ON THE CONVOLUTION ALGEBRAS OF H-INARIANT MEASURES

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The totality $M(eSe/H)$ of bounded regular Borel measures on the orbit space $eSe/H$, where $S$ is a locally compact semigroup and $H$ is a compact subgroup with the identity $e$, forms a Banach space; however, its closed subspace $M_H(ESe/H)$ of $H$-invariant measures forms even a Banach algebra under a suitable convolution. Furthermore, if $w$ is an idempotent probability measure with compact support on $S$, then $w * M(S) * w \equiv w_H * M(S) * w_H \equiv M_H(eSe/H)$ algebraically and in various topologies, where $w_H$ is the normalized Haar measure on some compact subgroup $H$.

1. Introduction. We denote the Banach space of bounded regular Borel measures and the totality of probability measures on a locally compact (Hausdorff) space $X$ by $M(X)$ and $P(X)$, respectively. Beside the norm topology, $M(X)$ may be equipped with the weak, weak* and vague topologies, which are the topologies of pointwise convergence on $C^b(X)$, $C_0(X)$ and $K(X)$, respectively, where $C^b(X)$ denotes the totality of bounded continuous functions on $X$, $C_0(X)$ and $K(X)$ the subspaces of functions vanishing at $\infty$ and functions with compact supports, respectively. In $P(X)$, the weak, weak* and vague topologies coincide (p. 59, [2]; [7]). Let $S$ be a locally compact semigroup, then $M(S)$ is a Banach algebra and $P(S)$ a topological (Hausdorff) semigroup under the convolution $\ast$. We refer to [7] for the continuity of $\ast$ in the weak, weak* and vague topologies.

**Lemma 1.1.** Let $S$ be a locally compact semigroup. Then $\text{supp}(\mu * \nu) \subseteq (\text{supp}(\mu) \text{supp}(\nu))^\circ$ for $\mu, \nu \in M(S)$, and equality holds for $\mu, \nu \equiv 0$, where $\text{supp}(\mu)$ denotes the support of $\mu$.

**Proof.** (Cf. 1.1, p. 686, [5]).

**Lemma 1.2.** Let $\alpha: X \to Y$ be a continuous map (resp. morphism) between locally compact spaces (resp. semigroups). Then $M(\alpha): M(X) \to M(Y)$ given by

$$[M(\alpha)(\mu)](f) = \mu(f \circ \alpha), \quad f \in C^b(Y)$$
is a norm-decreasing linear morphism (resp. algebra morphism) continuous in the weak topology. Moreover, if \( \alpha \) is proper, then \( M(\alpha) \) is also continuous in both weak* and vague topologies.

**Proof.** Straightforward.

**Lemma 1.3.** Let \( Y \) be a closed subspace of a locally compact space \( X \). Then every \( f \in K(Y) \) (resp. \( f \in C_0(Y) \)) has an extension \( F \in K(X) \) (resp. \( F \in C_0(X) \)).

**Proof.** This follows from (7.40, p. 99, [1]) and the following commutative diagram:

\[
\begin{array}{ccc}
X & \to & X \cup \{\infty\} \\
\uparrow & & \uparrow \\
Y & \to & Y \cup \{\infty\} \to C \\
\uparrow & & \uparrow \\
Y & \to & C \\
\end{array}
\]

\[ (f(\infty) = 0). \]

**Proposition 1.4.** Let \( S \) be a locally compact semigroup and \( e^2 = e \in S \). Then \( \delta_e * M(S) * \delta_e = M(eSe) \) is a Banach subalgebra of \( M(S) \). In fact, if \( i : eSe \to S \) is the inclusion map, then \( M(i) : M(eSe) \to M(S) \) is an embedding. (Note that, unless mentioned otherwise, our statements are to apply to each of the topologies mentioned before.)

**Proof.** We first observe from Lemma 1.1 that \( \delta_e * M(S) * \delta_e \subseteq M(eSe) \) and that \( \delta_e \) is the identity for \( M(eSe) \), whence \( M(eSe) = \delta_e * M(eSe) * \delta_e \subseteq \delta_e * M(S) * \delta_e \) and thus \( \delta_e * M(S) * \delta_e = M(eSe) \). Since \( \mu \mapsto \delta_e * \mu * \delta_e \) is a Banach space linear retraction, \( M(eSe) \) is a linear closed norm retract of \( M(S) \). As to the others, we will show the weak embedding only. Let \( M(i) (\mu_o) \to M(i)(\mu) \) in \( M(S) \) and \( f \in C^b(eSe) \); then \( f \) has an extension \( F \in C^b(S) \) given by \( F(s) = f(es) \) and thus \( \mu_o(f) = [M(i)(\mu_o)](F) \to [M(i)(\mu)](F) = \mu(f) \). Hence \( M(i) \) is an embedding.

For the purpose of this paper it is therefore no loss of generality to assume that \( S \) is a monoid with the identity \( e \).

**Proposition 1.5.** Suppose that \( S \) acts on the left on a locally compact space \( X \). If \( \mu \in M(X) \) and \( f \in C^b(X) \), then \( f_\mu \in C^b(S) \) is well defined by \( f_\mu(s) = \int f(sx)\mu(dx) \).
Proof. Let $\varepsilon > 0$ be given. By the regularity of $|\mu|$, there exists a compact subset $K \subseteq X$ so that $|\mu|(X \setminus K) < \varepsilon$. For this $K$ and a given $s \in S$, let

$$\varphi(t) = \sup\{|f(tx) - f(sx)| : x \in K\}.$$ 

Then $\varphi(t) \to 0$ as $t \to s$; otherwise, there exist nets $t_n \to s$, and $x_n \to x_0$ in $K$ so that $|f(t_n x_n) - f(sx_0)| > \varepsilon$ which contradicts to the continuity of $f$ at $sx_0$. Hence

$$|f_\mu(t) - f_\mu(s)| \leq \int_K \varphi(t) |\mu| (dx) + \int_{X \setminus K} 2\|f\| |\mu| (dx)$$

$$\leq \varphi(t) |\mu|(K) + 2\|f\| \varepsilon \leq 3\|f\| \varepsilon$$

whenever $t$ is close enough to $s$. Hence $f_\mu \in C^b(S)$.

2. $H$-invariant measures. Let $H$ be any compact group acting on the left on a locally compact space $X$. A $\mu \in M(X)$ is called $H$-invariant if $\int f(hx) \mu(dx) = \int f(x) \mu(dx)$ for all $f \in C^b(X)$, $h \in H$.

For convenience, we will denote by $M_H(X)$ the Banach subspace of all $H$-invariant measures in $M(X)$. We now assume that $S$ acts on the left on $X$ and $H$ is a compact subgroup of units in $S$. Suppose now that $f \in C^b(X)$ and $\mu \in M_H(X)$. By Proposition 1.4, $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx) \mu(dx)$. If we set $(fs)(x) = f(sx)$, then we note that $f_\mu(sh) = \int (fs)(hx) \mu(dx) = \mu(fs) = \int f(sx) \mu(dx) = f_\mu(s)$ for all $h \in H$. Hence $f_\mu$ is constant on left cosets $sH$ in $S$. If $S/H = \{sH : s \in S\}$ and $p : S \to S/H$ is given by $p(s) = sH$, then $F \mapsto F \circ p : C^b(S/H) \to C^b(S)$ is an isometry onto $C^b_h(S)$ of all functions which are constant on orbits $sH$. Hence there is a unique function $\mu * v \in C^b(S/H)$ such that $\tilde{\mu} \circ p = f_\mu$. If now $\mu \in M_H(S/H)$ and $\nu \in M_H(X)$, then we define

$$\mu * \nu(f) = \mu(\tilde{\nu})$$

on $C^b(X)$, which we will write

$$\mu * \nu(f) = \int f(sx) \mu(d\tilde{s}) \nu(dx), \quad \tilde{s} = p(s).$$

As $(fh)_\mu = (f_\mu)h$, we have $\mu * \nu(fh) = \mu((fh)_\mu) = \mu(f_\mu) = \mu(f)$, whence $\mu * \nu \in M_H(X)$. In particular, if $\mu$, $\nu \in M_H(S/H)$, then $\mu * \nu \in M_H(S/H)$. 

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**Lemma 2.1.** $M(p): M(S) \rightarrow M(S/H)$ is a norm-decreasing continuous linear morphism mapping $w_H^* M(S)$ into $M_H(S/H)$ where $w_H$ is the normalized Haar measure on $H$.

**Proof.** We observe first that $w_H^* M(S) \subseteq M_H(S)$ by invariance of $w_H$, and that $M(p)$ maps $M_H(S)$ into $M_H(S/H)$. And since $M(p)$ is continuous in various topologies, then so is any restriction and corestriction of $M(p)$.

**Lemma 2.2.** $M(p)$ induces norm-preserving bijections $M(S)^* w_H \rightarrow M(S/H)$ and $w_H^* M(S)^* w_H \rightarrow M_H(S/H)$.

**Proof.** It suffices to show bijections only (cf. 2.45, p. 20, [6]).

1. **Surjectivity:** Let $f \in C^b(S)$ and set $f_H = \int f(s h) w_H(dh)$. Then $f_H \in C_H^b(S)$ and hence defines a unique $f_H \in C^b(S/H)$ such that $f_H \circ p = f_H$. If now $\nu \in M(S/H)$, then $f \mapsto \nu'(f_H)$ is a bounded linear functional. Hence there is a $\nu \in M(S)$ with $\nu(f) = \nu'(f_H)$. Now $\nu * w_H(f) = \nu(f_H) = \nu'(f_H) = \nu'(f_H) = \nu(f)$. Thus $\nu * w_H = \nu$, i.e. $\nu \in M(S)^* w_H$. Now suppose that even $\nu' \in M_H(S/H)$. Then

$$w_H^* \nu(f) = \int f(hx) w_H(dh) \nu(dx) = \int \nu(fh) w_H(dh)$$

$$= \int \nu'(fh) w_H(dh) = \int \nu'(f_H) w_H(dh) = \nu'(f_H)$$

since $\nu' \in M_H(S/H)$. The last term equals $\nu(f_H) = \nu(f)$. Thus $w_H^* \nu = \nu$, i.e. $\nu \in w_H^* M(S)^* w_H$. Now, for $f \in C^b(S/H)$, $[M(p)(\nu)](f) = \nu(f \circ p) = \nu'(f_H) \mu$. But $(f \circ p) \circ p = (f \circ p)_H = f \circ p$, whence $f = (f \circ p)_H$; thus $\nu'(f_H) = \nu'(f)$. This shows $M(p)(\nu) = \nu'$ in both cases, i.e. $M(S/H)$ is in the image of $M(S)^* w_H$ and $M_H(S/H)$ is in the image of $w_H^* M(S)^* w_H$ under $M(p)$.

2. **Injectivity:** For $\mu, \nu \in M(S)^* w_H$, we note that $M(p)(\mu) = M(p)(\nu)$ implies $\mu(f) = [M(p)(\mu)](f_H) = [M(p)(\nu)](f_H) = \nu(f)$ for $f \in C^b(S)$, hence $\mu = \nu$.

**Lemma 2.3.** $M(p): w_H^* M(S)^* w_H \rightarrow M_H(S/H)$ is an algebra morphism.

**Proof.** First of all, we observe the following facts: (1) For $\mu \in w_H^* M(S)^* w_H$ and $f \in C^b(S)$, $\mu(f) = [M(p)(\mu)](f_H)$. (2) For $\nu \in w_H^* M(S)^* w_H$ and $f \in C^b(S/H)$, $f \circ C_H^b(S)$ is well defined by
\[ f_v(x) = \int f(xy) [M(p)(\nu)](dy) = \int f(xy) \nu(dy) = \int f \circ p(xy) \nu(dy), \text{ with } \nu = M(p)(\nu). \]

Then, if \( \mu, \nu \in w_H * M(S) * w_H \) and \( f \in C^*(S/H) \), we have

\[
[M(p)(\mu * \nu)](f) = \mu * \nu (f \circ p) = \int f \circ p(xy) \mu(dx) \nu(dy)
\]

\[
= \int f(xy) \mu(dx) [M(p)(\nu)](dy)
\]

\[
= \mu(f_v) = [M(p)(\mu)]((f_v)_H)
\]

\[
= [M(p)(\mu)](\tilde{f}_v) = [M(p)(\mu) * M(p)(\nu)](f).
\]

**Proposition 2.4.** \( M(p): w_H * M(S) * w_H \rightarrow M_H(S/H) \) is a norm-preserving algebra isomorphism.

**Proof.** It remains to show that \( M(p)|w_H * M(S) * w_H \) is open which follows from the facts that \( \mu(f) = [M(p)(\mu)](f_H) \) for all \( \mu \in w_H * M(S) * w_H \), and that \( f \in K(S) \) (resp. \( f \in C_0(S) \)) implies \( f_H \in K(S) \) (resp. \( f_H \in C_0(S) \)) and thus \( f_H \in K(S/H) \) (resp. \( f_H \in C_0(S/H) \)).

**Corollary 2.5.** Let \( H \) be normal in \( S \) (2.1, p. 17, [3]). Then \( M(p): M(S) \rightarrow M(S/H) \) is a continuous algebra morphism mapping \( w_H * M(S) * w_H \) isomorphically onto \( M_H(S/H) \).

**Corollary 2.6.** Let \( P_H(S/H) \) denote the totality of \( H \)-invariant probability measures in \( P(S/H) \). Then \( M(p): w_H * P(S) * w_H \rightarrow P_H(S/H) \) is an isomorphism.

In the remainder, we assume that \( w \) is an idempotent probability measure with compact support on \( S \); then \( w = \mu_e * w_H * \mu_F \) [4].

**Lemma 2.7.** The maps \( w * M(S) * w \xrightarrow{\alpha} w_H * M(S) * w_H \) defined via \( \alpha(\mu) = w_H * \mu * w_H \) and \( \beta(\nu) = w * \nu * w \) are mutually inverse norm-preserving continuous algebra morphisms so that \( \alpha(w) = w_H \) and \( \beta(w_H) = w \).

**Proof.** The proof in (3.1–2, [8]) yields this.
Proposition 2.8.

\[ w * M(S) * w = w_H * M(S) * w_H = M_H(S/H) \]

algebraically and topologically.

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