \| + \text{Lip} f)/r) > 1.
\]
Thus by the inverse function theorem and the proposition, \( G = \text{Id} - g \) is a homeomorphism of \( \mathcal{U}_0 \) onto \( G(\mathcal{U}_0) \), an open subset of \( c_0(E) \), and \( G^{-1} \) is of the same class of \( f \) and varies with \( f \) according to Table 3. Let \( I_1: U_i \rightarrow c_0(E) \) be defined by \( I_1(x_o) = ((x_0, 0), (0, 0), \cdots) \). Observe that if we let \( \mathcal{X}_0 = I_1(x_0) \) and \( \mathcal{X}_{n+1} = \mathcal{X}_0 + g(\mathcal{X}_0) \) then \( \mathcal{X}_n \in \mathcal{U}_0 \) for all \( n \) and \( \lim_n \mathcal{X}_n = \mathcal{X} \) exists and satisfies \( G \circ \mathcal{X} = \mathcal{X}_0 \). Thus \( I_1(x_0) \in \text{Range}(G) \) so we can define \( w(x) = G^{-1} \circ I_1(x) \) on \( U_i \). The equation \( G \circ G^{-1}(I_1(x)) = I_1(x) \) is equivalent to \( w(x) = I_1(x) + g(w(x)) \) and writing this out gives

\[
\begin{align*}
\Pi_1 \circ w_0(x) &= x + 0 \\
\Pi_2 \circ w_0(x) &= r^{-1}L_2(\Pi_2 \circ w_1(x)) + \Pi_2 \circ \tilde{f}(w_1(x)/r) \\
\Pi_1 \circ w_i(x) &= r \cdot L_1(\Pi_1 \circ w_{i-1}(x)) - r \cdot L_1(\Pi_1 \circ \tilde{f}(w_i(x)/r)) \\
\Pi_2 \circ w_i(x) &= (1/r) \cdot (L_2(\Pi_2 \circ w_{i+1}(x)) + r \cdot \Pi_2 \circ \tilde{f}(w_{i+1}(x)/r^{-1}) \\
\end{align*}
\]

Multiply the equations with \( \Pi_1 \) and \( i \geq 1 \) by \( L_1 \) and then move the 2nd term on the right to the left to get

\[
(3) \quad r \cdot \Pi_1 \circ f(w_i(x)/r) = r \cdot \Pi_1 \circ w_{i-1}(x)
\]

The terms involving \( \Pi_2 \) give

\[
(3') \quad \Pi_2 \cdot w_i(x) = r \cdot \Pi_2 \circ f(w_{i+1}(x)/r^{-1})
\]

Thus

\[
(4) \quad f(w_{i+1}(x)/r^{-1}) = \Pi_2 \circ f(w_i(x)/r) \quad i \geq 1
\]

Since \( w_i(x) \in c_0(E) \), \( w_i(x)/r^i = o(r^{-i}) \). Therefore letting \( g_i(x) = \Pi_2 \circ w_0(x) \) we have \( (x, g_i(x)) = w_0(x) \in \mathcal{W}_{E_i} \) and \( g_i \) is of the same class as \( f \) and varies with \( f \) as in the Table 3. On the other hand if \( (x, y) \in \mathcal{W}_{E_i} \) then there exists \( \tilde{w} = (\tilde{w}_0, \tilde{w}_1, \cdots) \in c_0(E) \) s.t. \( \tilde{w}_0 = (x, y) \) and \( f(\tilde{w}_{i+1}/r^{-1}) = w_i/r \circ \tilde{w} \) satisfies equations (4) and hence equations (3) and by the 1–1 ness of \( L_1 \) equation (2). But \( G \) is 1–1 so \( \tilde{w} = w(x) \) and \( y = g_i(x) \). Thus \( \mathcal{W}_{E_i} = \{(x, g_i(x)) | x \in U_i \} \). If \( r^i \) also satisfies the condi-
tions and \( r < r' \) then defining \( \tilde{w}(x) = (r'/r'')\tilde{w}(x) \) we have that \( \tilde{w}(x) \) satisfies equations 4, 3, 2 with \( r \) replaced by \( r' \). By the uniqueness of \( G \), \( \tilde{w}(x) = w(x) \) (using \( r' \)). Thus \( g_1 \) is independent of \( r \).

(Case 2) This case follows along the same lines as Case 1. Define \( g: \mathcal{U}_0 \to c_0(E) \) by \( g = \mathcal{L} + h \) where \( \mathcal{L}(\mathcal{X}) = ((r \cdot L'_1(x_{i+1}), L_2(y_{i-1}))/r), i \geq 1 \) and \( \mathcal{L}(\mathcal{X})_0 = (r \cdot L'_1(x_1), 0) \) and

\[
\begin{align*}
  h(\mathcal{X})_i &= (1/r')[-L'_i \circ \Pi_1 \circ \mathcal{F}\left((x_n, y_i) \cdot r'\right)], \quad i \geq 1 \\
  h(\mathcal{X})_0 &= (-L'_1 \circ \Pi_1 \circ \mathcal{F}(x_0, y_0), 0)
\end{align*}
\]

Then \( \text{Lip}(g) \leq \max(r \cdot \|L'_1\| + \|L'_i\| \cdot \text{Lip}(\mathcal{F}), (\|L_2\| + \text{Lip}(\mathcal{F}))/r) < 1 \). So \( G = \text{Id} - g \) and \( G^{-1} \) are of the same class as \( f \). \( G \) is a homeomorphism of \( \mathcal{U}_0 \) onto \( G(\mathcal{U}_0) \) and \( G^{-1} \) varies with \( f \) as in the Table 3. Let \( I_2: U_2 \to c_0(E) \) be defined by \( I_2(y) = ((0, y), (0, 0), \cdots) \) and let \( w(y) = G^{-1} \circ I_2 \). As before \( w(y) \) is defined on all of \( U_2 \). The equation \( G \circ G^{-1}(I_2(y)) = I_2(y) \) is equivalent to

\[
w(y) = ((0, y), (0, 0), \cdots) + g(w(y))
\]

which is equivalent to

\[
\begin{align*}
  \Pi_1 \circ w_0(y) &= r \cdot L'_1((\Pi_1 \circ w_1(y)) - L'_i \circ \Pi_1 \circ \mathcal{F}(w(y)) \\
  \Pi_2 \circ w_0(y) &= y \\
  \cdots & \cdots \cdots \\
  \Pi_1 \circ w_i(y) &= r \cdot L'_i \circ \Pi_1 \circ w_{i+1}(y) - r^{-1}L'_i \circ \Pi_1 \circ \mathcal{F}(r'w_i(y)) \\
  \Pi_2 \circ w_i(y) &= r^{-1} \cdot L_2 \circ \Pi_2 \circ w_{i-1}(y) + r^{-1}L_2 \circ \Pi_2 \circ \mathcal{F}(r^{-1}w_{i-1}(y))
\end{align*}
\]

which is equivalent to

\[
f(r^{-1}w_{i-1}(y)) = r'w_i(y) \quad i \geq 1.
\]

Letting \( g_2(y) = \Pi_1 \circ w_0(y) \) we have \( \{(g_2(y), y) | y \in U_2\} = W_{E_2} \) as before and \( g_2 \) is of the same class as \( f \) and varies with \( f \) as in Table 3. The independence of \( g_2 \) and \( W_{E_2} \) from \( r \) follows as before.

Remark. As noted in the proof, \( w(x) = \lim w^*(x) \) where \( w^*(x) = I_i(\text{or} \ I_2)(x) + g \circ w^{*-1}(x), n \geq 1, w^0 = 0 \). Thus

\[
g_1 \text{ or } g_2 = \lim \Pi_2 \circ w^*_n(x) \left( \text{or } \lim \Pi_2 \circ w^*_n(x) \right)
\]
and the evaluation of the right hand side for any value of $n$ involves $2n - 1$ evaluations of $f$. This gives an effective iterative method for numerically determining the invariant manifolds.

**Counterexample.** The continuity in the theorem of $g_1$ and $g_2$ as functions of $f$ for $f$ in $C^1$ cannot be sharpened from $\| \|_0$ continuity to $\| \|_1$ continuity, as the following example of a $C^1$ map from $l_2$ to $l_2$ shows.

Let $s: R^1 \to R^1$ be defined by

$$s(x) = x, \ 0 \leq x \leq 1/4; \quad 1/2 - x, \ 1/4 \leq x \leq 3/4; \quad = x - 1, \ 3/4 \leq x \leq 1.$$  

$s(x)$ outside of $[0, 1]$ is defined such that $s$ is periodic with period one. Let $a_n(x) = s(nx) \cdot \lfloor n |x| / n \rfloor, \ |x| \leq 2; = 0, \ |x| > 2$ and $\tilde{a}_n(x) = a_n(x), \ |x| \leq 1; = 0, \ |x| > 1$. Then $a_n, \tilde{a}_n$ are continuous and $|a_n(x)|, |\tilde{a}_n(x)| \leq |x| / 4$ for all $n$. Let $b_n(x) = \int_0^x a_n(t) dt$ and $\tilde{b}_n(x) = \int_0^x \tilde{a}_n(t) dt$. Define $A$ and $\tilde{A}: l_2 \to l_2$ by $A(x) = \Sigma a_n(x)e_n$ and $A(x) = \Sigma \tilde{a}_n(x)e_n$ where $e_n$ is an orthonormal basis. Define $B$ and $\tilde{B}: l_2 \to R^1$ by $B(x) = \Sigma b_n(x)$ and $\tilde{B}(x) = \Sigma \tilde{b}_n(x_n)$. Then it is not hard to show that $B$ and $\tilde{B} \in C^1(l_2, R^1)$ that $DB = A$ and $D\tilde{B} = \tilde{A}$. $b_n(x)$ is depicted in the figure.

To construct $f$ we let $E = l_2 \oplus R$ with $\| (x, y) \|_E = \max (\| x \|, | y |)$ and $f(x, y) = (2x, (y - 1/10 \cdot \tilde{B}(x))/2 + 1/10 \cdot B(2x))$. On

$$U = \{(x, y) | \| (x, y) \|_E \leq 4\}, \quad \text{Lip} \left( f - \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \right) < 1/2.$$  

This $f$ will satisfy the conditions of theorem 1 and $g(x) = B(x)$ when $\| x \| \leq 2$. Here $E_1 = l_2$ and $E_2 = R^1$. Now define perturbations of $f$,

$$f_n(x, y) = f(x, y) + 3/2n \cdot (1 - 3/4n)^2 \cdot (\max((x_n - 1), 0))^2.$$
Then $f_n \to f$ in $\|\|_2$. On the other hand the effect of this perturbation is to shift $g_f$ along the segment $\{x \mid x = te^n, 2 \leq t \leq 4\}$ in such way that $\sup_x \|Dg_n - Dg_f\|$ stays bounded away from zero.

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