

Pacific Journal of Mathematics

UNIQUENESS THEOREMS FOR TAUT SUBMANIFOLDS

MICHAEL FREEDMAN

UNIQUENESS THEOREMS FOR TAUT SUBMANIFOLDS

MICHAEL FREEDMAN

1. Introduction and statements of theorems. Given two closed smooth manifolds, how do you tell if they are diffeomorphic? If you start out with a homotopy equivalence, Browder–Novikoff Theory breaks the problem up into: (1) finding all self-equivalences, (2) finding a normal bordism, and (3) the surgery obstruction on the normal bordism. In applications, however, one may encounter manifolds suspected of being diffeomorphic, where no obvious homotopy equivalence is present.

We describe such a situation: Let $\begin{matrix} \xi \\ | \\ M \end{matrix}$ be a simply connected, finite, simplicial complex with linear bundle. Let $K_i^{2n} \xrightarrow{f_i} M$, $i = 0$ or 1 , $n \geq 3$, be normal maps from closed smooth manifolds, i.e. $f_i^*(\zeta) = \nu(K_i)$: Suppose that f_1 and f_2 are normally bordant, f_i is n -connected, and that $B_n(K_0) = B_n(K_1)$. B_n here denotes the n -th Betti number. It follows from Poincaré's Duality and the universal coefficient theorem that K_0 and K_1 have isomorphic integral homology groups, but a map inducing this isomorphism is lacking. However,

THEOREM 1. *If n is odd, K_0 and K_1 are diffeomorphic.*

THEOREM 2. *If n is even, but not 2, and the intersection pairings on*

$$\begin{aligned} &(\text{Ker } f_0: H_n(K_0; Z) \rightarrow H_n(M; Z))/\text{torsion} \quad \text{and} \\ &(\text{Ker } f_1: H_n(K_1; Z) \rightarrow H_n(M; Z))/\text{torsion} \end{aligned}$$

are isometric and nonsingular, then K_0 and K_1 are diffeomorphic.

COROLLARY 1. *If M^{2n+2} is a compact, simply connected, smooth $2n + 2$ -manifold, n odd, and $K_0^{2n} \xrightarrow{i_0} M^{2n+2}$ and $K_1^{2n} \xrightarrow{i_1} M^{2n+2}$ are n -connected inclusions of closed submanifolds with $i_0.[K_0] = i_1.[K_1] \in H_{2n}(M^{2n+2}; Z)$, then if $B_n(K_0) = B_n(K_1)$, K_0 is diffeomorphic to K_1 .*

COROLLARY 2. *Assume M^{2n+2} is a simply connected smooth $2n + 2$ -manifold, n even ($n \neq 2$), with $H_n(M; Z) = 0$. If i_0 and i_1 are as above, then if the intersection pairings on $H_n(K_0; Z)/\text{torsion}$ and $H_n(K_1; Z)/\text{torsion}$ are isometric, K_1 is diffeomorphic to K_2 .*

REMARK 1. The above corollaries are specialized by replacing the

connectivity assumptions with the assumption that the submanifolds are taut (definition: $\pi_i(M\text{-neighborhood}(K), \partial) = 0, i \leq n$).

REMARK 2. It follows from Corollary 1 and [2] that if n is odd and $\pi_1(M^{2n+2}) = 0$, every homology class $X \in H_{2n}(M^{2n+2}; Z)$ is represented by a simplest submanifold with an n -connected inclusion map, K^0 , and every other submanifold with an n -connected inclusion map is diffeomorphic to $K^t = K^0 \# \underbrace{S^n \times S^n \times \cdots \times S^n \times S^n}_t, t \geq 0$.

t -copies

THEOREM 3. If $\pi_1(M^{2n+2}) = 0$, and if K_0 and K_1 are submanifolds representing $X \in H_{2n}(M^{2n+2}; Z)$ with n -connected inclusion maps, then there are integers j, k such that $K_0 \# \underbrace{S^n \times S^n \times \cdots \times S^n \times S^n}_j$ is diffeomor-

j -copies

phic to $K_1 \# \underbrace{S^n \times S^n \times \cdots \times S^n \times S^n}_k$.

k -copies

2. Proofs. (All coefficients will be integral unless stated.)

Thm 1 \Rightarrow Cor 1 and Thm 2 \Rightarrow Cor 2: Let N be a very large integer. $i: CP^{N-1} \rightarrow CP^N$ is an isomorphism on $H_2(\ ; Z)$ so there is a unique 2-plane bundle ζ extending $\nu_{CP^{N-1} \rightarrow CP^N}$. Since $i_1[K_1] = i_2[K_2]$, there is a homotopy $h: M \times I \rightarrow CP^N$ with $h_0^{-1}(CP^{N-1}) = K_0$ and $h_1^{-1}(CP^{N-1}) = K_1$. $h^*(\zeta)$ extends $\nu_{K_0 \times 0 \rightarrow M \times 0}$ and $\nu_{K_1 \times 1 \rightarrow M \times 1}$. Furthermore, the inclusions $K_0 \times 0 \rightarrow \begin{matrix} \nu_{(M) \oplus h^*(\zeta)} \\ | \\ M \times 0 \end{matrix}$ and $K_1 \times 1 \rightarrow \begin{matrix} \nu_{(M) \oplus h^*(\zeta)} \\ | \\ M \times 1 \end{matrix}$ are normal maps and if h is transverse to CP^{N-1} , $h^{-1}(CP^{N-1})$ provides a normal bordism. This together with the hypothesis of Corollary 1 (or Corollary 2) is the hypothesis of Theorem 1 (or Theorem 2). Corollary 1 (or Corollary 2) follows.

Proof of Theorem 1. Let $L, \partial L = K_0 \cup -K_1$, be a normal bordism. By Theorem 1.2 [4] we may assume $f: L \rightarrow M$ is n -connected. It follows that $H_*(L, K_0) = 0$ for $* \neq n, n + 1$. Our plan is to perform some normal n -surgery on interior (L) to produce L' with $H_n(L', K_0) = 0$ (this will leave $H_*(L, K_0)$ unchanged $* \leq n - 1$). By duality and U.C.T. $H_*(L', K_0) = 0$, and $L \cong^{diff} K_0 \times I \Rightarrow K_1 \cong^{diff} K_0$ by the h -cobordism theorem.

Let $i_0: K_0 \rightarrow L$ and $i_1: K_1 \rightarrow L$ be the inclusions. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & & \frac{H_{n+1}(M, L)}{i_{0*}(H_{n+1}(M, K_0))} & \xrightarrow{\cong} & H_n(L, K_0) \longrightarrow 0 \\
 & & 0 & \longrightarrow & & & \\
 & & \uparrow & & & & \uparrow \\
 H_{n+1}(L) & \xrightarrow{f_{n+1}} & H_{n+1}(M) & \longrightarrow & H_{n+1}(M, L) & \longrightarrow & H_n(L) \longrightarrow H_n(M) \xrightarrow{f_n} 0 \\
 \uparrow i_{0*} & & \uparrow id_* & & \uparrow i_{0*} & & \uparrow i_{0*} & & \uparrow id_* \\
 H_{n+1}(K_0) & \xrightarrow{f_{0n+1}} & H_{n+1}(M) & \longrightarrow & H_{n+1}(M, K_0) & \longrightarrow & H_n(K_0) \xrightarrow{f_{0n}} & H_n(M) & \longrightarrow 0
 \end{array}$$

Since f_0 factors through i_0 , $\text{coker}(f_{0n+1}) \xrightarrow{id_*} \text{coker}(f_{n+1})$ is epi. There is a short exact sequence

$$\begin{array}{ccccccc}
 0 \rightarrow & \text{coker}(f_{n+1})/\text{coker}(f_{0n+1}) & \rightarrow & \frac{H_{n+1}(M, L)}{i_{0*}H_{n+1}(M, K_0)} & \rightarrow & \text{Ker } f_{n_{i_{0*}(\text{Ker } f_{0n})}} & \rightarrow 0 \\
 & \parallel & & & & & \\
 & 0 & & & & &
 \end{array}$$

so there is a natural isomorphism

$$\text{Ker } f_{n_{i_{0*}(\text{Ker } f_{0n})}} \cong H_n(L, K_0).$$

We will consider the two modules identified.

Duality and the U.C.T. show $H_n(L, K_0)$ and $H_n(L, K_1)$ are isomorphic. So we have a noncommutative diagram:

$$\begin{array}{ccc}
 & \text{epi } \pi_0 & \text{ker } f_{n_{i_{0*}(\text{Ker } f_{0n})}} = C_0 \\
 A = \text{ker } f_n & \nearrow & \parallel \\
 & \text{epi } \pi_1 & \text{ker } f_{n_{i_{1*}(\text{Ker } f_{0n})}} = C_1
 \end{array}$$

We need an algebraic fact about such diagrams.

LEMMA 1. *If F is a field, there are elements $a_1, \dots, a_k \in A$, such that $\{\pi_\epsilon(a_1) \otimes 1, \dots, \pi_\epsilon(a_k) \otimes 1\}$ is a basis for $C_\epsilon \otimes_Z F$, $\epsilon = 0$ or 1 .*

Proof. By induction. $\dim_F(C_0 \otimes_Z F) = \dim_F(C_1 \otimes_Z F)$. Suppose $a_1 \dots a_j$ are already chosen so that $\pi_\epsilon(a_1) \otimes 1, \dots, \pi_\epsilon(a_j) \otimes 1$ are independent. Let the spans of these vectors be $\text{span}_{j,0}$ and $\text{span}_{j,1}$. If these spans are proper subspaces of $C_0 \otimes_Z F$ and $C_1 \otimes_Z F$, then $\pi_\epsilon^{-1}(\text{span}_{j,\epsilon})$ are proper subspaces of $A \otimes_Z F$. Let a_{j+1} be any element of A such that $a_{j+1} \otimes 1 \in (A \otimes_Z F - (\bigcup_{\epsilon=0,1} \pi_\epsilon^{-1}(\text{span}_{j,\epsilon}))) \neq \emptyset$.

In order to compute the effect on $H_n(L, K_0)$ of normal n -surgeries along L , we use a diagram adapted from Lemma IV.3.2 [1]. (At this

point it may be helpful for the reader to review the proof that the odd dimensional simply connected surgery obstruction vanishes, see Ch. IV, §3 of [1]. This argument is originally due to J. Milnor and M. Kervaire, see [3].)

LEMMA 2. *If L' results from a normal surgery on $S^n \hookrightarrow L$, $[S^n] = x \in \ker f_n$, we have the following diagram:*

$$\begin{array}{ccccccc}
 (*) & & & H_{n+1}(L', K_0) & & & \\
 & & & \downarrow & & & \\
 & & & Z \in \lambda' & & & \\
 & & & \downarrow d' & & & \\
 H_{n+1}(L, K_0) & \xrightarrow[\theta]{\pi_1(x) \cdot} & Z & \xrightarrow{d} & H_n(L_0, K_0) & \xrightarrow{i_*} & H_n(L, K_0) \longrightarrow 0 \\
 & & & & \downarrow i'_* & & \\
 & & & & H_n(L', K_0) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

with: (1) $\pi_0(x) = x_0 = i_* d(\lambda')$, where λ' is the appropriate generator above. (2) The map $\theta: H_{n+1}(L, K_0) \rightarrow Z$ is given by intersection with $\pi_1(x) \in H_n(L, K_1)$.

Proof. (1) See Lemma IV.3.2 [1].

(2) $\theta: H_{n+1}(L, K_0) \rightarrow Z$ may be thought of as:

$$\begin{array}{c}
 H_{n+1}(L/K_0) \xrightarrow{j} H_{n+1}(L/K_0, L_0/K_0) \xrightarrow{\text{exc}^{-1}} H_{n+1}(S^n \times D^{n+1}, S^n \times S^n) \xrightarrow{U(\)} Z \\
 \underbrace{\hspace{10em}}_{\bar{j}}
 \end{array}$$

where the last map is evaluation of the Thom class $U \in H^{n+1}(S^n \times D^{n+1}, S^n \times S^n)$. Let $a \in H_{n+1}(L/K_0)$. $j^*(\text{exc}^{-1}U)(a) = U(\bar{j}_* a) = \theta(a)$.

$$\begin{array}{ccccc}
 H^{n+1}(L, K_0) & \xrightarrow{j^*} & H^{n+1}(L, L_0) & \xleftarrow{\text{exc}^{-1}} & H^{n+1}(S^n \times D^{n+1}, S^n \times S^n) \\
 \downarrow \cap [L, K_0 K_1] & & \downarrow & & \downarrow \\
 \cap [L, K_0 \cup K_1] & & H^{n+1}(L, K_0 \cup K_1) & & \cap [S^n \times D^{n+1}, S^n \times S^n] \\
 \downarrow & & \downarrow & \swarrow U \in & \downarrow \\
 H_n(L, K_1) & \xleftarrow{p_1} & H_n(L) & \xleftarrow{\exists x} & H_n(S^n \times S^n)
 \end{array}$$

$j^*(\text{exc}^{-1}U) \cap [L, K_0 \cup K_1] = \pi_1(x) \in H_n(L, K_1)$, so

$$\theta(a) = j^*(\text{exc}^{-1}U)(a) = \pi_1(x) \cdot (a),$$

where \cdot denotes algebraic intersection.

LEMMA 3. *If we set $y = i'_* d(\lambda)$, then:*

(1) $H_n(L', K_0)/\langle y \rangle \cong H_n(L, K_0)/\langle x_0 \rangle$, $\langle \rangle$ means “subgroup generated by”.

(2) Suppose (x_0) has finite order s , then for some $t \in \mathbb{Z}$ $sd'(\lambda') + td(\lambda) = 0$. Order $(y) = \infty$ iff $t = 0$. Order $y = t$ iff $t \neq 0$.

(3) If n is even (as in Theorem 2), $t = 0$.

(4) If n is odd (as in Theorem 1), L' is not uniquely determined, and may be varied to L'_m by changing the framing of the surgery along x by m times the generator of $\ker(\pi_n(SO(n+1)) \rightarrow \pi_n(SO)) \cong \mathbb{Z}$. So t is a function of L'_m . We have the relation: $t(L'_m) = t(L') + 2ms$.

Proof. These facts correspond to Lemmas 3.2, 3.5, 3.8 and 3.11 respectively in Chapter IV of [1].

LEMMA 4. $L^{2n} \xrightarrow[\text{M}]{\xi}$ is normally bordant rel ∂ to L' satisfying $H_*(L', K_0) = 0$, $* \leq n - 1$ and $H_n(L', K_0) = \text{torsion}$. (This lemma holds for n even or odd.)

Proof. In Lemma 1 set $F = Q$, normal surgery on the classes $\{a_1, \dots, a_k\}$ affords the desired L' . More precisely, assume $\text{Rank}_{\mathbb{Z}}(H_n(L, K_0)) > 0$. By setting $F = Q$ and $a_1 = x$ in Lemma 1, $\pi_0(x) = x_0$ and $\pi_1(x) = x_1$ are infinite order. Therefore, by diagram (*) $\langle x_0 \rangle = \mathbb{Z}$ and $\langle y \rangle = \text{torsion}$. Let L' result from surgery on x . Now (1) of Lemma 3 implies $\text{Rank}_{\mathbb{Z}}(H_n(L', K_0)) = \text{Rank}_{\mathbb{Z}}(H_n(L, K_0)) - 1$. Proceed inductively.

REMARK 3. It may not be possible to do the preceding construction without increasing the order of torsion $(H_n(L, K_0))$. The reason is that there are diagrams, for example:

$$\begin{array}{ccc} & & \mathbb{Z} \\ & \nearrow^{4a+b} & \\ \mathbb{Z} + \mathbb{Z} & & \\ & \searrow_{a+4b} & \\ & & \mathbb{Z} \end{array} \quad \begin{array}{c} \pi_0 \\ \pi_1 \end{array}$$

so that $\exists x \in \mathbb{Z} + \mathbb{Z}$ with $\pi_\epsilon(x)$ generating a free summand. So unlike the classical case, killing the free part may increase the torsion!

LEMMA 4'. Given a prime, p , $L^{2n} \xrightarrow[\text{M}]{\xi}$ is normally bordant rel ∂ to L' such that $H_*(L', K_0) = 0$, $* \leq n - 1$, and $H_n(L', K_0) \otimes \mathbb{Z}_p = 0$. (This lemma holds for n even or odd.)

Proof. Assume L satisfies the conclusion of Lemma 4. In Lemma 1 set $F = Z_p$ and $a_1 = x$. Now consider diagram (*) with Z_p coefficient. The mod p reductions $(\pi_0(x))_p = (x_0)_p \in H_n(L, K_0; Z_p)$ and $(\pi_1(x))_p = (x_1)_p \in H_n(L, K_1; Z_p)$ are nonzero. d is the zero map so by (1) of Lemma 3:

$$\text{Rank}_{Z_p}(H_n(L, K_0; Z_p)) = \text{Rank}_{Z_p}(H_n(L', K_0; Z_p)) + 1.$$

Using the U.C.T. and $H_{n+1}(L', K_0; Z) = 0$ we have:

$$\text{Rank}_{Z_p}(H_n(L, K_0; Z) \otimes Z_p) = \text{Rank}_{Z_p}(H_n(L', K_0) \otimes Z_p) + 1.$$

Lemma 4' now follows by induction on the above rank.

The argument is now restricted to $n = \text{odd}$. The prime 2 will play a special role.

Let L satisfy $H_*(L, K_0) = 0$, $* \leq n - 1$ and $H_n(L, K_0) \otimes Z_2 = 0$ (i.e., $H_n(L, K_0)$ is odd torsion).

LEMMA 5. *If n is odd and L is as above, and if a prime p /order $(H_n(L, K_0))$ there is a normal bordism rel ∂L to L' (consisting of either one or two normal n -surgeries) with: (1) $H_*(L', K_0) = 0$, $* \leq n - 1$, (2) $H_n(L', K_0)$ odd torsion, and either: (3) order $H_n(L', K_0) \leq \text{order } H_n(L, K_0)$, and (4) $\text{Rank}_{Z_p}(H_n(L', K_0) \otimes Z_p) < \text{Rank}_{Z_p}(H_n(L, K_0) \otimes Z_p)$ or (3') order $H_n(L', K_0) < \text{order } H_n(L, K_0)$.*

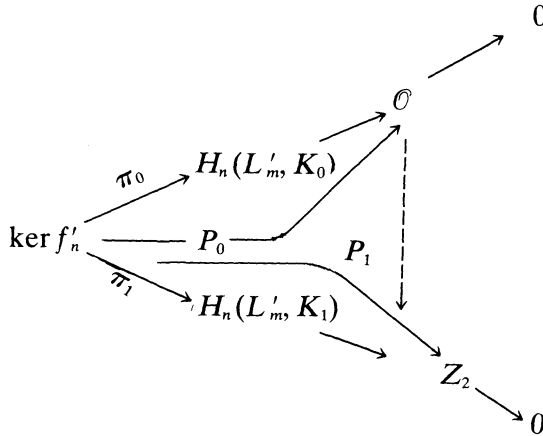
Proof. In Lemma 1 set $F = Z_p$ and $a_1 = x$. As in the proof of Lemma 4', any normal surgery based on x results in an L' with $\text{Rank}_{Z_p}(H_n(L', K_0) \otimes Z_p) < \text{Rank}_{Z_p}(H_n(L, K_0) \otimes Z_p)$. By (4) of Lemma 3 $\forall m, t(L'_m) = t(L') + 2ms$, where $s = \text{order } x_0 \in H_n(L, K_0)$. By (1) and (2) of Lemma 3, $t(L'_m) \neq 0$ iff $H_n(L'_m, K_0) = \text{torsion}$, and order (torsion $H_n(L'_m, K_0)) \leq \text{order (torsion } H_n(L_m, K_0))$ iff $-s \leq t(L'_m) \leq s$. If (and only if) $t(L'_m) = 2s$, order $H_n(L', K_0) = 2(\text{order } H_n(L, K_0))$. We may choose m so that: *Case A.* $t(L'_m) \neq 0$ and $-s \leq t(L'_m) \leq s$ or *Case B.* $t(L'_m) = 2s$.

In each case diagram (*) with Z_2 coefficients shows $\text{Rank}_{Z_2}(H_n(L', K_0) \otimes Z_2) = 0$ or 1. In *Case A* we are finished if the above rank is zero; assume it is one. If $t(L'_m) = \pm s$, order $H_n(L', K_0) = \text{order } H_n(L, K_0)$, so (2) and (3) above are satisfied and we are finished. So add the assumption: $-s < t(L'_m) < s$. Then order $H_n(L', K_0) < \text{order } H_n(L, K_0)$. Now Lemma 1 with $F = Z_2$ provides $x' = a_1 \in \ker f'_n: H_n(L') \rightarrow H_n(M)$, so that if L'' is the result of a normal surgery on x' then $H_n(L'', K_0) \otimes Z_2 = 0$. $\exists r \in Z$ such that $-\text{order } x' \leq t(L'') \leq \text{order } x'$. However, $t(L'') \neq 0$ as this would imply order $H(L'', K_0) = \infty$, contradicting $H_n(L'', K_0) \otimes Z_2 = 0$. So order $(H_n(L'', K_0)) \leq \text{order } (H_n(L'_m, K_0)) < \text{order } (H_n(L, K_0))$ so (3') above is satisfied. This completes *Case A*.

In Case B order $(H_n(L'_m, K_0)) = 2(\text{order } (H_n(L, K_0)))$.

LEMMA 6. $\exists x' \in \ker f'_n: H_n(L'_m) \rightarrow H_n(M)$ such that $\pi_0(x')$ is the unique element of order 2, δ , in $H_n(L'_m, K_0)$ and $(\pi_1(x'))_2 \neq 0$.

Proof. $H_n(L'_m, K_0) \cong H_n(L'_m, K_1) \cong Z_2 + \mathcal{O}$, where \mathcal{O} is odd torsion. Consider compositions P_0, P_1



$\text{Ker } P_0 \not\subset \text{ker } P_1$, otherwise P_1 could be factored through P_0 by an epimorphism (dotted arrow). Let $x \in \text{ker } P_0 - \text{ker } P_1$. $\pi_0(x) = 0$ or δ , $(\pi_1(x))_2 \neq 0$. If $\pi_0(x) = \delta$, set $x' = x$ and we are finished. If $\pi_0(x) = 0$ let $y \in \text{ker } f'_n$ satisfy $\pi_0(y) = \delta$. If $(\pi_1(y))_2 \neq 0$, set $x' = y$ and we are finished. If $(\pi_1(y))_2 = 0$, set $x' = x + y$. Now $\pi_0(x') = \pi_0(x + y) = 0 + \delta = \delta$ and $(\pi_1(x'))_2 = (\pi_1(x + y))_2 = (\pi_1(x))_2 + (\pi_1(y))_2 = 1 + 0 = 1$.

Let L'' be the result of a normal surgery along x' . Again looking at Rank_{Z_2} tells us $t(L'') \neq 0, 2$. Therefore we can choose r so that $t(L'') = \pm 1$. It follows from diagram (*) that $H_n(L'', K_0) \cong H_n(L'_m, K_0)/Z_2$. L'' satisfies (3) above, but

$$\text{Rank}_{Z_p}(H_n(L'', K_0) \otimes Z_p) = \text{Rank}_{Z_p}(H_n(L'_m, K_0)) < \text{Rank}_{Z_p}(H_n(L, K_0))$$

so L'' also satisfies (4) above. This completes Case B.

$\text{Rank}_{Z_p}(H_n(L, K_0) \otimes Z_p)$ is finite so after a finite number of applications of Lemma 5, it is no longer possible to reduce the Z_p -rank without increasing order $(H_n(L, K_0))$. So we can find a normal cobordism rel ∂ of L to L' with (1), (2) and (3') satisfied. By inducting on order $(H_n(L, K_0))$, we have a normal bordism rel boundary to L' with $H_*(L', K_0) = 0$, all values of $*$. Theorem 1 now follows from the h -cobordism theorem.

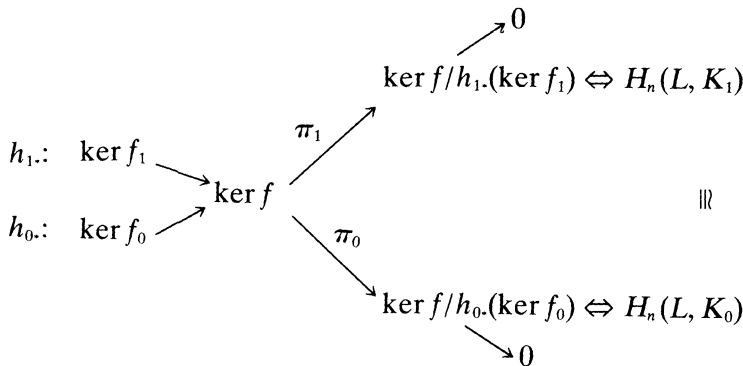
Proof of Theorem 2. If n is even, surgery on a torsion class of $H_n(L, K_0)$ will always increase the $\text{rank}_Z(H_n(L, K_0))$ (see Lemma IV.3.8 of [1]). Remark 3 shows that it may be impossible to do surgery to lower $\text{Rank}_Z(H_n(L, K_0))$ without increasing order (torsion $H_n(L, K_0)$). This prevents the usual inductive argument on order torsion $H_n(L, K_0)$, and is the reason that Theorem 2 requires an additional assumption.

Let $\{\alpha_1^0, \dots, \alpha_k^0\} \subset \ker f_0: H_n(K_0) \rightarrow H_n(M)$ be a basis for $(\ker f_0; Z)/\text{torsion}$ and let $\{\alpha_1^1, \dots, \alpha_k^1\} \subset (\ker f_1, Z)$ be the isometric image of $\{\alpha_1^0, \dots, \alpha_k^0\}$. By the relative Hurewicz Theorem, the α 's are spherical. The classes $\{i_0(\alpha_1^0) - i_1(\alpha_1^1), \dots, i_0(\alpha_k^0) - i_1(\alpha_k^1)\}$ are represented by framed, imbedded n -spheres, s_1, \dots, s_k . Let L' be the result of a normal surgery on $\{s_1, \dots, s_k\}$. Let h_0, h_1 denote the inclusions $K_0 \rightarrow L', K_1 \rightarrow L'$.

LEMMA 7. $h_0(\ker f_0; Z) = h_1(\ker f_1; Z)$ in the quotient $H_n(L'; Z)/\text{torsion}$.

Proof. By induction on the number of surgeries. We will denote the inclusions at any stage by inc_0 and inc_1 . Surgery on s_1 makes $\text{inc}_0(\alpha_1^0) = \text{inc}_1(\alpha_1^1)$. Assume that after surgery on s_1, \dots, s_j , $\text{inc}_0(\alpha_i^0) = \text{inc}_1(\alpha_i^1)$, $1 \leq i \leq j$. Let b_1, \dots, b_i be simplicial chains with $\partial b_i = \alpha_i^0 \cup -\alpha_i^1$. Let \cap denote algebraic intersection. $\cap_L(s_{j+1}, b_i) = \cap_{K_0}(\alpha_{j+1}^0, \alpha_i^0) + \cap_{K_1}(-\alpha_{j+1}^1, \alpha_i^1) = 0$ (using the isometry). As a result, $\text{inc}_0(\alpha_i^0)$ and $\text{inc}_1(\alpha_i^1)$ remain equal after surgery on s_{j+1} and $\text{inc}_0(\alpha_{j+1}^0) = \text{inc}_1(\alpha_{j+1}^1)$. The lemma follows.

Consider the following diagram with all kernels interpreted by applying the function $H_n(\ ; Z)$.



Lemma 7 implies $h_1(\ker f_1) = h_0(\ker f_0)$ in the quotient $(\ker f)/\text{torsion}$.

Assuming the hypothesis of Theorem 2, we have:

LEMMA 8. Suppose $h_1(\ker f_1) = h_0(\ker f_0)$ in the quotient $(\ker f)/\text{torsion}$. If $\text{Rank}_Z H_n(L, K_0) > 0 \exists x \in \ker f$ such that

- (1) $\pi_\epsilon(x) = x_\epsilon$ generates a free summand of $H_n(L, K_\epsilon)$, for $\epsilon = 0, 1$, and
- (2) If L' is the result of surgery on x , $h_1(\ker f_1) = h_0(\ker f_0)$ in the quotient $(\ker f')/\text{torsion}$.
- (3) $H_n(L, K_0) \cong H_n(L', K_0) \oplus Z$.

Proof. Let $y \in H_n(L, K_0)$ generate a free summand. Let $x' \in \ker f$ satisfy $\pi_0(x') = y$. $\pi_1(x')$ generates a free summand of $H_n(L, K_1)$ (Proof: Consider the above diagram modulo torsion.). Let b_i , $1 \leq i \leq k$, be as before. Let $a_i = \cap(x', b_i)$. Since $(\ker f_0, \cap)$ is nonsingular, $\exists \beta \in \ker f_0$ such that $\cap_{K_0}(\beta, \alpha_i) = -a_i$, $\forall 1 \leq i \leq k$. Set $x = x' + i_0(\beta)$. $\cap(x', b_i) = 0 \forall b_i$. (2) now follows. It is easy to check (3) using diagram (*) of Lemma 2. (Compare with Remark 3.)

Let $(L'; K_0, K_1)$ be a normal bordism satisfying the conclusion of Lemma 7. By virtue of Lemma 8, we may do surgery to eliminate $H_n(L', K_0)$. The argument for this coincides with "the proof of Theorem IV.2.1 for $m = 2q + 1$, q even", page 104, [1]. As before, L' becomes an h -cobordism completing the proof of Theorem 2.

REMARK 4. If $n = 2$, this argument may be used to give another proof of: homotopy equivalence $\Rightarrow h$ -cobordism for simply connected 4-manifolds, see Wall, [4].

The Proof of Theorem 3. We may again assume $H_*(L, K_0) = 0$, $* \neq n, n + 1$. L may be described as $K_0 \times I \cup n, n + 1$ -handles. Let D_j^+ be the cores of the n -handles. Disks $D_j^- \rightarrow K_0 \times I$ may be chosen so that $f_\# [D_j^+ \cup D_j^-] = 0 \in \pi_n(M)$ as $f_\#: \pi_{n-1}(K_0) \rightarrow \pi_{n-1}(M)$ is an isomorphism and $f_\#: \pi_n(K_0) \rightarrow \pi_n(M)$ is surjective. Let H be the level set of L after the n -handles have been attached. The preceding statement implies that H is obtained from K_0 by a sequence of surgeries on $(n - 1)$ -spheres, ∂D_j^- , trivialized by some null-homotopy (D_j^-) in $K_0 \times I$. By general position $\{\partial D_j^-\}$ is isotopically trivial (we check $(n - 1) + 1 < \frac{1}{2}(2n + 1)$).

Therefore, $H \cong^{\text{diff}} K_0 \#_j (S^n \times S^n)_j$. By turning L upside down, we obtain $H \cong^{\text{diff}} K \#_k (S^n \times S^n)_k$. This proves Theorem 3.

REFERENCES

1. W. Browder, *Surgery on simply connected manifolds*, Springer-Verlag, 1972.
2. M. Kato and Y. Matsumoto, *Simply connected surgery of submanifolds in co-dimension 2*, I, J. Math. Soc. Japan, **24** (72), 586-608.
3. M. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Ann. Math., **77** (1963), 504-537.
4. C. T. C. Wall, *On simply connected 4-manifolds*, J. London Math. Soc., **39** (1964), 141-149.

Received May 19, 1975. Partially supported by NSF Grant.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1976 Pacific Journal of Mathematics
All Rights Reserved

Pacific Journal of Mathematics

Vol. 62, No. 2

February, 1976

Allan Russell Adler and Catarina Isabel Kiefe, <i>Pseudofinite fields, procyclic fields and model-completion</i>	305
Christopher Allday, <i>The stratification of compact connected Lie group actions by subtori</i>	311
Martin Bartelt, <i>Commutants of multipliers and translation operators</i>	329
Herbert Stanley Bear, Jr., <i>Ordered Gleason parts</i>	337
James Robert Boone, <i>On irreducible spaces. II</i>	351
James Robert Boone, <i>On the cardinality relationships between discrete collections and open covers</i>	359
L. S. Dube, <i>On finite Hankel transformation of generalized functions</i>	365
Michael Freedman, <i>Uniqueness theorems for taut submanifolds</i>	379
Shmuel Friedland and Raphael Loewy, <i>Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue</i>	389
Theodore William Gamelin, <i>Uniform algebras spanned by Hartogs series</i>	401
James Guyker, <i>On partial isometries with no isometric part</i>	419
Shigeru Hasegawa and Ryōtarō Satō, <i>A general ratio ergodic theorem for semigroups</i>	435
Nigel Kalton and G. V. Wood, <i>Homomorphisms of group algebras with norm less than $\sqrt{2}$</i>	439
Thomas Laffey, <i>On the structure of algebraic algebras</i>	461
Will Y. K. Lee, <i>On a correctness class of the Bessel type differential operator S_μ</i>	473
Robert D. Little, <i>Complex vector fields and divisible Chern classes</i>	483
Kenneth Louden, <i>Maximal quotient rings of ring extensions</i>	489
Dieter Lutz, <i>Scalar spectral operators, ordered l^p-direct sums, and the counterexample of Kakutani-McCarthy</i>	497
Ralph Tyrrell Rockafellar and Roger Jean-Baptiste Robert Wets, <i>Stochastic convex programming: singular multipliers and extended duality singular multipliers and duality</i>	507
Edward Barry Saff and Richard Steven Varga, <i>Geometric overconvergence of rational functions in unbounded domains</i>	523
Joel Linn Schiff, <i>Isomorphisms between harmonic and P-harmonic Hardy spaces on Riemann surfaces</i>	551
Virinda Mohan Sehgal and S. P. Singh, <i>On a fixed point theorem of Krasnoselskii for locally convex spaces</i>	561
Lewis Shilane, <i>Filtered spaces admitting spectral sequence operations</i>	569
Michel Smith, <i>Generating large indecomposable continua</i>	587
John Yuan, <i>On the convolution algebras of H-invariant measures</i>	595