

Pacific Journal of Mathematics

**SCALAR SPECTRAL OPERATORS, ORDERED l^p -DIRECT
SUMS, AND THE COUNTEREXAMPLE OF
KAKUTANI-MCCARTHY**

DIETER LUTZ

SCALAR SPECTRAL OPERATORS, ORDERED l^p -DIRECT SUMS, AND THE COUNTEREXAMPLE OF KAKUTANI — MC CARTHY

DIETER LUTZ

Contrary to the situation on Hilbert space, the sum and product of two commuting scalar spectral operators on a Banach space X need not be spectral, even if X is reflexive. This has been shown by Kakutani and Mc Carthy. In this note, order-theoretic methods are used to discuss Mc Carthy's construction. To this end, a special class of scalar spectral operators is introduced.

In what follows, X always denotes a Banach space over \mathbf{R} , $B(X)$ the Banach algebra of bounded linear operators on X , $\sigma(T)$, $\rho(T)$ and $R(z, T) = (T - zI)^{-1}$ for $T \in B(X)$ the spectrum, the resolvent set and the resolvent operator for $z \in \rho(T)$ (taken as usual with respect to the complexification of X).

In the first section we introduce a class \mathcal{S} of C^m -selfadjoint operators ([2]) defined on an ordered Banach space X . In Theorem 1, conditions are given which assure that the elements of \mathcal{S} are, in fact, scalar spectral operators. In the counterexample of Mc Carthy ([1]), which improves an example constructed by Kakutani ([4]), the underlying Banach space X is the l^p -direct sum of finite-dimensional spaces. So in the second section, some order properties of l^p -direct sums of ordered Banach spaces are considered. The last section is devoted to Mc Carthy's construction. It is shown that the natural order of the space he used is not normal. This is remarkable since this order is induced by the natural \mathbf{R}^n -order of the summands. Therefore, a direct proof of the nonnormality is added.

1. A class of scalar spectral operators. A proper convex cone $K \subset X$ induces an ordering on $B(X)$ by

$$T \geq 0 \quad \text{iff} \quad TK \subset K.$$

Then $\mathcal{K} = \{T \in B(X), T \geq 0\}$ is a convex cone. \mathcal{K} is proper if K generates X , that is, if $K - K = X$.

Let $\mathcal{S} \subset B(X)$ consist of all operators A with the following properties:

1. $\sigma(A) \subset \mathbf{R}$.
2. There exist constants $m, M \in \mathbf{R}$ with

$$\|R(z, A)\| \leq M \cdot |\operatorname{Im} z|^{-m}$$

for all $z \in \mathbf{C}$ with $\operatorname{Im} z \neq 0$.

3. $R(z, A)R(\bar{z}, A) \in \mathcal{H}$ for every $z \in \rho(A)$ (\bar{z} denotes the complex conjugate of z).

If $A \in B(X)$ obeys 1 and 2, A is a $C^{m+1}(I)$ -selfadjoint operator in the sense of Colojoara and Foias [2] for every interval $I \subset \mathbf{R}$ which contains $\sigma(A)$ in its interior. According to Tillmann [6], the uniquely determined $C^{m+1}(I)$ -spectral function

$$U: C^{m+1}(I) \rightarrow B(X)$$

for A is given by

$$U(f) = \lim_{\epsilon \rightarrow 0} (2\pi i)^{-1} \int_I f(x) \{R(x + i\epsilon, A) - R(x - i\epsilon, A)\} dx.$$

From the resolvent equation

$$R(z, A) - R(z', A) = (z - z')R(z, A)R(z', A)$$

it follows that

$$(1) \quad U(f) = \lim_{\epsilon \rightarrow 0} \epsilon \pi^{-1} \cdot \int_I f(x) \{R(x + i\epsilon, A)R(x - i\epsilon, A)\} dx.$$

U is a continuous homomorphism from $C^{m+1}(I)$ into $B(X)$, if $C^{m+1}(I)$ is endowed with the topology of uniform convergence of all derivatives up to order $m+1$. Now let M_λ for $\lambda \in \mathbf{R}$ denote the interval $(-\infty, \lambda)$. We set

$$C_0^\infty(M_\lambda) := \{f \in C^\infty(\mathbf{R}), \operatorname{supp} f \subset M_\lambda \text{ compact}\}.$$

If χ_λ denotes the characteristic function of M_λ , we can choose a monotonely increasing sequence $\{f_{\lambda, n}\}_{n \in \mathbf{N}}$ of functions in $C_0^\infty(M_\lambda)$ such that the sets

$$M_{\lambda, n} = \{x \in \mathbf{R}, f_{\lambda, n}(x) = 1\}$$

exhaust M_λ and $\{f_{\lambda,n}\}_n$ converges pointwise to χ_λ . Then for every $g \in C_0^\infty(M_\lambda)$ with $g \leq \chi_\lambda$ there is a $n_0 \in \mathbf{N}$ with $g \leq f_{\lambda,n}$ for every $n \in \mathbf{N}$ with $n \geq n_0$.

Further for every n there exists a m with

$$(2) \quad f_{\lambda,n}^2 \leq f_{\lambda,n} \leq f_{\lambda,m}^2.$$

Clearly the order of functions is taken pointwise.

If in addition A obeys 3, substituting $f_{\lambda,n}$ into (1) we get a monotonely increasing sequence $\{U(f_{\lambda,n})\}_n$ of operators with

$$0 \leq U(f_{\lambda,n}) \leq I$$

for every $n \in \mathbf{N}$.

If $\{U(f_{\lambda,n})x\}_n$ converges for every $x \in X$, then

$$E_\lambda := \lim_n U(f_{\lambda,n})$$

is a continuous linear operator. E_λ is positive and idempotent by (2). If K is closed it follows that

$$E_\lambda = \sup_n U(f_{\lambda,n}) \quad (\text{Schaefer [5]}).$$

The cone K is called normal if there exists a norm $\|\cdot\|^*$ on X equivalent to the initial norm $\|\cdot\|$ such that $0 \leq x \leq y$ implies that $\|x\|^* \leq \|y\|^*$. Such a norm is said to be monotone.

Now we can state

THEOREM 1. *Let X be weakly sequentially complete and ordered by a normal closed and generating cone K . If $A \in \mathcal{S}$, then for every $\lambda \in \mathbf{R}$ the sequence $\{U(f_{\lambda,n})\}_{n \in \mathbf{N}}$ converges in the strong operator topology to a continuous positive operator E_λ . Moreover, we have*

1. $E_\lambda = \sup_n U(f_{\lambda,n})$.
2. $E_\lambda^2 = E_\lambda$.
3. $E_\lambda = \sup\{U(f), f \in C^\infty(\mathbf{R}), \text{supp } f \subset M_\lambda, f \leq \chi_\lambda\}$.
4. $E_\lambda = 0$ for $\lambda < -r(A)$ ($r(A)$ denotes the spectral radius of A).
5. $E_\lambda = I$ for $\lambda > r(A)$.
6. $E_\lambda = \lim_{\mu < \lambda} E_\mu$.

7. $A = \int_{\mathbf{R}} \lambda dE_\lambda$ (where the integral exists in the strong operator topology).

Since E_λ is a left continuous spectral family, A is a scalar spectral operator in the sense of Dunford-Schwartz [3]. The continuous homomorphism

$$f \rightarrow f(A) = \int f(\lambda) dE_\lambda$$

of $C(\mathbf{R})$ into $B(X)$ is an extension of the $C^{m+1}(I)$ -spectral function U of A .

Proof. We only prove the convergence of $\{U(f_{\lambda,n})\}_n$; then the other assertions can be shown following Tillmann [6]. Since K is normal, the dual cone

$$K' := \{x' \in X', x'(x) \geq 0 \text{ for every } x \in K\}$$

generates X' . If $x \in K$, $\{x'(U(f_{\lambda,n})x)\}_n$ is monotonely increasing and bounded by $x'(x)$ and consequently convergent for every $x' \in K'$. Since K and K' generate X and X' and since X is weakly sequentially complete $\{U(f_{\lambda,n})\}_n$ thus converges weakly to an operator E_λ . Moreover, since K is normal, the convergence follows in the strong operator topology from the generalized theorem of Dini ([5]).

To see that for general Banach spaces and generating closed cones K Theorem 1 need not be true, let $X = C^k[0, 1]$, $K = \{f \in X, f(x) \geq 0 \text{ for all } x \in [0, 1]\}$ and define $A \in B(X)$ by

$$(Af)(t) := t \cdot f(t), \quad t \in [0, 1].$$

Then

1. $\sigma(A) = [0, 1]$.

2.
$$\|R(z, A)f\| = \sup_{0 \leq t \leq k} \sup_{t \in [0, 1]} \left| \left(\frac{f(t)}{z - t} \right)^{(k)} \right|$$

$$\leq \|f\| \cdot M \cdot |\operatorname{Im} z|^{-k} \quad \text{for all } z \in \rho(A) \text{ and } f \in C^k[0, 1],$$

if M is appropriately chosen.

3.
$$(R(z, A)R(\bar{z}, A)f)(t) = (z - t)^{-1}(\bar{z} - t)^{-1}f(t)$$

$$= |z - t|^{-2}f(t) \geq 0, \quad \text{if } f(t) \geq 0.$$

Thus we have $A \in \mathcal{S}$, but it is well known that A is not a spectral operator.

2. Ordered l^p -direct sums. Let I be an index set, let $(X_i, \|\cdot\|_i)$ be Banach spaces for $i \in I$, and define for $1 \leq p < \infty$

$$X^p := l^p((X_i, \|\cdot\|_i)_{i \in I} := \left\{ x = \{x_i\}_i, x_i \in X_i, \sum_i \|x_i\|_i^p < \infty \right\}.$$

Then X^p is a Banach space with respect to the norm

$$\|x\|^p = \left(\sum_i \|x_i\|_i^p \right)^{1/p}.$$

X^p is reflexive for $p > 1$, if the X_i are reflexive.

If for every $i \in I$ X_i is ordered by a convex cone K_i , a natural order on X^p is defined by the cone K with

$$x = \{x_i\}_i \in K \quad \text{iff} \quad x_i \in K_i \quad \text{for every} \quad i \in I.$$

We recall that a cone K in a normed space X is called M -generating if for every $x \in X$ there is a decomposition $x = x^+ - x^-$ with $x^+, x^- \in K$ and $\|x^+\|, \|x^-\| \leq M \cdot \|x\|$.

Then one can easily prove the following

PROPOSITION 2.

1. K is a proper cone iff K_i is proper for every $i \in I$.
2. If K generates X then K_i generates X_i for every $i \in I$.
3. K is M -generating iff K_i is M -generating for every $i \in I$.
4. K is normal iff for every $i \in I$ K_i is normal and there is a monotone norm $\|\cdot\|_i^*$ on X_i equivalent to $\|\cdot\|_i$, such that

$$m \cdot \|\cdot\|_i^* \leq \|\cdot\|_i \leq M \cdot \|\cdot\|_i^*$$

with constants $m, M > 0$ independent of i .

3. The counterexample of Kakutani–Mc Carthy. The sum and product of two commuting scalar spectral operators defined on a Banach space X need not be spectral as was shown by an example of Kakutani in [4]. Mc Carthy’s modification of this example in [1] led to a counterexample even on a separable reflexive Banach space, namely the l^2 -direct sum of finite-dimensional spaces. This space a natural order can be given by the summands. We start this section by showing that from Mc Carthy’s result the nonnormality of this ordering follows. Since we want to give some explicit calculations we have to recall in short the construction of Mc Carthy.

For $n = 1, 2, \dots$ be $Y_n = Y'_n = \mathbf{R}^{2^n}$ with the sup-norm, and X_n the space of all $2^n \times 2^n$ -matrices with real coefficients. On X_n the “projective tensor norm” q_n is defined by

$$q_n(x) := \inf \sum_{i=1}^l \|y_i\|_\infty \|z_i\|_\infty,$$

where we take the infimum over all representations of $x \in X_n$ of the form

$$x(i, j) = \sum_{k=1}^l y_k(i) z_k(j), \quad i, j = 1, 2, \dots, 2^n,$$

with

$$\begin{aligned} y_k &= (y_k(1), \dots, y_k(2^n)) \in Y_n, \\ z_k &= (z_k(1), \dots, z_k(2^n)) \in Y'_n, \end{aligned} \quad k = 1, \dots, l, l \in \mathbf{N}.$$

We define matrices $s_n \in X_n$ by setting

$$(3) \quad s_n(i, j) := (-1)^{\sum_{k=1}^n e_k(i) e_k(j)}, \quad i, j = 1, \dots, 2^n,$$

where the $e_k(i) \in \{0, 1\}$ are uniquely determined by

$$(4) \quad i = e_1(i)2^{n-1} + e_2(i)2^{n-2} + \dots + e_{n-1}(i)2 + e_n(i) + 1.$$

According to [1] and [4], for every $u \in X_n$ we have

$$(5) \quad q_n(u) \cong 2^{n/2} \cdot \frac{1}{2^{2^n}} \cdot \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} s_n(i, j) u(i, j).$$

Now we put $X = l^2((X_n, q_n)_{n \in \mathbf{N}})$. X is a reflexive and therefore weakly sequentially complete Banach space.

We define $S, T \in B(X)$ by

$$T(\{x_n\}_n) := \{T_n x_n\}_n \quad \text{with} \quad (T_n x_n)(i, j) := 2^{-2^n} 3^{-i} x_n(i, j),$$

$$S(\{x_n\}_n) := \{S_n x_n\}_n \quad \text{with} \quad (S_n x_n)(i, j) := 5^{-j} x_n(i, j).$$

T and S are commuting scalar spectral operators, but neither TS nor $T + S$ is spectral.

We show that TS belongs to the family \mathcal{S} introduced in the first chapter. Indeed,

1. $\sigma(TS)$ is the closure of the set of eigenvalues $2^{-2^n} 3^{-i} 5^{-j}$, $1 \leq i, j \leq 2^n$, $n \in \mathbf{N}$, and is therefore contained in \mathbf{R} .

2. T and S are commuting scalar spectral operators. By [2], TS (and $T + S$) is a generalized scalar operator and thus the resolvent of TS obeys a growth condition.

3. If X_n is given its natural order, defined by

$$x_n \in K_n \quad \text{iff} \quad x_n(i, j) \geq 0 \quad \text{for} \quad i, j = 1, \dots, 2^n,$$

each of the cones K_n is proper, closed and generates X_n . Moreover, K_n is normal since q_n is equivalent to a monotone norm on X_n , e.g. to the sup-norm. If $K \subset X$ is defined as above, K is also a proper and closed cone. K generates X , since if $x_n \in X_n$ is represented by

$$x_n(i, j) = \sum_{k=1}^l y_k(i)z_k(j)$$

with $y_k \in Y_n, z_k \in Y'_n$, we put

$$y_k^+(i) = \begin{cases} y_k(i), & \text{if } y_k(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$y_k^-(i) = \begin{cases} -y_k(i), & \text{if } y_k(i) < 0 \\ 0 & \text{otherwise} \end{cases},$$

and define z_k^+, z_k^- similarly.

We obtain

$$\begin{aligned} x_n(i, j) &= \left(\sum_{k=1}^l (y_k^+(i)z_k^+(j) + y_k^-(i)z_k^-(j)) \right) \\ &\quad - \left(\sum_{k=1}^l (y_k^-(i)z_k^+(j) + y_k^+(i)z_k^-(j)) \right) \\ &= x_n^+ - x_n^- \end{aligned}$$

with $x_n^+, x_n^- \in K_n, q_n(x_n^+), q_n(x_n^-) \leq 2q_n(x_n)$.

Thus the cones K_n are 2-generating in X_n and by proposition 2 the same is true for K in X .

The assertion that $x \in K$ implies $R(z, TS)R(\bar{z}, TS)x \in K$ for $z \in \rho(TS)$ can now be proven in the equivalent form, that

$$(TS - zI)(TS - \bar{z}I)x \in K \quad \text{implies that} \quad x \in K.$$

Indeed, we have

$$(TS - zI)(TS - \bar{z}I)x = \{(T_n S_n - zI)(T_n S_n - \bar{z}I)x_n\}_n$$

and

$$\begin{aligned} ((T_n S_n - zI)(T_n S_n - \bar{z}I)x_n)(i, j) &= ((2^{-2^n} 3^{-1} 5^{-1})^2 + |z|^2 \\ &\quad - 2 \cdot 2^{-2^n} 3^{-1} 5^{-1} \cdot \text{Re } z)x_n(i, j) \geq 0 \end{aligned}$$

gives $x_n(i, j) \geq 0$ since the scalar factor at the right side is positive.

This would imply

$$\|t_n\|_n \leq \|v_n\|_n \leq \frac{1}{m} q_n(v_n) = \frac{1}{m}$$

and therefore $q_n(t_n) \leq M/m$ in contrary to (6).

REFERENCES

1. C. A. Mc Carthy, *Commuting Boolean algebras of projections*, Pacific J. Math., **11** (1961), 295–307.
2. I. Colojoara and C. Foias, *Theory of generalized spectral operators*, New York (1968).
3. N. Dunford and J. T. Schwartz, *Linear operators, Part III: Spectral operators*, New York (1971).
4. S. Kakutani, *An example concerning uniform boundedness of spectral measures*, Pacific J. Math., **4** (1954), 363–372.
5. H. H. Schaefer, *Topological vector spaces*, New York (1966).
6. H. G. Tillmann, *Eine Erweiterung des Funktionalkalküls für lineare Operatoren*, Math. Ann., **151** (1963), 424–430.

Received August 18, 1975 and in revised form January 8, 1976.

FACHBEREICH MATHEMATIK
DER UNIVERSITÄT
7750 KONSTANZ
POSTFACH 7733

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1976 Pacific Journal of Mathematics
All Rights Reserved

Allan Russell Adler and Catarina Isabel Kiefe, <i>Pseudofinite fields, pro-cyclic fields and model-completion</i>	305
Christopher Allday, <i>The stratification of compact connected Lie group actions by subtori</i>	311
Martin Bartelt, <i>Commutants of multipliers and translation operators</i>	329
Herbert Stanley Bear, Jr., <i>Ordered Gleason parts</i>	337
James Robert Boone, <i>On irreducible spaces. II</i>	351
James Robert Boone, <i>On the cardinality relationships between discrete collections and open covers</i>	359
L. S. Dube, <i>On finite Hankel transformation of generalized functions</i>	365
Michael Freedman, <i>Uniqueness theorems for taut submanifolds</i>	379
Shmuel Friedland and Raphael Loewy, <i>Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue</i>	389
Theodore William Gamelin, <i>Uniform algebras spanned by Hartogs series</i>	401
James Guyker, <i>On partial isometries with no isometric part</i>	419
Shigeru Hasegawa and Ryōtarō Satō, <i>A general ratio ergodic theorem for semigroups</i>	435
Nigel Kalton and G. V. Wood, <i>Homomorphisms of group algebras with norm less than $\sqrt{2}$</i>	439
Thomas Laffey, <i>On the structure of algebraic algebras</i>	461
Will Y. K. Lee, <i>On a correctness class of the Bessel type differential operator S_μ</i>	473
Robert D. Little, <i>Complex vector fields and divisible Chern classes</i>	483
Kenneth Loudon, <i>Maximal quotient rings of ring extensions</i>	489
Dieter Lutz, <i>Scalar spectral operators, ordered l^p-direct sums, and the counterexample of Kakutani-McCarthy</i>	497
Ralph Tyrrell Rockafellar and Roger Jean-Baptiste Robert Wets, <i>Stochastic convex programming: singular multipliers and extended duality singular multipliers and duality</i>	507
Edward Barry Saff and Richard Steven Varga, <i>Geometric overconvergence of rational functions in unbounded domains</i>	523
Joel Linn Schiff, <i>Isomorphisms between harmonic and P-harmonic Hardy spaces on Riemann surfaces</i>	551
Virinda Mohan Sehgal and S. P. Singh, <i>On a fixed point theorem of Krasnoselskii for locally convex spaces</i>	561
Lewis Shilane, <i>Filtered spaces admitting spectral sequence operations</i>	569
Michel Smith, <i>Generating large indecomposable continua</i>	587
John Yuan, <i>On the convolution algebras of H-invariant measures</i>	595