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ON IRREDUCIBLE SPACES. II

JAMES ROBERT BOONE

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JAMES R. BOONE

A topological space is said to be *irreducible* if every open covering has an open refinement that covers the space minimally. Irreducibility is a fundamental property related to cardinality conditions for open coverings. In this paper, a constructive proof is presented to establish that the weak $\bar{\theta}$ -refinable spaces of Smith are irreducible. Various results concerning cardinality conditions for open coverings follow as corollaries. Some examples are included.

1. Introduction. Recently Smith [11] introduced the notion of a weak $\bar{\theta}$ -refinable space and obtained a variety of applications of this topological structure. Smith defines a space to be *weak $\bar{\theta}$ -refinable* if every open covering has a weak θ -refinement $\cup \{\mathcal{G}_i : i \in N\}$ such that $\{\mathcal{G}_i^* : i \in N\}$ is point-finite. For a collection of sets \mathcal{M} , \mathcal{M}^* denotes the union of the sets in \mathcal{M} . We define a space to be *weakly θ -refinable of finite type* if every open covering has a weak θ -refinement $\cup \{\mathcal{G}_i : i \in N\}$ such that all but finitely many \mathcal{G}_i are empty. Also we call a weakly θ -refinable space which is *not* of finite type, *strictly weakly θ -refinable*. The variations of Bing's Example G which are presented in [2], [6], [7] and [12] as presented in [4], are weakly θ -refinable of finite type. Clearly, metacompact spaces are weakly θ -refinable of finite type. Weakly θ -refinable spaces of finite type and the θ -refinable spaces of Wicke and Worrell [13] (As Smith proves in [11]) are weak θ -refinable.

The primary purpose of this paper is to establish that the weak $\bar{\theta}$ -refinable spaces of Smith have the fundamental property of irreducibility. A space is *irreducible* if every open covering has an open refinement that covers the space minimally. In [5] a constructive proof that every θ -refinable space is irreducible is presented. Also, the following characterization of irreducibility is established.

THEOREM 1.1. *A nonempty space X is irreducible if and only if for each open covering $\{V_\alpha : \alpha \in A\}$ of X there exists a discrete collection of nonempty closed sets $\{T_\beta : \beta \in B\}$ such that $B \subset A$, $T_\beta \subset V_\beta$ for each $\beta \in B$ and $\{V_\beta : \beta \in B\}$ covers X .*

In this paper the cardinality of set γ will be denoted by $|\gamma|$ and the natural numbers will be denoted by N . Also, if $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ is a collection of subsets of a space where A is well-ordered and a collection

of nonempty subsets $\mathcal{L} = \{L_\gamma: \gamma \subset A, \gamma \neq \emptyset\}$ refines \mathcal{V} in such a way that $L_\gamma \subset \bigcap \{V_\alpha: \alpha \in \gamma\}$, for each $\gamma \subset A, \gamma \neq \emptyset$, then we will adopt the following convention in the proof of Theorem 2.1, that $\text{st}(\mathcal{L}, \mathcal{V})$ is the union of the sets V_α where α is the least element in γ for some γ in the indexing set of L . When denoting a collection of sets \mathcal{V} as the range $\{V_\alpha: \alpha \in A\}$ of some function, we mean that this function is a bijection.

The main results of this study are contained in §2. Some examples relating to weak $\bar{\theta}$ -refinable spaces are in §3 and §4 is the proof of the main theorem, Theorem 2.1.

2. Main theorem and some corollaries.

THEOREM 2.1. *Every weak $\bar{\theta}$ -refinable space is irreducible.*

Proof. Section 4.

A space is said to have *property* (δ) [11] if discrete collections in X are countable. The following corollary to Theorem 2.1 is a special case of Theorem 2.5.

COROLLARY 2.2 [11]. *In a weak $\bar{\theta}$ -refinable space the following are equivalent:*

- (a) *property* (δ)
- (b) *Lindelöf*
- (c) \aleph_1 -*compact*.

Since the essential property of metacompact spaces used by Arens and Dugundji in [1] was irreducibility, we have the following corollary.

COROLLARY 2.3. [11] *A weak $\bar{\theta}$ -refinable space is compact if and only if it is countably compact.*

Also since irreducibility of spaces with *property* (δ) implies these spaces are Lindelöf, and regular Lindelöf spaces have the star-finite property [9, Theorem 10], Theorem 4.2 of [11] can be generalized in the following manner.

COROLLARY 2.4. *Every regular irreducible space with property* (δ) *has the star-finite property.*

The results of Corollary 2.2 can be extended to arbitrary infinite cardinal numbers. For instance, if m is any infinite cardinal we say a space has *property* (m) if every discrete collection has cardinality $\leq m$. Also, a space is *m-Lindelöf* if every open covering has a subcovering of cardinality $\leq m$.

THEOREM 2.5. *Let X be an irreducible space. If X has property (m) , then X is m -Lindelöf.*

Proof. Suppose X is not m -Lindelöf. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open covering of X such that every subcovering of \mathcal{U} has cardinality $> m$. Since X is irreducible, by Theorem 1.1, there exists a discrete collection of nonempty closed sets $\{T_\beta : \beta \in B\}$ such that $B \subset A$, $T_\beta \subset U_\beta$ for each $\beta \in B$ and $\{U_\beta : \beta \in B\}$ covers X . Since $\{U_\beta : \beta \in B\}$ is a subcovering of \mathcal{U} , the cardinality of $\{T_\beta : \beta \in B\}$ is strictly greater than m . Thus X does not have property m . This completes the proof.

Further, for an infinite cardinal m a space is called m -compact if every subset of cardinality m has a cluster point. Let m^+ be the first cardinal exceeding m . In the following corollary, which is an extension of Corollary 2.2, the irreducibility of the space is needed only in (a) \Rightarrow (b). The implications (a) \Leftrightarrow (c) and (b) \Rightarrow (a) are easily verified.

COROLLARY 2.6. *In an irreducible T_1 -space the following are equivalent:*

- (a) *property (m)*
- (b) *m -Lindelöf*
- (c) *m^+ -compact.*

Every discrete collection of subsets of the ordinal space $[0, \Omega)$, where Ω is the first uncountable ordinal, has finite cardinality. Thus $[0, \Omega)$ has property (\aleph_0) (\equiv property (δ)). However, $[0, \Omega)$ is not \aleph_0 -Lindelöf (\equiv Lindelöf). Thus, (a) does not imply (b), if the space is not irreducible.

3. Examples.

EXAMPLE 3.1. *A space which is weakly θ -refinable of finite type which is not countably θ -refinable.*

This is Example 2 of [10]. It is presented here to correct an inconsistency between the statement in [10] that this space is θ -refinable but not countably metacompact and the theorem of Gittings [8] which establishes that countable metacompactness is equivalent to countable θ -refinability.

Let $G = \{p_i : i \in N\}$ be any countably infinite set of objects which are not real numbers, and let R be the set of real numbers. Let $X = R \cup G$. A neighborhood base at $r \in R$ is a usual neighborhood base in R . A neighborhood base at $p_i \in G$ consists of all sets of the form $\{p_i\} \cup (R - C)$ where C is any closed countable subset of R . Let $\mathcal{U} = \{U_n : n \in N\}$ be any countable open covering of X by basic open

sets. Then every open σ -refinement of \mathcal{U} , $\mathcal{V} = \cup\{\mathcal{V}_j: j \in N\}$, which covers X on every level must contain on every level j , sets of the form $V'_j = \{p_i\} \cup (R - C'_j)$, for each $p_i \in G$. Since $R \cap (\cap\{V'_j: i, j \in N\}) = R - (\cup\{C'_j: i, j \in N\}) \neq \emptyset$, there are uncountably many $r \in R$ such that $r \in V'_j$ for each $i \in N$ and on each level j . Accordingly, X is not countably θ -refinable. The space X is T_1 , Lindelöf and weakly θ -refinable of finite type. Thus X is irreducible. The closed set G is not a G_δ set, which is precisely the reason that X is not θ -refinable [2].

EXAMPLE 3.2. *A strictly weakly θ -refinable space which is weak $\bar{\theta}$ -refinable.*

All examples of weakly θ -refinable, but not θ -refinable spaces that appear in the papers cited in the introduction are of finite type. Thus these examples are trivially weak $\bar{\theta}$ -refinable. This example is built on the disjoint union of a countably infinite number of copies of the real line.

Let $X = R \times N = \cup\{R_n: n \in N\}$ where $R_n = \{(r, n): r \in R\}$. For each $(r, n) \in X$ a basic open neighborhood of (r, n) is any set of the form $U(r, n, \lambda, \mathcal{F}_n) = \{(p, n): p \in (r - \lambda, r + \lambda)\} \cup \cup\{R_j - F_j: j > n\}$, where $F_j \in \mathcal{F}_n$, F_j is any finite subset of R_j , and $\lambda > 0$.

This space is hereditarily weak $\bar{\theta}$ -refinable, T_1 and Lindelöf. It is not of finite type. That is, it is strictly weakly θ -refinable.

4. Proof of Theorem 2.1.

Proof. Let X be a weak $\bar{\theta}$ -refinable space, and let $\mathcal{V} = \{V_\alpha: \alpha \in A\}$ be any open covering of X . Consider A to be well ordered. Recall, all stars of collections of subsets are taken according to the criterion described in the introduction. Let $\mathcal{U} = \cup\{\mathcal{U}_i: i \in N\}$ be a σ -precise weak $\bar{\theta}$ -refinement of \mathcal{V} , where $\mathcal{U}_i = \{U_\alpha^i: \alpha \in A_i \subset A, U_\alpha^i \neq \emptyset\}$.

The discrete collection of closed sets, indicated in Theorem 1.1, is constructed by induction with the sets defined in the following paragraph.

For each $n, k \in N$, let $K_{nk} = \{p \in X: p \text{ is in exactly } k \text{ sets in } \mathcal{U}_n\}$ and $Z_n = X - \mathcal{U}_n^*$. Let $\mathcal{F}(n, k) = \{F(n, \gamma): \gamma \subset A_n, |\gamma| = k\}$, where $F(n, \gamma) = K_{nk} \cap (\cap\{U_\alpha^i: \alpha \in \gamma\})$. Then $\mathcal{F}(n, k)$ is a collection of subsets of $X - (Z_n \cup \cup\{\mathcal{F}^*(n, j): j < k\})$, which are closed in this subspace. Also, $\mathcal{F}(n, k)$ is discrete in this same subspace. Let $\mathcal{N}_k = \{\Gamma \subset N: |\Gamma| = k\}$, and let $H_k = \{p \in X: p \text{ is an element of exactly } k \text{ sets in } \{\mathcal{U}_i^*: i \in N\}\}$. Let $\mathcal{H}_k = \{H(k, \Gamma): \Gamma \in \mathcal{N}_k, H(k, \Gamma) \neq \emptyset\}$, where

$$H(k, \Gamma) = H_k \cap (\cap\{\mathcal{U}_i^*: i \in \Gamma\}),$$

for each $\Gamma \in \mathcal{N}_k$. Then \mathcal{H}_k is a collection of subsets of $X - \cup\{\mathcal{H}_j^*: j < k\}$,

which are closed in this subspace. Also, \mathcal{H}_k is discrete in this same subspace. Note that: $X = \cup\{\mathcal{H}_k^*: k \in N\}$, and $X = \cup\{\mathcal{F}^*(n, k): n, k \in N\}$.

The desired discrete collection of nonempty closed subsets of X which are constructed by induction is denoted by \mathcal{T} and will consist of unions of the families denoted $\mathcal{C}(n, j, k, \Gamma)$ and $\mathcal{F}(k, \Gamma)$. The verification that \mathcal{T} is a discrete collection of nonempty closed sets is presented at the end of the proof.

We begin the construction in the set \mathcal{H}_1 . (Every point in \mathcal{H}_1^* is covered by at most a finite number of sets in one level of the collection \mathcal{U} .) For each $H(1, \Gamma) \in \mathcal{H}_1$, $\Gamma = \{n\}$ for some $n \in N$, and $\mathcal{U}_i^* \cap H(1, \Gamma) = \emptyset$, if $i \neq n$. Let $\mathcal{C}(n, 1, 1, \Gamma) = \{C(n, \gamma, 1, \Gamma): \gamma \subset A_n, |\gamma| = 1, \Gamma = \{n\}\}$, where $C(n, \gamma, 1, \Gamma) = F(n, \gamma) \cap H(1, \Gamma)$.

Then for each $k > 1$, let $\mathcal{C}(n, k, 1, \Gamma) = \{C(n, \gamma, 1, \Gamma): \gamma \subset A_n, |\gamma| = k, \Gamma = \{n\}\}$, where

$$C(n, \gamma, 1, \Gamma) = (F(n, \gamma) \cap H(1, \Gamma)) - st\left(\bigcup_{j < k} \mathcal{C}(n, j, 1, \Gamma), \mathcal{V}\right).$$

Hence $\mathcal{T}_1 = \cup\{\mathcal{F}(1, \Gamma): H(1, \Gamma) \in \mathcal{H}_1\}$, where $\mathcal{F}(1, \Gamma) = \cup\{\mathcal{C}(n, j, 1, \Gamma): \Gamma = \{n\}, j \in N\}$ is a discrete collection of closed subsets of X and $\mathcal{T}_1^* \subset \mathcal{H}_1^* \subset st(\mathcal{T}_1, \mathcal{V})$. This completes the construction for points in one set in $\{\mathcal{U}_i^*: i \in N\}$.

Suppose this process has been continued for the sets $\mathcal{H}_j, j < k$. By this we mean, discrete collections of closed subsets of $X, \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{k-1}$ have been defined such that $\mathcal{T}_j^* \subset \mathcal{H}_j^* \subset st(\bigcup_{i \leq j} \mathcal{T}_i, \mathcal{V})$ for each $j \leq k - 1$.

Now consider the collection \mathcal{H}_k . Let $H(k, \Gamma) \in \mathcal{H}_k$. Say $\Gamma = \{n(1), n(2), \dots, n(k)\}$, where $n(1) < n(2) < \dots < n(k)$. Let $M(k, \Gamma) = H(k, \Gamma) - st(\cup\{\mathcal{T}_j: j = 1, 2, \dots, k - 1\}, \mathcal{V})$, for each $H(k, \Gamma) \in \mathcal{H}_k$. Note that $\{M(k, \Gamma): \Gamma \in \mathcal{N}_k\}$ is a discrete collection of closed subsets of X and every point in $M(k, \Gamma)$ is covered by a finite number of sets on at least one of the levels $n(1), n(2), \dots, n(k)$.

The construction of the discrete collections of closed subsets of $M(k, \Gamma)$ will proceed in the following order:

$$\begin{aligned} &\mathcal{F}(n(1), 1), \mathcal{F}(n(2), 1), \dots, \mathcal{F}(n(k), 1), \\ &\mathcal{F}(n(1), 2), \dots, \mathcal{F}(n(k), 2), \dots, \\ &\mathcal{F}(n(1), m), \mathcal{F}(n(2), m), \dots, \mathcal{F}(n(k), m), \dots \end{aligned}$$

Let $\mathcal{C}(n(1), 1, k, \Gamma) = \{C(n(1), \gamma, k, \Gamma): \gamma \subset A_{n(1)}, |\gamma| = 1\}$, where

$$C(n(1), \gamma, k, \Gamma) = F(n(1), \gamma) \cap M(k, \Gamma).$$

Also let $\mathcal{C}(n(2), 1, k, \Gamma) = \{C(n(2), \gamma, k, \Gamma): \gamma \subset A_{n(2)}, |\gamma| = 1\}$, where $C(n(2), \gamma, k, \Gamma) = F(n(2), \gamma) \cap M(k, \Gamma) - st(\mathcal{C}(n(1), 1, k, \Gamma), \mathcal{V})$. For $2 < j \leq k$, let $\mathcal{C}(n(j), 1, k, \Gamma) = \{C(n(j), \gamma, k, \Gamma): \gamma \subset A_{n(j)}, |\gamma| = 1\}$, where

$$C(n(j), \gamma, k, \Gamma) = F(n(j), \gamma) \cap M(k, \Gamma) - st(\cup\{\mathcal{C}(n(i), 1, k, \Gamma): i = 1, 2, \dots, j - 1\}, \mathcal{V}).$$

This completes the steps through $\mathcal{F}(n(k), 1)$.

Let $m > 1$. Suppose discrete collections of closed subsets of X , $\mathcal{C}(n(j), t, k, \Gamma)$ have been defined for $1 \leq j \leq k$ and $t < m$. Let $\mathcal{C}(n(1), m, k, \Gamma) = \{C(n(1), \gamma, k, \Gamma): \gamma \subset A_{n(1)}, |\gamma| = m\}$, where $C(n(1), \gamma, k, \Gamma) = F(n(1), \gamma) \cap M(k, \Gamma) - st(\cup\{\mathcal{C}(n(j), t, k, \Gamma): 1 \leq j \leq k, t < m\}, \mathcal{V})$. For $n(k) \geq n(r) > n(1)$ and $\gamma \subset A_{n(r)}$ such that $|\gamma| = m$, let $C(n(r), \gamma, k, \Gamma) = F(n(r), \gamma) \cap M(k, \Gamma) - st(\cup\{\mathcal{C}(n(j), t, k, \Gamma): 1 \leq j \leq k, t < m \text{ or } t = m \text{ if } j < r\}, \mathcal{V})$. Then let $\mathcal{C}(n(r), m, k, \Gamma) = \{C(n(r), \gamma, k, \Gamma): \gamma \subset A_{n(r)}, |\gamma| = m\}$. If $\mathcal{T}(k, \Gamma) = \cup\{\mathcal{C}(n(j), n, k, \Gamma): j = 1, 2, \dots, k, n \in N\}$, then $\mathcal{T}^*(k, \Gamma) \subset M(k, \Gamma) \subset st(\mathcal{T}(k, \Gamma), \mathcal{V})$.

Since $H(k, \Gamma)$ was any set in \mathcal{H}_k , there is a discrete collection of closed sets

$$\mathcal{T}_k = \cup\{\mathcal{T}(k, \Gamma): H(k, \Gamma) \in \mathcal{H}_k\} \text{ such that } \mathcal{T}_k^* \subset \mathcal{H}_k^* \subset st(\mathcal{T}_k, \mathcal{V}).$$

Thus the discrete collections of closed sets \mathcal{T}_k are defined for each $k \in N$, consisting of sets of the type $C(n, \gamma, k, \Gamma)$. Let

$$\mathcal{T} = \{C(n, \gamma, k, \Gamma) \in \mathcal{T}_k: C(n, \gamma, k, \Gamma) \neq \emptyset, k \in N\}.$$

\mathcal{T} is a discrete collection of nonempty closed sets. It is easily verified that for each $n, k \in N$, $\mathcal{F}(n, k)$ is a collection of subsets of $X - (Z_n \cup \cup\{\mathcal{F}^*(n, j); j < k\})$ which is discrete in this subspace and the sets $F(n, \gamma) \in \mathcal{F}(n, k)$ are closed in this subspace. Also, for each $k \in N$, \mathcal{H}_k is a collection of subsets of $X - \cup\{\mathcal{H}_j^*; j < k\}$ which is discrete in this subspace and the sets $H(k, \Gamma) \in \mathcal{H}_k$ are closed in this subspace. Since $X = \cup\{\mathcal{H}_k^*: k \in N\}, \cup\{\mathcal{H}_j^*: j < k\} \subset st(\cup\{\mathcal{T}_j: j < k\}, \mathcal{V})$ and $M(k, \Gamma) = H(k, \Gamma) - st(\cup\{\mathcal{T}_j: j < k\}, \mathcal{V})$ for each $H(k, \Gamma) \in \mathcal{H}_k, \{M(k, \Gamma): H(k, \Gamma) \in \mathcal{H}_k, k \in N\}$ is a discrete collection of closed sets in X . Further, since for each $|\gamma'| < m, M(k, \Gamma) \cap F(n, \gamma') \subset st(\cup\{\mathcal{C}(n', t, k, \Gamma): t < m \text{ or } t = m \text{ if } n' < n, n' \in \Gamma\}, \mathcal{V}), \{M(k, \Gamma) \cap F(n, \gamma) - st(\cup\{\mathcal{C}(n', t, k, \Gamma): n' < n, n' \in \Gamma, t < m \text{ or } t = m \text{ if } n' < n\}, \mathcal{V}): |\gamma| = m\}$ is a discrete collection of closed sets in X . Since $\mathcal{F}(n, k)$ covers $H(k, \Gamma)$ for $k \in N$ and $n \in \Gamma$, it follows from the definition of $C(n, t, k, \Gamma)$ that \mathcal{T} is a discrete collection of

nonempty closed sets in X since its members are the intersections of closed sets from two discrete collections.

Let $B = \{\beta \in A : \beta \text{ is the least element in } \gamma \text{ for some } C(n, \gamma, k, \Gamma) \in \mathcal{T}\}$. By the convention in the introduction for forming the star of a collection of subsets, and the use of the stars in the construction, $\{V_\beta : \beta \in B\}$ covers X . By Theorem 1.1, X is irreducible.

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