ON A FIXED POINT THEOREM OF KRASNOSELSKII FOR
LOCALLY CONVEX SPACES

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Let $\mathcal{U}$ be a neighborhood basis of the origin consisting of absolutely convex open subsets of a separated locally convex topological vector space $E$ and $S$ a subset of $E$. Let a mapping $f: S \to E$ satisfy the condition: for each $U \in \mathcal{U}$ and $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, U) > 0$ such that if $x, y \in S$ and $x - y \in (\epsilon + \delta)U$, then $f(x) - f(y) \in \epsilon U$. In the present paper, sufficient conditions are given for the mapping $f$ to have a fixed point in $S$. The result is extended to the sum of two mappings of Krasnoselskii type.

In a recent paper, Meir and Keeler [8] gave an interesting generalization of the Banach's contraction principle. Following [8], a self mapping $f$ of a metric space $(X, d)$ is an $(\epsilon, \delta)$ contraction iff for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in X$ with $\epsilon \leq d(x, y) \leq \epsilon + \delta$ implies $d(f(x), f(y)) < \epsilon$. The $(\epsilon, \delta)$ contraction mappings clearly contain the class of strict contractions ($d(f(x), f(y)) \leq \lambda d(x, y), 0 < \lambda < 1$) and the nonlinear contractions investigated by Boyd and Wong [4]. In this paper, we consider mappings defined on a subset $S$ of a locally convex vector space $E$ with values in $E$ (not necessarily $S$) and satisfy a certain condition similar to $(\epsilon, \delta)$ contraction. The main result here generalizes a result of Cain and Nashed [5] and a recent result of Assad and Kirk [2] and provides a further generalization of a well-known result of Krasnoselskii [7].

Throughout this paper, $E$ is a separated locally convex topological vector space and $\mathcal{U}$ is a neighborhood basis of the origin consisting of absolutely convex open subsets of $E$. For each $U \in \mathcal{U}$, let $p_U$ be the Minkowski's functional of $U$. Further, if $x, y \in E$ let

$$(x, y) = \{z \in E: z = \lambda x + (1 - \lambda)y, 0 < \lambda < 1\}$$

and $[x, y] = \{x\} \cup (x, y)$. For a set $A \subseteq E$, $\partial(A)$ denotes the boundary of $A$ and $\text{cl}(A)$ the closure of $A$ in $E$. Also for $A, B \subseteq E$, $A - B = \{x - y: x \in A, y \in B\}$.

Let $S$ be a nonempty subset of $E$. A mapping $f: S \to E$ is a $U$-contraction ($U \in \mathcal{U}$) iff for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon, U) > 0$ such that if $x, y \in S$ and if

$$(1) \quad x - y \in (\epsilon + \delta)U, \quad \text{then} \quad f(x) - f(y) \in \epsilon U.$$
If \( f: S \rightarrow E \) is a \( U \)-contraction for each \( U \in \mathcal{U} \), then \( f \) is a \( \mathcal{U} \)-contraction.

Note that if \( f \) is a \( \mathcal{U} \)-contraction, then \( f \) is continuous. (For a related definition of \( \mathcal{U} \)-contraction, see Taylor [11].)

It may be remarked that if \( E \) is a normed space with \( \mathcal{U} = \{ x \in E : \| x \| < \epsilon, \epsilon > 0 \} \) then (1) is equivalent to \( (\epsilon, \delta) \) contraction [8].

The following lemma simplifies the proof of next theorem.

**Lemma 1.** Let \( f: S \rightarrow E \) be a \( \mathcal{U} \)-contraction, then \( f \) is \( \mathcal{U} \)-contractive, that is for each \( U \in \mathcal{U} \), \( p_U(f(x)-f(y)) < p_U(x-y) \) if \( p_U(x-y) \neq 0 \) and 0 otherwise.

**Proof.** Let \( x, y \in S \) and suppose \( p = p_U, \ p(x-y) = \epsilon > 0 \). Then \( x-y \in (\epsilon + \delta)U \) for each \( \delta > 0 \) and in particular \( x-y \in (\epsilon + \delta_0)U \) where \( \delta_0 = \delta(U, \epsilon) \). Therefore by (1) \( (f(x)-f(y)) \in \epsilon U \). Since \( U \) is open, this implies that \( p(f(x)-f(y)) < \epsilon = p(x-y) \). If \( \epsilon = 0 \), then \( x-y \in \epsilon U \) for each \( \epsilon > 0 \) and hence by (1) \( (f(x)-f(y)) \in \epsilon U \) which implies that \( p(f(x)-f(y)) = 0 \).

**Theorem 1.** Let \( S \) be a sequentially complete subset of \( E \) and \( f: S \rightarrow E \) be a \( \mathcal{U} \)-contraction. If \( f \) satisfies the condition:

\[
(2) \quad \text{for each } x \in S \text{ with } f(x) \notin S, \text{ there is a } z \in (x, f(x)) \cap S \text{ such that } f(z) \in S
\]

then \( f \) has a unique fixed point in \( S \).

**Proof.** Let \( x_0 \in S \) and choose a sequence \( \{x_n\} \subseteq S \) defined inductively as follows: for each \( n \in I \) (positive integers) if \( f(x_n) \in S \), set \( x_{n+1} = f(x_n) \) and if \( f(x_n) \notin S \), let \( x_{n+1} \) be any element of \( (x_n, f(x_n)) \cap S \) such that \( f(x_{n+1}) \in S \) (such \( x_{n+1} \) exists by (2)). It then follows that for each \( n \in I \), there is a \( \lambda_n \in [0, 1) \) satisfying

\[
(3) \quad x_{n+1} = \lambda_n x_n + (1 - \lambda_n) f(x_n).
\]

We show that the sequence \( \{x_n\} \) so constructed satisfies

\[
(4) \quad (a) \quad x_{n+1} - x_n \rightarrow 0 \quad (b) \quad x_n - f(x_n) \rightarrow 0
\]

To establish (4), note that by (3)

\[
(5) \quad x_{n+1} - x_n = (1 - \lambda_n) (f(x_n) - x_n), \quad \text{and}
\]

\[
(6) \quad f(x_n) - x_{n+1} = \lambda_n (f(x_n) - x_n).
\]

Therefore, for a \( U \in \mathcal{U} \) with \( p = p_U \), it follows by the above lemma that
Thus by (5) \( p(f(x_{n+1}) - x_{n+1}) \leq p(f(x_n) - x_n) + p(f(x) - x_n) \)

\[ \leq p(x_{n+1} - x_n) + \lambda_n(f(x) - x_n). \]

Thus by (5) \( p(f(x_{n+1}) - x_{n+1}) \leq p(f(x_n) - x_n) \) for each \( n \in I \), that is \( \{p(f(x_n) - x_n)\} \) is a nonincreasing sequence of nonnegative reals and hence for each \( p = p_U, U \in \mathcal{U} \), there is a \( r(U) \geq 0 \) with

\[ (7) \quad r(U) = p(f(x_n) - x_n) \rightarrow r(U) \geq 0. \]

We claim that \( r(U) \equiv 0 \). Suppose \( r(U) > 0 \). Choose a \( \delta = \delta(r(U), U) > 0 \) satisfying (1). Then by (7) there is a \( n_0 \in I \) such that \( p(f(x_n) - x_n) < r(U) + \delta \) for all \( n \geq n_0 \). Now choose an \( m \in I, m \geq n_0 \) such that \( x_{m+1} = f(x_m) \), (let \( m = n_0 \) if \( f(x_{n_0}) \in S \), otherwise let \( m = n_0 + 1 \), then \( x_{m+1} = f(x_m) \in S \)). Thus for this \( m \),

\[ p(x_m - x_{m+1}) = p(x_m - f(x_m)) < r(U) + \delta. \]

and hence by (1)

\[ p(x_{m+1} - f(x_{m+1})) = p(f(x_m) - f(x_{m+1})) < r(U), \]

which contradicts (7). Thus \( r(U) = 0 \) for each \( U \in \mathcal{U} \) and this implies that the sequence \( x_n - f(x_n) \rightarrow 0 \). This establishes 4(b) and 4(a) now, follows by (5).

We assert that \( \{x_n\} \) is a Cauchy sequence in \( E \). Suppose not. Let for each \( k \in I, A_k = \{x_n: n \geq k\} \). Then by assumption there is \( U \in \mathcal{U} \) such that \( A_k - A_k \not\subseteq U \) for any \( k \in I \). Choose an \( \epsilon \) with \( 0 < \epsilon < 1 \) and a \( \delta \) with \( 0 < \delta < \delta(\epsilon, U) \) satisfying \( \epsilon + \delta < 1 \). It follows that \( A_k - A_k \not\subseteq (\epsilon + \delta/2)U \) for any \( k \in I \). Thus for each \( k \in I \), there exist integers \( n(k) \) and \( m(k) \) with \( k \leq n(k) < m(k) \) such that

\[ (8) \quad x_{n(k)} - x_{m(k)} \not\subseteq (\epsilon + \delta/2)U. \]

Let \( m(k) \) be the least integer exceeding \( n(k) \) satisfying (8). Then by (8)

\[ (9) \quad x_{n(k)} - x_{m(k)} = (x_{n(k)} - x_{m(k)-1}) + (x_{m(k)-1} - x_{m(k)}) \]

\[ \subseteq (x_{m(k)-1} - x_{m(k)}) + (\epsilon + \delta/2)U. \]

Now by (4) there is a \( k_0 \in I \) such that \( x_k - f(x_k) \in (\delta/4)U \) and \( x_{k-1} - x_k \in (\delta/4)U \) whenever \( k \geq k_0 \), and hence by (9)

\[ x_{n(k)} - x_{m(k)} \subseteq (\epsilon + \delta)U, \quad k \geq k_0. \]

It follows, that for all \( k \geq k_0 \)
\[ f(x_{n(k)}) - f(x_{m(k)}) \in \epsilon U. \]

However, for \( k \geq k_0 \),

\[ x_{n(k)} - x_{m(k)} = (x_{n(k)} - f(x_{n(k)})) + (f(x_{n(k)}) - f(x_{m(k)})) + (f(x_{m(k)}) - x_{m(k)}) \]

and therefore,

\[ x_{n(k)} - x_{m(k)} \in \left( \frac{\delta}{4} U + \epsilon U + \frac{\delta}{4} U \right) \subseteq \left( \epsilon + \frac{\delta}{2} \right) U, \quad k \geq k_0, \]

which contradicts (8). Thus \( \{x_n\} \) is a Cauchy sequence in \( S \) and the sequential completeness implies that there is a \( u \in S \) such that \( x_n \to u \). Since \( f \) is continuous, it follows by (4b) that \( u = f(u) \). This proves the existence of the fixed point of \( f \). Since \( E \) is separated, the uniqueness is an immediate consequence of the Lemma 1.

The following result was proven in [10] and its proof here is given for completeness.

**Lemma 2.** Let \( S \) be a closed or sequentially complete subset of \( E \). If \( x \in S \) and \( y \not\in S \) then there is a \( \lambda \in [0, 1] \) such that \( z = (1 - \lambda)x + \lambda y \in \partial(S) \). Further, if \( x \not\in \partial(S) \) then \( 0 < \lambda < 1 \).

**Proof.** Let \( A = \{\mu \geq 0 : (1 - \alpha)x + \alpha y \in S \text{ for all } \alpha \text{ with } 0 \leq \alpha \leq \mu\} \). Since \( x \in S \), \( A \neq \emptyset \). The hypothesis \( y \not\in S \) implies that \( \lambda = \sup\{\mu : \mu \in A\} \leq 1 \). Now if \( S \) is closed or sequentially complete, it follows that \( z = (1 - \lambda)x + \lambda y \in S \) and hence \( \lambda < 1 \). To show that \( z \in \partial(S) \), it suffices to show that for each \( U \in \mathcal{U} \), \( (z + U) \cap c(S) \neq \emptyset \), where \( c(S) \) is the complement of \( S \) in \( E \). Choose a \( \beta_0 > \lambda \) with \( (\beta_0 - \lambda)p(x - y) < 1 \) where \( p = p_U \). By definition of \( \lambda \), there is a \( \beta \) with \( \lambda < \beta \leq \beta_0 \) such that \( z_1 = (1 - \beta)x + \beta y \not\in S \). Since \( p(z_1 - z) = (\beta - \lambda)p(x - y) < 1 \), it follows that \( z_1 \in (z + U) \) and hence \( z \in \partial(S) \). If \( x \not\in \partial(S) \) but \( x \in S \), then clearly \( 0 < \lambda < 1 \).

The following is now an immediate consequence of Theorem 1.

**Theorem 2.** Let \( S \) be sequentially complete subset of \( E \) and \( f : S \to E \) be a \( \mathcal{U} \)-contraction. If \( f(S \cap \partial(S)) \subseteq S \), then \( f \) has a unique fixed point.

It may be noted that if \( S \) is closed then \( S \cap \partial(S) = \partial(S) \).

In the following, let \( \mathcal{P} = \{p = p_U \text{ for some } U \in \mathcal{U}\}, R^+ \) the nonnegative reals and \( \Psi \) a family of mappings defined as \( \Psi = \{\phi : R^+ \to R^+ : \phi \text{ is continuous and } \phi(t) < t \text{ if } t > 0\} \). A mapping \( f : S \to E \) is a nonlinear \( \mathcal{P} \) contraction (see also Boyd and Wong [4]) iff for each \( p \in \mathcal{P} \), there is a \( \phi_p \in \Psi \) such that \( p(f(x) - f(y)) \leq \phi_p(p(x - y)) \) for all \( x, y \in S \). If this
inequality holds with \( \phi_p(t) = \alpha_p t, \ 0 < \alpha_p < 1 \), then \( f \) is called \( \mathcal{P} \)-contraction (see [5]). Since a nonlinear \( \mathcal{P} \)-contraction is a \( \mathcal{U} \)-contraction, the following result immediately follows by Theorem 1 and provides an extension of a result in [5], (see also Assad [11]).

**Theorem 3.** Let \( S \) be a sequentially complete subset of \( E \) and \( f: S \rightarrow E \) be a nonlinear \( \mathcal{P} \) contraction. If \( f \) satisfies (2) then \( f \) has a unique fixed point in \( S \).

As an application of Theorem 3, we give here a generalization of a well-known result of Krasnoselskii [7] which has been extended recently to locally convex spaces in [5]. The following extension of Tychonoff’s theorem [12] is due to Singball [3] (see also Himmelberg [6]) and is used in the proof of Theorem 5.

**Theorem 4.** Let \( S \) be a closed and convex subset of \( E \) and \( f: S \rightarrow S \) be a continuous mapping such that the range \( f(S) \) is contained in a compact set. Then \( f \) has fixed point.

In the rest of this paper, a mapping \( f: S \rightarrow E \) is completely continuous if it is continuous and \( f(S) \) is contained in a compact subset of \( E \). Further, if \( A: S \rightarrow E \) is a nonlinear \( \mathcal{P} \) contraction and \( B: S \rightarrow E \) is completely continuous, then for each fixed \( x \in S \), the mapping \( f_x: S \rightarrow E \) is defined by \( f_x(y) = A(y) + x \). Note that since \( E \) is separated, the mapping \( (I - A): S \rightarrow E \) is one-to-one, where \( I \) is the identity map of \( S \).

The following lemma follows immediately from Theorem 3.

**Lemma 3.** Let \( S \) be a sequentially complete subset of \( E \) and \( A: S \rightarrow E \) be a nonlinear \( \mathcal{P} \) contraction. Suppose for a \( x \in E \), the mapping \( f: S \rightarrow E \) defined by \( f(y) = A(y) + x \) satisfies (2), then there exists a unique \( u(x) \in S \) with \( f(u(x)) = u(x) \), that is \( (I - A)^{-1} x = u(x) \in S \).

**Theorem 5.** Let \( S \) be a convex and complete subset of \( E \). Let \( A: S \rightarrow E \) be a nonlinear \( \mathcal{P} \) contraction and \( B: S \rightarrow E \) be completely continuous. If for each \( x \in S \), the mapping \( f_x: S \rightarrow E \) satisfies (2) and \( (I - A)^{-1} B(S) \) is a bounded subset of \( S \), then there is a \( u \in S \) satisfying \( A(u) + B(u) = u \).

**Proof.** For each fixed \( x \in S \), the mapping \( f_x \) satisfies the conditions of Lemma 3 and hence there is a unique \( u_x \in S \) with \( f_x(u_x) = u_x \). Define a mapping \( L: S \rightarrow S \) by

\[
L(x) = u_x = A(L(x)) + B(x), \quad x \in S.
\]
Then, for each \( x \in S \), \( L(x) = (I - A)^{-1}B(x) \). If follows by hypothesis that \( L(S) \) is a bounded subset of \( E \). We show that \( L \) in (10) is continuous. Let \{\( x_\alpha : \alpha \in \Gamma \} \subseteq S \) be a net such that \( x_\alpha \rightarrow x \in S \) and suppose \( L(x_\alpha) \) does not converge to \( L(x) \). Then there is a \( p \in \mathcal{P} \) and an \( \epsilon > 0 \) and a subnet \{\( p(L(x_\alpha) - L(x)) : \alpha \in \Gamma^1 \} \) of the net \{\( p(L(x_\alpha) - L(x)) : \alpha \in \Gamma \} \) such that

\[
(11) \quad p(L(x_\alpha) - L(x)) > \epsilon \quad \text{for each} \quad \alpha \in \Gamma^1.
\]

Since \{\( p(L(x_\alpha) - L(x)) : \alpha \in \Gamma^1 \} \) is a bounded subset of the reals, it has a subnet \{\( p(L(x_\alpha) - L(x)) : \alpha \in \Gamma^2 \subseteq \Gamma^1 \} \rightarrow r \equiv 0. \) However, by (10) for any \( \alpha \in \Gamma^2 \)

\[
p(L(x_\alpha) - L(x)) \leq p(B(x_\alpha) - B(x)) + \phi_p(p(L(x_\alpha) - L(x))),
\]

which implies that \( r = 0 \). This contradicts (11) and consequently \( L \) is continuous. We now show that \( L(S) \) is relatively compact in \( S \). If \{\( L(x_\alpha) : \alpha \in \Gamma \} \) is a net in \( L(S) \), then there is a net \{\( B(x_\alpha) : \alpha \in \Gamma^1 \} \) which is convergent. We assert that \{\( L(x_\alpha) : \alpha \in \Gamma^1 \} \) is a Cauchy subnet. Suppose not. Then there is a \( p \in \mathcal{P} \) and an \( \epsilon > 0 \) such that for each \( \alpha \in \Gamma^1 \) there are elements \( n(\alpha) \) and \( m(\alpha) \) in \( \Gamma^1 \) with \( n(\alpha) \geq \alpha, m(\alpha) \geq \alpha \), satisfying

\[
(12) \quad r_\alpha = p(L(x_{n(\alpha)}) - L(x_{m(\alpha)})) > \epsilon, \quad \alpha \in \Gamma^1.
\]

Since \{\( B(x_\alpha) : \alpha \in \Gamma^1 \} \) is a Cauchy net, there is an \( \alpha_0 \in \Gamma^1 \) such that \( p(B(x_\alpha) - B(x_\beta)) < \epsilon \) for all \( \alpha, \beta \geq \alpha_0, \alpha, \beta \in \Gamma^1 \). However, \{\( r_\alpha : \alpha \in \Gamma^1 \} \) being a bounded subset of reals has a convergent subnet \{\( r_\alpha : \alpha \in \Gamma^2 \} \rightarrow r \equiv 0. \) The same argument as above implies that \( r = 0 \) and this contradicts (12). This proves the assertion. It now follows by Theorem 4, that \( L(u) = u \) for some \( u \in S \) and hence by (10) \( A(u) + B(u) = u \).

The following consequence of Theorem 5 appears new and generalizes a result of Nashed and Wong (Theorem 1 [9]). Note that in a normed linear space \( E \) a mapping \( f : S \rightarrow E \) is a nonlinear contraction (see [4]) if there exists a \( \phi \in \Psi \) such that \( \|f(x) - f(y)\| \leq \phi(\|x - y\|) \) for all \( x, y \in S \).

**Corollary 1.** Let \( S \) be a closed, bounded and convex subset of a Banach space \( E \). If \( A : S \rightarrow E \) is a nonlinear contraction and \( B : S \rightarrow E \) is completely continuous such that for each \( x \in \partial(S) \), \( f_x(\partial(S)) \subseteq S \), then \( A(u) + B(u) = u \) for some \( u \in S \).

As another consequence, we have the following extension of a result of Cain and Nashed [5].
Corollary 2. Let $S$ be a convex and complete subset of $E$. Let $A: S \to E$ be a $\mathcal{P}$ contraction and $B: S \to E$ be a completely continuous mapping. If for each $x \in S$, $f_x$ satisfies (2) then $A(u) + B(u) = u$ for some $u \in S$.

Proof. It suffices to show that for each $p \in \mathcal{P}$, $p((I-A)^{-1}B(S))$ is a bounded subset of reals. Now it follows by (10) that for all $x, y \in S$

$$p(L(x) - L(y)) \leq p(B(x) - B(y)) + \alpha_p p(L(x) - L(y)),$$

which implies that $p(L(x) - L(y)) \leq (1 - \alpha_p)^{-1}p(B(x) - B(y))$ and hence $L(S) = (I - A)^{-1}B(S)$ is bounded.

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