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A GENERAL RATIO ERGODIC THEOREM FOR SEMIGROUPS

SHIGERU HASEGAWA AND RYŌTARŌ SATŌ

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The purpose of this note is to prove a ratio ergodic theorem, which is a continuous parameter version of Chacon's general ergodic theorem.

Let (X, \mathcal{F}, μ) be a σ -finite measure space and $L_1 = L_1(X, \mathcal{F}, \mu)$ the Banach space of equivalence classes of integrable complex-valued functions on X . Let $\Gamma = \{T_t; t > 0\}$ be a strongly continuous semigroup of linear contractions on L_1 . It then follows (cf. [6, §4]) that for any $f \in L_1$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and \mathcal{F} , such that $T_t f(x)$ belongs to the equivalence class of Tf for each $t > 0$. Moreover there exists a set $N(f)$ with $\mu(N(f)) = 0$, dependent on f but independent of t , such that if $x \notin N(f)$, then $T_t f(x)$ is integrable on every finite interval (a, b) and the integral $\int_a^b T_t f(x) dt$, as a function of x , belongs to the equivalence class of $\int_a^b T_t f dt$.

THEOREM. *Let $p_t(x)$ be a nonnegative function on $(0, \infty) \times X$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and \mathcal{F} , such that $f \in L_1$ and $|f| \leq p_s$ for some s imply $|T_t f| \leq p_{s+t}$ for all $t > 0$. Then for any $f \in L_1$ the limit*

$$\lim_{b \rightarrow \infty} \int_0^b T_t f(x) dt / \int_0^b p_t(x) dt$$

exists and is finite a.e. on $\left\{x; \int_0^\infty p_t(x) dt > 0\right\}$.

LEMMA. *Let T be a linear contraction on L_1 and $\{p_n; n \geq 0\}$ a sequence of nonnegative measurable functions on X such that $f \in L_1$ and $|f| \leq p_n$ for some n imply $|Tf| \leq p_{n+1}$. If $g \in L_1$, then*

$$\lim_n p_n(x) / \sum_{i=0}^{n-1} p_i(x) = 0$$

a.e. on

$$\left\{x; \sum_{i=0}^\infty p_i(x) > 0 \text{ and } \lim_n \left| \sum_{i=0}^n T^i g(x) / \sum_{i=0}^n p_i(x) \right| > 0\right\}.$$

Proof. By the Chacon theorem, $\lim_n \Sigma_{i=0}^n T^i g(x) / \Sigma_{i=0}^n p_i(x)$ exists and is finite a.e. on $\{x; \Sigma_{i=0}^\infty p_i(x) > 0\}$. Using this and the linear modulus [3] of T , the desired conclusion follows easily from the Chacon-Ornstein lemma [5, Theorem 2.4.2].

Proof of the Theorem. Write $q_n(x) = \int_n^{n+1} p_t(x) dt$. Then $\{q_n; n \geq 0\}$ is a sequence of nonnegative measurable functions on X . We first prove that $f \in L_1$ and $|f| \leq q_n$ for some n imply $|T_1 f| \leq q_{n+1}$. For this purpose, let $\epsilon > 0$ be given, and choose a nonnegative function $h \in L_1$ such that if we set $p'_i(x) = \min(p_i(x), h(x))$, $q'_n(x) = \int_n^{n+1} p'_i(x) dt$ and $A = \{x; f(x) < q'_n(x)\}$, then $\|f 1_{X-A}\| < \epsilon$. Define $s(x) = f(x)/q'_n(x)$ if $x \in A$, and $s(x) = 0$ if $x \notin A$. It follows that $\|f - sq'_n\| = \|f 1_{X-A}\| < \epsilon$, and

$$T_1(sq'_n) = T_1\left(\int_n^{n+1} s(x)p'_i(x) dt\right) = \int_n^{n+1} T_1(sp'_i) dt.$$

(Here we note that $t \rightarrow sp'_i$ is strongly integrable over the interval $(n, n + 1)$). Since $|sp'_i| \leq p_i$, $|T_1(sp'_i)| \leq p_{i+1}$ for all $t > 0$. Now let $r_t(x)$ be a scalar function on $(n, n + 1) \times X$, measurable with respect to the product of the Lebesgue measurable subsets of $(n, n + 1)$ and \mathcal{F} , such that for almost all t , $r_t(x)$ belongs to the equivalence class of $T_1(sp'_i)$ [4, Theorem III.11.17]. Then we have

$$\begin{aligned} \left| \int_n^{n+1} T_1(sp'_i) dt \right| &= \left| \int_n^{n+1} r_t(x) dt \right| \leq \int_n^{n+1} |r_t(x)| dt \\ &\leq \int_n^{n+1} p_{t+1}(x) dt = q_{n+1}(x) \quad \text{a.e.,} \end{aligned}$$

and hence $|T_1 f| \leq q_{n+1}$.

Next, for any $f \in L_1$, put $f' = \int_0^1 T_t f dt$. If $b > 0$, write $b = n + a$, where $n = [b]$ and $0 \leq a < 1$. Then, as in [7],

$$\begin{aligned} &\int_0^b T_t f(x) dt / \int_0^b p_t(x) dt \\ &= \left(\frac{\sum_{i=0}^{n-1} T_i f'(x)}{\sum_{i=0}^{n-1} q_i(x)} + \frac{\int_n^b T_t f(x) dt}{\sum_{i=0}^{n-1} q_i(x)} \right) / \left(1 + \frac{\int_n^b p_t(x) dt}{\sum_{i=0}^{n-1} q_i(x)} \right), \end{aligned}$$

and

$$\frac{\left| \int_n^b T_i f(x) dt \right|}{\sum_{i=0}^{n-1} q_i(x)} \leq \frac{\tau^n \left(\int_0^1 |T_i f| dt \right)(x)}{\sum_{i=0}^{n-1} q_i(x)} \rightarrow 0$$

a.e. on $\{x; \sum_{i=0}^{\infty} q_i(x) > 0\}$ by the Chacon-Ornstein lemma, where τ denotes the linear modulus of T_1 . Hence the Chacon theorem and our lemma complete the proof.

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