

# Pacific Journal of Mathematics

## **COMPLEX VECTOR FIELDS AND DIVISIBLE CHERN CLASSES**

ROBERT D. LITTLE

# COMPLEX VECTOR FIELDS AND DIVISIBLE CHERN CLASSES

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**This paper contains two theorems which relate the maximal number of independent sections of a complex bundle over a manifold to the Chern classes of the bundle and certain functional cohomology operations. The main theoretical result of the paper is a formula which relates the obstruction to a lifting in a fibration and a functional cohomology operation.**

**1. Introduction.** Let  $M$  be a connected, closed, orientable, smooth manifold of dimension  $2n$ . If  $\omega$  is a complex  $n$ -plane bundle over  $M$ , the complex span of  $\omega$  is the maximal number of cross-sections of  $\omega$  which are linearly independent over the complex numbers. In this paper, we consider the following question: when is complex span  $\omega \geq q$ ? Hopf's theorem says that complex span  $\omega > 0$  if and only if  $\omega$  has vanishing Euler class and the theorems of Thomas ([10] and [11]) give an effective answer in the case  $q = 2$ . We study this problem in the cases  $q = 3, 4$  and establish theorems which give necessary and sufficient conditions for complex span  $\omega \geq q$  in terms of the Chern classes of  $\omega$  and certain functional cohomology operations. The Chern class of  $\omega$  in  $H^{2i}(M; \mathbf{Z})$  is denoted by  $c_i(\omega)$ . If  $\delta_p P^1$  denotes the Steenrod  $p$ -power  $P^1$  followed by the Bockstein associated with reduction mod  $p$ ,  $\delta_p P^1_\omega(c(i - p + 1))$  denotes a subset of a functional operation defined on the universal Chern class  $c(i - p + 1)$  and contained in  $H^{2i}(M; \mathbf{Z})$ . This subset will be described in the second section of this paper. If  $p$  is a prime,  $M$  is  $j$ -connected mod  $p$  if  $H_i(M; \mathbf{Z}_p) = 0, 1 \leq i \leq j$ . In both theorems below,  $M$  is 1-connected mod 2 and 3.

**THEOREM 1.** *If  $n$  is even,  $n \geq 6$ , then complex span  $\omega \geq 3$  if and only if  $c_i(\omega) = 0, n - 2 \leq i \leq n$ , and  $0 \in \delta Sq^2_\omega(c(n - 2))$ .*

**THEOREM 2.** *If  $n$  is odd,  $n \not\equiv 1 \pmod{3}$ ,  $n \geq 9$ , and  $M$  is 3-connected mod 2, then complex span  $\omega \geq 4$  if and only if  $c_i(\omega) = 0, n - 3 \leq i \leq n$ ,  $0 \in \delta Sq^2_\omega(c(n - 3))$ , and  $0 \in \delta_3 P^1_\omega(c(n - 3))$ .*

In Theorem 1, if  $n \not\equiv 0 \pmod{3}$ , and  $n \equiv 2 \pmod{4}$ , the connectedness hypothesis can be dropped. Thomas and Gilmore [11] show that if  $M$  is 3-connected, then for every  $n$ , complex span  $\omega \geq 3$  if and only if  $c_{n-2}(\omega) = 0$  and  $c_n(\omega) = 0$ . Gilmore [2] proves theorems similar to Theorems 1 and 2 in which he assumes that  $H_2(M; \mathbf{Z})$  and  $H_4(M; \mathbf{Z})$ ,

respectively, have no elements of order 2. For the specified values of  $n$ , our theorems contain the results of Thomas and Gilmore, because the operation  $\delta_p P^1_\omega(c(i - p + 1))$  is a set of elements of order  $p$  and hence must vanish if  $H^{2i}(M; \mathbf{Z})$  has no  $p$ -torsion. If  $M$  is an almost-complex manifold with almost-complex structure  $\omega$ , our theorems relate the complex span of  $M$  to the Chern classes of  $M$  and the functional operations.

**2. Obstruction formulas.** Let the map  $f: M \rightarrow BU(n)$  classify  $\omega$ . It is clear that complex span  $\omega \cong q$  if and only if  $f$  lifts to the total space of the fibration  $W_{n,q} \rightarrow BU(n - q) \rightarrow BU(n)$  where  $W_{n,q}$  is the Stiefel variety of complex  $q$ -frames in complex  $n$ -space. We will apply two obstruction formulas to this lifting problem. The first formula is due to Olum [8]. If  $\pi: E \rightarrow B$  is a fibration with fiber  $F$ ,  $c$  in  $H^{n-1}(F; G)$  a class transgressing to  $d$  in  $H^n(B; G)$ ,  $X$  a CW complex and  $f: X \rightarrow B$  a map such that the lifting obstruction  $O^n(f)$  is nonvoid, then

$$(2.1) \quad -c_* O^n(f) = f^* d,$$

where  $c_*: H^n(X; \pi_{n-1}(F)) \rightarrow H^n(X; G)$  is induced by  $c_\#: \pi_{n-1}(F) \rightarrow G$ , the composite of the Hurewicz homomorphism and evaluation.

The second formula is the main theoretical result of this paper. We assume that there is a class  $a$  in  $H^{n-2p+1}(F; \mathbf{Z})$  such that  $\delta P^1 a = 0$  and  $a$  is the only class transgressing to  $b$ , where  $f^* b = 0$  and  $P^1 b \equiv 0$  (mod integral classes in kernel  $\pi^* \cap$  kernel  $f^*$ ). Under these hypotheses, there are liftings  $f_a: F \rightarrow K(\mathbf{Z}, n - 2p + 1; \mathbf{Z}, n - 1, \delta P^1)$  and  $f_b: B \rightarrow K(\mathbf{Z}, n - 2p + 2; \mathbf{Z}, n, \delta P^1)$  of  $a$  and  $b$ , where the range spaces are two-stage Postnikov systems induced by  $\delta P^1$ . In [3], we show that the set  $\{f_{a\#}: \pi_{n-1}(F) \rightarrow \mathbf{Z}\}$  is a congruence class modulo the image of the Hurewicz homomorphism and for every  $[g]$  in  $\pi_{n-1}(F)$ ,  $f_{a\#}[g] \in \delta P^1_g(a)$ , where  $\delta P^1_g$  is the standard functional operation. (See [6], p. 157.) Therefore the induced homomorphism  $f_{a*}: H^n(X; \pi_{n-1}(F)) \rightarrow H^n(X; \mathbf{Z})$  can be effectively computed in some cases. In the proposition below,  $\iota$  denotes the fundamental class of  $K(\mathbf{Z}, n - 2p + 2; \mathbf{Z}, n, \delta P^1)$  and we assume that  $H^n(E; \mathbf{Z})$  and  $H^n(B; \mathbf{Z})$  are torsion free.

**PROPOSITION 2.2.** *If  $\pi: E \rightarrow B$  is a fibration satisfying the above hypotheses and  $p$  is a prime such that  $n \geq 2(2p - 1)$ , then we have containments*

$$(2.3) \quad f_{a*} O^n(f) \equiv \delta P^1_{f_a}(\iota) \equiv \delta P^1_{f_b}(b).$$

Note that (2.3) shows that the obstruction is contained in the

operation  $\delta P_f^1(b)$ . A general result of this kind was obtained by Meyer. (See [5], §13.) It will be clear in the proof of (2.2) that the indeterminacy of  $\delta P_{f,f}^1$  is image  $\delta P^1$ , and so (2.3) expresses the obstruction as an operation with smaller indeterminacy than the indeterminacy of  $\delta P_f^1$ , image  $(\delta P^1 + f^*)$ . Proposition 2.2 will follow from the next lemma. In the proof of the lemma, the reduction mod  $p$  of an integral class  $\theta$  will be denoted by  $\bar{\theta}$ .

LEMMA 2.4. *The composite  $f_b\pi$  is homotopically trivial.*

*Proof.* Let  $K = K(\mathbf{Z}, n - 2p + 2; \mathbf{Z}, n, \delta P^1)$ . If  $n \geq 2(2p - 1)$ ,  $H^n(K; \mathbf{Z})$  is isomorphic to  $\mathbf{Z}$  modulo finite groups with a free generator  $\theta$  such that  $\theta_{\#}: \pi_n(K) \rightarrow \mathbf{Z}$  is multiplication by  $p$  and  $\bar{\theta} = P^1\iota$ . These facts follow immediately from the long exact homotopy and Serre cohomology sequences for the fibration  $K \rightarrow K(\mathbf{Z}, n - 2p + 2)$  and the Hurewicz theorem modulo finite groups. Since  $(f_b\pi)^*\iota = \pi^*b = 0$ , because  $b$  is in the image of the transgression,  $O^n(f_b\pi, *)$  is nonvoid. It follows from the hypotheses that  $f_b^*\theta \equiv 0 \pmod{\text{integral classes in kernel } \pi^* \cap \text{kernel } f^*}$  and from the properties of  $\theta$  and (4.4) in [8] that  $f_b^*\theta - f'_b\theta = pO^n(f_b, f'_b)$  for any two liftings of  $b$ . Therefore, after alteration by an  $n$ -cocycle, we may assume  $f_b^*\theta \in \text{kernel } \pi^* \cap \text{kernel } f^*$  and so  $\pi^*f_b^*\theta = 0 = pO^n(f_b\pi, *)$ , and this implies  $f_b\pi$  is homotopically trivial since  $H^n(E; \mathbf{Z})$  has no torsion.

Lemma 2.4 implies the existence of a map of fibrations from the fibration  $\pi$  into the path space fibration over  $K$  with fiber  $\Omega K = K(\mathbf{Z}, n - 2p + 1; \mathbf{Z}, n - 1, \delta P^1)$ . The map of fibers is a lifting of  $a, f_a: F \rightarrow K(\mathbf{Z}, n - 2p + 1; \mathbf{Z}, n - 1, \delta P^1)$ , because we are assuming that  $a$  is the only class transgressing to  $b$ . The map  $f_b f$  lifts to the path space if and only if it is homotopic to the constant map. We have taken care that  $f_b^*\theta \in \ker f^*$  and so  $(f_b f)^*$  has image zero in dimension  $n$ . Therefore the obstruction to a homotopy of  $f_b f$  to point is precisely  $\delta P_{f_b f}^1(\iota)$  (10.8 [7]) and the indeterminacy of  $\delta P_{f_b f}^1$  is image  $\delta P^1$ . Formula (2.3) now follows immediately from standard naturality properties of obstructions and functional operations ([6], p. 157).

**3. The proofs of Theorems 1 and 2.** We begin with some general remarks. The group  $H^*(W_{n,q}; \mathbf{Z})$  is an exterior algebra with generators  $a_k$  in  $H^{2k-1}(W_{n,q}; \mathbf{Z})$ ,  $n - q + 1 \leq k \leq n$  ([1], p. 444). The space  $W_{n,q}$  is  $2(n - q)$ -connected and  $\pi_i(W_{n,q})$  is  $\mathbf{Z}$  if  $i$  is odd and a finite group if  $i$  is even as long as  $n$  is large,  $n - q$  is odd,  $2 \leq q \leq 4$ , and  $2(n - q) + 1 \leq i \leq 2n - 1$ . The necessary size of  $n$  is indicated in the theorems. The group  $\pi_{2(n-q)+2}(W_{n,q})$  is zero and the other finite groups are 2 or 3 torsion groups in this range of dimensions, [2]. Since the

Hurewicz homomorphism  $h: \pi_i(W_{n,q}) \rightarrow H_i(W_{n,q}; \mathbf{Z})$  is an isomorphism modulo finite abelian groups if  $i \leq 4(n - q)$ , it sends a generator in  $\pi_{2k-1}(W_{n,q})$  into an integer  $m_k \neq 0$ ,  $n - q + 1 \leq k \leq n$ . These integers can be computed using an inductive argument based on the fibration  $W_{n-1,q-1} \rightarrow W_{n,q} \rightarrow S^{2n-1}$ . In particular,  $m_{n-q+1} = 1$ ,  $m_{n-q+2} = 2$  and the prime divisors of the others are either 2 or 3, [4]. Since  $a_k$  transgresses to  $c(k)$ , it follows from (2.1) that  $-m_k O^{2k}(f) = c_k(\omega)$ . Thomas' theorem [10] follows from this formula and the first two values of  $m_k$ : if  $M$  is arbitrary and  $n$  is odd, complex span  $\omega \cong 2$  if and only if  $c_i(\omega) = 0$ ,  $n - 1 \leq i \leq n$ .

Consider the lifting obstruction  $O^{2k}(f)$ , where  $n - q + p \leq k \leq n$ . In this range of dimensions,  $c(k - p + 1)$  is the image of a unique class  $a_{k-p+1}$  under transgression and if  $c_k(\omega) = 0$ ,  $P^1 c(k - p + 1) \equiv 0 \pmod{\text{integral classes in kernel } \pi^* \cap \text{kernel } f^*}$ , ([1], p. 429). Let  $f': BU(n) \rightarrow K(\mathbf{Z}, 2(k - p + 1); \mathbf{Z}, 2k, \delta P^1)$  be a lifting of  $c(k - p + 1)$ ,  $f_{k-p+1}: W_{n,q} \rightarrow K(\mathbf{Z}, 2k - 2p + 1; \mathbf{Z}, 2k - 1, \delta P^1)$  a lifting of  $a_{k-p+1}$ , and set  $\delta P^1_\omega(c(k - p + 1)) = \delta P^1_{f'}(u)$ . If  $[g]$  in  $\pi_{2k-1}(W_{n,q})$  is a generator, then  $f_{k-p+1}[g] \in \delta P^1_g(a_{k-p+1})$ , [3]. Since  $P^1 a_{k-p+1} = \mu_p(k) a_k$ , where  $\mu_p(k) \equiv k \pmod{p}$ , ([1], p. 429), a direct computation of the operation  $\delta P^1_g(a_{k-p+1})$  yields the equation  $f_{k-p+1}[g] \equiv \mu_p(k) p^{-1} m_k \pmod{m_k}$ . If  $c_k(\omega) = 0$ , then  $m_k O^{2k}(f) = 0$  and the action of the homomorphisms  $f_{k-p+1*}$  on the obstruction is independent of the lifting. The next proposition follows immediately from the above remarks and Proposition 2.2. The assumption  $p \leq q$  forces the inequality of the proposition because we have  $2q \leq n$  in our theorems.

PROPOSITION 3.1. *Let  $p$  be a prime such that  $p \leq q$ . If  $c_k(\omega) = 0$  and  $m_k \equiv 0 \pmod{p}$ , then*

$$(3.2) \quad \mu_p(k) p^{-1} m_k O^{2k}(f) \equiv \delta P^1_\omega(c(k - p + 1)) \equiv \delta P^1_f(c(k - p + 1)).$$

We now turn to the proof of Theorem 2. The proof of Theorem 1 will be discussed later. In addition to the properties of  $W_{n,4}$  mentioned above, we will need the fact that the 3-component of  $\pi_{2n-4}(W_{n,4})$  is zero if  $n \not\equiv 1 \pmod{3}$  [2] and some more precise information on the image of the Hurewicz homomorphism  $h: \pi_{2n-3}(W_{n,4}) \rightarrow H_{2n-3}(W_{n,4}; \mathbf{Z})$ ,  $n$  odd and  $n \geq 9$ :  $m_{n-1} \equiv 0 \pmod{3}$  and  $m_{n-1} \not\equiv 0 \pmod{9}$  if  $n \not\equiv 1 \pmod{3}$ , [4].

The conditions in Theorem 2 are clearly necessary. The hypothesis  $c_{n-3}(\omega) = 0$  implies that  $O^{2n-4}(f)$  is nonvoid and because  $n$  is odd and  $m_{n-2} = 2$ , Proposition 3.1 implies that  $O^{2n-4}(f) = \delta S q^2_\omega(c(n - 3))$ . This equation is an actual equality because the indeterminacy of  $O^{2n-4}(f)$  is image  $\delta S q^2$  ([10], p. 191) which is the indeterminacy of  $\delta S q^2_\omega(c(n - 3))$ . Therefore the hypothesis  $0 \in \delta S q^2_\omega(c(n - 3))$  is enough to imply that  $O^{2n-2}(f)$  is nonvoid since  $n \not\equiv 1 \pmod{3}$  means that  $\pi_{2n-4}(W_{n,4})$  has no

3-component and  $M$  is 3-connected mod 2. Since  $c_{n-1}(\omega) = 0$  and  $m_{n-1} \equiv 0 \pmod{3}$ , we have  $\mu_3(n-1)3^{-1}m_{n-1}O^{2n-2}(f) \equiv \delta P^1_*(c(n-3))$ , where  $\mu_3(n-1) \neq 0$  because  $n \not\equiv 1 \pmod{3}$ . The proof of Theorem 2 will be complete when we have shown that this equation is an actual equality, because the other obstructions vanish by connectivity, Poincaré duality, and  $c_n(\omega) = 0$ .

To establish equality, we work with the equation in the form  $f_{n-3*}O^{2n-2}(f) \equiv O^{2n-2}(f'f)$ , where  $f_{n-3}: W_{n,4} \rightarrow K(\mathbf{Z}, 2n-7; \mathbf{Z}, 2n-3, \delta P^1)$  and  $f': BU(n) \rightarrow K(\mathbf{Z}, 2n-6; \mathbf{Z}, 2n-2, \delta P^1)$  are liftings of  $a_{n-3}$  and  $c(n-3)$ , respectively. Let  $h$  and  $\bar{h}$  be  $2n-3$ -liftings of  $f'f$  and  $f$ , respectively, and let  $\{c^{2n-2}(h)\}$  and  $\{c^{2n-2}(\bar{h})\}$  be the obstruction cohomology classes determined by these liftings. We assert that  $h$  and  $\bar{h}$  can be altered in such a way that the obstruction class of  $h$  is unchanged and  $f_{n-3*}\{c^{2n-2}(\bar{h})\} = \{c^{2n-2}(h)\}$ . The argument begins by altering  $h$  by a  $2n-7$ -cocycle  $\nu$  such that  $\{\nu\} = -3O^{2n-7}(f'\bar{h}, h)$ . The altered map  $h_\nu$  extends to a  $2n-3$  lifting of  $f'f$  because of the homotopy of  $K(\mathbf{Z}, 2n-7; \mathbf{Z}, 2n-3, \delta P^1)$  and  $\{c^{2n-2}(h)\} - \{c^{2n-2}(h_\nu)\} = \delta P^1 O^{2n-7}(h, h_\nu) = \delta P^1\{\nu\}$ , [9], so  $\{c^{2n-2}(h)\} = \{c^{2n-2}(h_\nu)\}$  by our choice of  $\nu$ . Note that

$$O^{2n-7}(f'\bar{h}, h_\nu) = O^{2n-7}(f'\bar{h}, h) + O^{2n-7}(h, h_\nu) = -2O^{2n-7}(f'\bar{h}, h).$$

The homomorphism  $f_{n-3*}: \pi_{2n-7}(W_{n,4}) \rightarrow \mathbf{Z}$  is the identity and so we may alter  $\bar{h}$  by a  $2n-7$ -cocycle  $\alpha$  such that  $f_{n-3*}\{\alpha\} = \{\alpha\} = O^{2n-7}(f'\bar{h}, h_\nu)$ . If  $\bar{h}_\alpha$  is the altered map,  $O^{2n-7}(\bar{h}, \bar{h}_\alpha) = -2O^{2n-7}(f'\bar{h}, h)$ , and so  $\{c^{2n-4}(\bar{h})\} - \{c^{2n-4}(\bar{h}_\alpha)\} = \delta Sq^2 O^{2n-7}(\bar{h}, \bar{h}_\alpha) = 0$ . Therefore,  $\bar{h}_\alpha$  extends to a  $2n-3$  lifting of  $f$  because  $\{c^{2n-4}(\bar{h})\} = 0$  and  $M$  is 3-connected mod 2. We have

$$O^{2n-7}(f'\bar{h}_\alpha, h_\nu) = O^{2n-7}(f'\bar{h}_\alpha, f'\bar{h}) + O^{2n-7}(f'\bar{h}, h_\nu) = 0$$

because  $O^{2n-7}(\bar{h}_\alpha, \bar{h}) = -\{\alpha\}$ , and so  $f'\bar{h}_\alpha$  is homotopic to  $h_\nu$  in dimensions less than  $2n-3$  and the difference formula for cocycles yields  $f_{n-3*}\{c^{2n-2}(\bar{h}_\alpha)\} = \{c^{2n-2}(h_\nu)\}$ . This completes the proof of equality and of Theorem 2. The proof of Theorem 1 is exactly the same except that there is no obstruction of order 3. The remark about the special case  $n \not\equiv 0 \pmod{3}$ ,  $n \equiv 2 \pmod{4}$  follows from [2].

The problem of computing the operations  $\delta P^1_*$  seems difficult. The following example shows that they are nontrivial invariants of the sectioning problem on the level of complexes<sup>1</sup>. Let  $n$  be even,  $n \geq 6$ , and let  $p: E \rightarrow BU(n)$  be the first stage in the Postnikov system for the fibration  $BU(n-3) \rightarrow BU(n)$  and let  $p_1: E_1 \rightarrow BU(n-2)$  be the first

<sup>1</sup> I am grateful to the referee for this example.

stage in the system of the fibration  $BU(n-3) \rightarrow BU(n-2)$ . Let  $X$  be the  $2n-1$ -skeleton of  $E_1$  and  $\omega$  the bundle classified by the inclusion  $BU(n-2) \rightarrow BU(n)$  composed with  $p_1$ . If  $k$  in  $H^{2n-2}(E; \mathbf{Z})$  is the  $k$ -invariant, then  $2k = p^*c(n-1)$  [10] and if  $s: E_1 \rightarrow E$  is the natural map, then  $s^*k$  is in the image of the Bockstein mod 2. With these observations, it is easy to see that  $\omega$  has the following properties:  $c_1(\omega) = 0$ ,  $n-2 \leq i \leq n$ , complex span  $\omega \cong 2$ , but  $0 \notin \delta Sq_\omega^2(c(n-2))$  because  $X$  does not lift to  $BU(n-3)$ .

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