

Pacific Journal of Mathematics

**ON THE CONVOLUTION ALGEBRAS OF H -INVARIANT
MEASURES**

JOHN YUAN

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The totality $M(eSe/H)$ of bounded regular Borel measures on the orbit space eSe/H , where S is a locally compact semigroup and H is a compact subgroup with the identity e , forms a Banach space; however, its closed subspace $M_H(eSe/H)$ of H -invariant measures forms even a Banach algebra under a suitable convolution. Furthermore, if w is an idempotent probability measure with compact support on S , then $w * M(S) * w \cong w_H * M(S) * w_H \cong M_H(eSe/H)$ algebraically and in various topologies, where w_H is the normalized Haar measure on some compact subgroup H .

1. Introduction. We denote the Banach space of bounded regular Borel measures and the totality of probability measures on a locally compact (Hausdorff) space X by $M(X)$ and $P(X)$, respectively. Beside the norm topology, $M(X)$ may be equipped with the weak, weak* and vague topologies, which are the topologies of pointwise convergence on $C^b(X)$, $C_0(X)$ and $K(X)$, respectively, where $C^b(X)$ denotes the totality of bounded continuous functions on X , $C_0(X)$ and $K(X)$ the subspaces of functions vanishing at ∞ and functions with compact supports, respectively. In $P(X)$, the weak, weak* and vague topologies coincide (p. 59, [2]; [7]). Let S be a locally compact semigroup, then $M(S)$ is a Banach algebra and $P(S)$ a topological (Hausdorff) semigroup under the convolution $*$. We refer to [7] for the continuity of $*$ in the weak, weak* and vague topologies.

LEMMA 1.1. *Let S be a locally compact semigroup. Then $\text{supp}(\mu * \nu) \subseteq (\text{supp}(\mu)\text{supp}(\nu))^-$ for $\mu, \nu \in M(S)$, and equality holds for $\mu, \nu \geq 0$, where $\text{supp}(\mu)$ denotes the support of μ .*

Proof. (Cf. 1.1, p. 686, [5]).

LEMMA 1.2. *Let $\alpha: X \rightarrow Y$ be a continuous map (resp. morphism) between locally compact spaces (resp. semigroups). Then $M(\alpha): M(X) \rightarrow M(Y)$ given by*

$$[M(\alpha)(\mu)][f] = \mu(f \circ \alpha), \quad f \in C^b(Y)$$

is a norm-decreasing linear morphism (resp. algebra morphism) continuous in the weak topology. Moreover, if α is proper, then $M(\alpha)$ is also continuous in both weak* and vague topologies.

Proof. Straightforward.

LEMMA 1.3. Let Y be a closed subspace of a locally compact space X . Then every $f \in K(Y)$ (resp. $f \in C_0(Y)$) has an extension $F \in K(X)$ (resp. $F \in C_0(X)$).

Proof. This follows from (7.40, p. 99, [1]) and the following commutative diagram:

$$\begin{array}{ccc}
 X & \longrightarrow & X \cup \{\infty\} \\
 \uparrow & & \uparrow \\
 Y & \longrightarrow & Y \cup \{\infty\} \longrightarrow C \\
 \parallel & & \parallel \\
 Y & \longrightarrow & C \quad (f(\infty) = 0).
 \end{array}$$

PROPOSITION 1.4. Let S be a locally compact semigroup and $e^2 = e \in S$. Then $\delta_e * M(S) * \delta_e = M(eSe)$ is a Banach subalgebra of $M(S)$. In fact, if $i: eSe \rightarrow S$ is the inclusion map, then $M(i): M(eSe) \rightarrow M(S)$ is an embedding. (Note that, unless mentioned otherwise, our statements are to apply to each of the topologies mentioned before.)

Proof. We first observe from Lemma 1.1 that $\delta_e * M(S) * \delta_e \subseteq M(eSe)$ and that δ_e is the identity for $M(eSe)$, whence $M(eSe) = \delta_e * M(eSe) * \delta_e \subseteq \delta_e * M(S) * \delta_e$ and thus $\delta_e * M(S) * \delta_e = M(eSe)$. Since $\mu \mapsto \delta_e * \mu * \delta_e$ is a Banach space linear retraction, $M(eSe)$ is a linear closed norm retract of $M(S)$. As to the others, we will show the weak embedding only. Let $M(i)(\mu_\alpha) \xrightarrow{w} M(i)(\mu)$ in $M(S)$ and $f \in C^b(eSe)$; then f has an extension $F \in C^b(S)$ given by $F(s) = f(ese)$ and thus $\mu_\alpha(f) = [M(i)(\mu_\alpha)](F) \rightarrow [M(i)(\mu)](F) = \mu(f)$. Hence $M(i)$ is an embedding.

For the purpose of this paper it is therefore no loss of generality to assume that S is a monoid with the identity e .

PROPOSITION 1.5. Suppose that S acts on the left on a locally compact space X . If $\mu \in M(X)$ and $f \in C^b(X)$, then $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx)\mu(dx)$.

Proof. Let $\epsilon > 0$ be given. By the regularity of $|\mu|$, there exists a compact subset $K \subseteq X$ so that $|\mu|(X \setminus K) < \epsilon$. For this K and a given $s \in S$, let

$$\varphi(t) = \sup\{|f(tx) - f(sx)| : x \in K\}.$$

Then $\varphi(t) \rightarrow 0$ as $t \rightarrow s$; otherwise, there exist nets $t_\alpha \rightarrow s$, and $x_\alpha \rightarrow x_0$ in K so that $|f(t_\alpha x_\alpha) - f(sx_\alpha)| > \epsilon$ which contradicts to the continuity of f at sx_0 . Hence

$$\begin{aligned} |f_\mu(t) - f_\mu(s)| &\leq \int_K \varphi(t) |\mu|(dx) + \int_{X \setminus K} 2\|f\| |\mu|(dx) \\ &\leq \varphi(t) |\mu|(K) + 2\|f\| \epsilon \leq 3\|f\| \epsilon \end{aligned}$$

whenever t is close enough to s . Hence $f_\mu \in C^b(S)$.

2. H -invariant measures. Let H be any compact group acting on the left on a locally compact space X . A $\mu \in M(X)$ is called H -invariant if $\int f(hx)\mu(dx) = \int f(x)\mu(dx)$ for all $f \in C^b(X)$, $h \in H$. For convenience, we will denote by $M_H(X)$ the Banach subspace of all H -invariant measures in $M(X)$. We now assume that S acts on the left on X and H is a compact subgroup of units in S . Suppose now that $f \in C^b(X)$ and $\mu \in M_H(X)$. By Proposition 1.4, $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx)\mu(dx)$. If we set $(fs)(x) = f(sx)$, then we note that $f_\mu(sh) = \int (fs)(hx)\mu(dx) = \mu(fs) = \int f(sx)\mu(dx) = f_\mu(s)$ for all $h \in H$. Hence f_μ is constant on left cosets sH in S . If $S/H = \{sH : s \in S\}$ and $p : S \rightarrow S/H$ is given by $p(s) = sH$, then $F \mapsto F \circ p : C^b(S/H) \rightarrow C^b(S)$ is an isometry onto $C^b_H(S)$ of all functions which are constant on orbits sH . Hence there is a unique function $\tilde{f}_\mu \in C^b(S/H)$ such that $\tilde{f}_\mu \circ p = f_\mu$. If now $\mu \in M_H(S/H)$ and $\nu \in M_H(X)$, then we define

$$\mu * \nu(f) = \mu(\tilde{f}_\nu)$$

on $C^b(X)$, which we will write

$$\mu * \nu(f) = \int f(sx)\mu(ds)\nu(dx), \quad s = p(s).$$

As $(fh)_\nu = (\tilde{f}_\nu)h$, we have $\mu * \nu(fh) = \mu((\tilde{f}_\nu)_h) = \mu(\tilde{f}_\nu h) = \mu(\tilde{f}_\nu) = \mu * \nu(f)$, whence $\mu * \nu \in M_H(X)$. In particular, if $\mu, \nu \in M_H(S/H)$, then $\mu * \nu \in M_H(S/H)$.

LEMMA 2.1. $M(p): M(S) \rightarrow M(S/H)$ is a norm-decreasing continuous linear morphism mapping $w_H * M(S)$ into $M_H(S/H)$ where w_H is the normalized Haar measure on H .

Proof. We observe first that $w_H * M(S) \subseteq M_H(S)$ by invariance of w_H , and that $M(p)$ maps $M_H(S)$ into $M_H(S/H)$. And since $M(p)$ is continuous in various topologies, then so is any restriction and corestriction of $M(p)$.

LEMMA 2.2. $M(p)$ induces norm-preserving bijections $M(S) * w_H \rightarrow M(S/H)$ and $w_H * M(S) * w_H \rightarrow M_H(S/H)$.

Proof. It suffices to show bijections only (cf. 2.45, p. 20, [6]).

(1) Surjectivity: Let $f \in C^b(S)$ and set $f_H = \int f(sh)w_H(dh)$. Then $f_H \in C^b(S/H)$ and hence defines a unique $\widetilde{f}_H \in C^b(S/H)$ such that $\widetilde{f}_H \circ p = f_H$. If now $\nu' \in M(S/H)$, then $f \mapsto \nu'(f_H)$ is a bounded linear functional. Hence there is a $\nu \in M(S)$ with $\nu(f) = \nu'(\widetilde{f}_H)$. Now $\nu * w_H(f) = \nu(f_H) = \nu'(\widetilde{(f_H)_H}) = \nu'(\widetilde{f}_H) = \nu(f)$. Thus $\nu * w_H = \nu$, i.e. $\nu \in M(S) * w_H$. Now suppose that even $\nu' \in M_H(S/H)$. Then

$$\begin{aligned} w_H * \nu(f) &= \int f(hx)w_H(dh)\nu(dx) = \int \nu(fh)w_H(dh) \\ &= \int \nu'(\widetilde{(fh)_H})w_H(dh) = \int \nu'(\widetilde{f_H}h)w_H(dh) = \nu'(\widetilde{f}_H) \end{aligned}$$

since $\nu' \in M_H(S/H)$. The last term equals $\nu(f_H) = \nu(f)$. Thus $w_H * \nu = \nu$, i.e. $\nu \in w_H * M(S) * w_H$. Now, for $f \in C^b(S/H)$, $[M(p)(\nu)](f) = \nu(f \circ p) = \nu'(\widetilde{(f \circ p)_H})$. But $(f \circ p)_H \circ p = (f \circ p)_H = f \circ p$, whence $f = \widetilde{(f \circ p)_H}$; thus $\nu'(\widetilde{(f \circ p)_H}) = \nu'(f)$. This shows $M(p)(\nu) = \nu'$ in both cases, i.e. $M(S/H)$ is in the image of $M(S) * w_H$ and $M_H(S/H)$ is in the image of $w_H * M(S) * w_H$ under $M(p)$. (2) Injectivity: For $\mu, \nu \in M(S) * w_H$, we note that $M(p)(\mu) = M(p)(\nu)$ implies $\mu(f) = [M(p)(\mu)](\widetilde{f}_H) = [M(p)(\nu)](\widetilde{f}_H) = \nu(f)$ for $f \in C^b(S)$, hence $\mu = \nu$.

LEMMA 2.3. $M(p): w_H * M(S) * w_H \rightarrow M_H(S/H)$ is an algebra morphism.

Proof. First of all, we observe the following facts: (1) For $\mu \in w_H * M(S) * w_H$ and $f \in C^b(S)$, $\mu(f) = [M(p)(\mu)](f_H)$. (2) For $\nu \in w_H * M(S) * w_H$ and $f \in C^b(S/H)$, $f_\nu \in C^b_H(S)$ is well defined by

$$\begin{aligned} f_{\dot{\nu}}(x) &= \int f(xy)[M(p)(\nu)](dy) = \int f(xy)\dot{\nu}(dy) \\ &= \int f \circ p(xy)\nu(dy), \text{ with } \dot{\nu} = M(p)(\nu). \end{aligned}$$

Then, if $\mu, \nu \in w_H * M(S) * w_H$ and $f \in C^b(S/H)$, we have

$$\begin{aligned} [M(p)(\mu * \nu)](f) &= \mu * \nu(f \circ p) = \int f \circ p(xy)\mu(dx)\nu(dy) \\ &= \int f(xy)\mu(dx)[M(p)(\nu)](dy) \\ &= \mu(f_{\dot{\nu}}) = [M(p)(\mu)](\widetilde{(f_{\dot{\nu}})}_H) \\ &= [M(p)(\mu)](\widetilde{f}_{\nu}) = [M(p)(\mu) * M(p)(\nu)](f). \end{aligned}$$

PROPOSITION 2.4. $M(p): w_H * M(S) * w_H \rightarrow M_H(S/H)$ is a norm-preserving algebra isomorphism.

Proof. It remains to show that $M(p)|_{w_H * M(S) * w_H}$ is open which follows from the facts that $\mu(f) = [M(p)(\mu)](f_H)$ for all $\mu \in w_H * M(S) * w_H$, and that $f \in K(S)$ (resp. $f \in C_0(S)$) implies $f_H \in K(S)$ (resp. $f_H \in C_0(S)$) and thus $\widetilde{f}_H \in K(S/H)$ (resp. $\widetilde{f}_H \in C_0(S/H)$).

COROLLARY 2.5. Let H be normal in S (2.1, p. 17, [3]). Then $M(p): M(S) \rightarrow M(S/H)$ is a continuous algebra morphism mapping $w_H * M(S) * w_H$ isomorphically onto $M_H(S/H)$.

COROLLARY 2.6. Let $P_H(S/H)$ denote the totality of H -invariant probability measures in $P(S/H)$. Then $M(p): w_H * P(S) * w_H \rightarrow P_H(S/H)$ is an isomorphism.

In the remainder, we assume that w is an idempotent probability measure with compact support on S ; then $w = \mu_E * w_H * \mu_F$ [4].

LEMMA 2.7. The maps $w * M(S) * w \xrightleftharpoons[\beta]{\alpha} w_H * M(S) * w_H$ defined via $\alpha(\mu) = w_H * \mu * w_H$ and $\beta(\nu) = w * \nu * w$ are mutually inverse norm-preserving continuous algebra morphisms so that $\alpha(w) = w_H$ and $\beta(w_H) = w$.

Proof. The proof in (3.1–2, [8]) yields this.

PROPOSITION 2.8.

$$w * M(S) * w \cong w_H * M(S) * w_H \cong M_H(S/H)$$

algebraically and topologically.

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