

# Pacific Journal of Mathematics

**A GENERAL RATIO ERGODIC THEOREM FOR SEMIGROUPS**

SHIGERU HASEGAWA AND RYŌTARŌ SATŌ

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**The purpose of this note is to prove a ratio ergodic theorem, which is a continuous parameter version of Chacon's general ergodic theorem.**

Let  $(X, \mathcal{F}, \mu)$  be a  $\sigma$ -finite measure space and  $L_1 = L_1(X, \mathcal{F}, \mu)$  the Banach space of equivalence classes of integrable complex-valued functions on  $X$ . Let  $\Gamma = \{T_t; t > 0\}$  be a strongly continuous semigroup of linear contractions on  $L_1$ . It then follows (cf. [6, §4]) that for any  $f \in L_1$  there exists a scalar function  $T_t f(x)$ , measurable with respect to the product of the Lebesgue measurable subsets of  $(0, \infty)$  and  $\mathcal{F}$ , such that  $T_t f(x)$  belongs to the equivalence class of  $T_t f$  for each  $t > 0$ . Moreover there exists a set  $N(f)$  with  $\mu(N(f)) = 0$ , dependent on  $f$  but independent of  $t$ , such that if  $x \notin N(f)$ , then  $T_t f(x)$  is integrable on every finite interval  $(a, b)$  and the integral  $\int_a^b T_t f(x) dt$ , as a function of  $x$ , belongs to the equivalence class of  $\int_a^b T_t f dt$ .

**THEOREM.** *Let  $p_t(x)$  be a nonnegative function on  $(0, \infty) \times X$ , measurable with respect to the product of the Lebesgue measurable subsets of  $(0, \infty)$  and  $\mathcal{F}$ , such that  $f \in L_1$  and  $|f| \leq p_s$  for some  $s$  imply  $|T_t f| \leq p_{s+t}$  for all  $t > 0$ . Then for any  $f \in L_1$  the limit*

$$\lim_{b \rightarrow \infty} \int_0^b T_t f(x) dt / \int_0^b p_t(x) dt$$

*exists and is finite a.e. on  $\left\{ x; \int_0^\infty p_t(x) dt > 0 \right\}$ .*

**LEMMA.** *Let  $T$  be a linear contraction on  $L_1$  and  $\{p_n; n \geq 0\}$  a sequence of nonnegative measurable functions on  $X$  such that  $f \in L_1$  and  $|f| \leq p_n$  for some  $n$  imply  $|Tf| \leq p_{n+1}$ . If  $g \in L_1$ , then*

$$\lim_n p_n(x) / \sum_{i=0}^{n-1} p_i(x) = 0$$

a.e. on

$$\left\{ x; \sum_{i=0}^\infty p_i(x) > 0 \text{ and } \lim_n \left| \frac{\sum_{i=0}^n T^i g(x)}{\sum_{i=0}^n p_i(x)} \right| > 0 \right\}.$$

*Proof.* By the Chacon theorem,  $\lim_n \sum_{i=0}^n T^i g(x) / \sum_{i=0}^n p_i(x)$  exists and is finite a.e. on  $\{x; \sum_{i=0}^\infty p_i(x) > 0\}$ . Using this and the linear modulus [3] of  $T$ , the desired conclusion follows easily from the Chacon-Ornstein lemma [5, Theorem 2.4.2].

*Proof of the Theorem.* Write  $q_n(x) = \int_n^{n+1} p_i(x) dt$ . Then  $\{q_n; n \geq 0\}$  is a sequence of nonnegative measurable functions on  $X$ . We first prove that  $f \in L_1$  and  $|f| \leq q_n$  for some  $n$  imply  $|T_1 f| \leq q_{n+1}$ . For this purpose, let  $\epsilon > 0$  be given, and choose a nonnegative function  $h \in L_1$  such that if we set  $p'_i(x) = \min(p_i(x), h(x))$ ,  $q'_n(x) = \int_n^{n+1} p'_i(x) dt$  and  $A = \{x; f(x) < q'_n(x)\}$ , then  $\|f 1_{X-A}\| < \epsilon$ . Define  $s(x) = f(x)/q'_n(x)$  if  $x \in A$ , and  $s(x) = 0$  if  $x \notin A$ . It follows that  $\|f - sq'_n\| = \|f 1_{X-A}\| < \epsilon$ , and

$$T_1(sq'_n) = T_1\left(\int_n^{n+1} s(x)p'_i(x) dt\right) = \int_n^{n+1} T_1(sp'_i) dt.$$

(Here we note that  $t \rightarrow sp'_i$  is strongly integrable over the interval  $(n, n + 1)$ ). Since  $|sp'_i| \leq p_i$ ,  $|T_1(sp'_i)| \leq p_{i+1}$  for all  $t > 0$ . Now let  $r_t(x)$  be a scalar function on  $(n, n + 1) \times X$ , measurable with respect to the product of the Lebesgue measurable subsets of  $(n, n + 1)$  and  $\mathcal{F}$ , such that for almost all  $t$ ,  $r_t(x)$  belongs to the equivalence class of  $T_1(sp'_i)$  [4, Theorem III.11.17]. Then we have

$$\begin{aligned} \left| \int_n^{n+1} T_1(sp'_i) dt \right| &= \left| \int_n^{n+1} r_t(x) dt \right| \leq \int_n^{n+1} |r_t(x)| dt \\ &\leq \int_n^{n+1} p_{i+1}(x) dt = q_{n+1}(x) \quad \text{a.e.,} \end{aligned}$$

and hence  $|T_1 f| \leq q_{n+1}$ .

Next, for any  $f \in L_1$ , put  $f' = \int_0^1 T f dt$ . If  $b > 0$ , write  $b = n + a$ , where  $n = [b]$  and  $0 \leq a < 1$ . Then, as in [7],

$$\begin{aligned} &\int_0^b T f(x) dt / \int_0^b p_i(x) dt \\ &= \left( \frac{\sum_{i=0}^{n-1} T_1 f'(x)}{\sum_{i=0}^{n-1} q_i(x)} + \frac{\int_n^b T f(x) dt}{\sum_{i=0}^{n-1} q_i(x)} \right) / \left( 1 + \frac{\int_n^b p_i(x) dt}{\sum_{i=0}^{n-1} q_i(x)} \right), \end{aligned}$$

and

$$\frac{\left| \int_n^b T_i f(x) dt \right|}{\sum_{i=0}^{n-1} q_i(x)} \leq \frac{\tau^n \left( \int_0^1 |T_i f| dt \right)(x)}{\sum_{i=0}^{n-1} q_i(x)} \rightarrow 0$$

a.e. on  $\{x; \sum_{i=0}^{\infty} q_i(x) > 0\}$  by the Chacon-Ornstein lemma, where  $\tau$  denotes the linear modulus of  $T_1$ . Hence the Chacon theorem and our lemma complete the proof.

#### REFERENCES

1. R. V. Chacon, *Operator averages*, Bull. Amer. Math. Soc., **68** (1962), 351–353.
2. ———, *Convergence of operator averages*, Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), 89–120, Academic Press, New York, 1963.
3. R. V. Chacon and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc., **15** (1964), 553–559.
4. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.
5. A. M. Garsia, *Topics in Almost Everywhere Convergence*, Markham, Chicago, 1970.
6. D. S. Ornstein, *The sums of iterates of a positive operator*, Advances in Probability and Related Topics, **2** (1970), 85–115.
7. S. Tsurumi, *An ergodic theorem for a semigroup of linear contractions*, Proc. Japan Acad., **49** (1973), 306–309.

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Allan Russell Adler and Catarina Isabel Kiefe, <i>Pseudofinite fields, pro-cyclic fields and model-completion</i> .....	305
Christopher Allday, <i>The stratification of compact connected Lie group actions by subtori</i> .....	311
Martin Bartelt, <i>Commutants of multipliers and translation operators</i> .....	329
Herbert Stanley Bear, Jr., <i>Ordered Gleason parts</i> .....	337
James Robert Boone, <i>On irreducible spaces. II</i> .....	351
James Robert Boone, <i>On the cardinality relationships between discrete collections and open covers</i> .....	359
L. S. Dube, <i>On finite Hankel transformation of generalized functions</i> .....	365
Michael Freedman, <i>Uniqueness theorems for taut submanifolds</i> .....	379
Shmuel Friedland and Raphael Loewy, <i>Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue</i> .....	389
Theodore William Gamelin, <i>Uniform algebras spanned by Hartogs series</i> .....	401
James Guyker, <i>On partial isometries with no isometric part</i> .....	419
Shigeru Hasegawa and Ryōtarō Satō, <i>A general ratio ergodic theorem for semigroups</i> .....	435
Nigel Kalton and G. V. Wood, <i>Homomorphisms of group algebras with norm less than <math>\sqrt{2}</math></i> .....	439
Thomas Laffey, <i>On the structure of algebraic algebras</i> .....	461
Will Y. K. Lee, <i>On a correctness class of the Bessel type differential operator <math>S_\mu</math></i> .....	473
Robert D. Little, <i>Complex vector fields and divisible Chern classes</i> .....	483
Kenneth Loudon, <i>Maximal quotient rings of ring extensions</i> .....	489
Dieter Lutz, <i>Scalar spectral operators, ordered <math>l^p</math>-direct sums, and the counterexample of Kakutani-McCarthy</i> .....	497
Ralph Tyrrell Rockafellar and Roger Jean-Baptiste Robert Wets, <i>Stochastic convex programming: singular multipliers and extended duality singular multipliers and duality</i> .....	507
Edward Barry Saff and Richard Steven Varga, <i>Geometric overconvergence of rational functions in unbounded domains</i> .....	523
Joel Linn Schiff, <i>Isomorphisms between harmonic and <math>P</math>-harmonic Hardy spaces on Riemann surfaces</i> .....	551
Virinda Mohan Sehgal and S. P. Singh, <i>On a fixed point theorem of Krasnoselskii for locally convex spaces</i> .....	561
Lewis Shilane, <i>Filtered spaces admitting spectral sequence operations</i> .....	569
Michel Smith, <i>Generating large indecomposable continua</i> .....	587
John Yuan, <i>On the convolution algebras of <math>H</math>-invariant measures</i> .....	595