

Pacific Journal of Mathematics

**ON THE CONVOLUTION ALGEBRAS OF H -INVARIANT
MEASURES**

JOHN YUAN

ON THE CONVOLUTION ALGEBRAS OF H -INVARIANT MEASURES

JOHN YUAN

The totality $M(eSe/H)$ of bounded regular Borel measures on the orbit space eSe/H , where S is a locally compact semigroup and H is a compact subgroup with the identity e , forms a Banach space; however, its closed subspace $M_H(ESe/H)$ of H -invariant measures forms even a Banach algebra under a suitable convolution. Furthermore, if w is an idempotent probability measure with compact support on S , then $w * M(S) * w \cong w_H * M(S) * w_H \cong M_H(eSe/H)$ algebraically and in various topologies, where w_H is the normalized Haar measure on some compact subgroup H .

1. Introduction. We denote the Banach space of bounded regular Borel measures and the totality of probability measures on a locally compact (Hausdorff) space X by $M(X)$ and $P(X)$, respectively. Beside the norm topology, $M(X)$ may be equipped with the weak, weak* and vague topologies, which are the topologies of pointwise convergence on $C^b(X)$, $C_0(X)$ and $K(X)$, respectively, where $C^b(X)$ denotes the totality of bounded continuous functions on X , $C_0(X)$ and $K(X)$ the subspaces of functions vanishing at ∞ and functions with compact supports, respectively. In $P(X)$, the weak, weak* and vague topologies coincide (p. 59, [2]; [7]). Let S be a locally compact semigroup, then $M(S)$ is a Banach algebra and $P(S)$ a topological (Hausdorff) semigroup under the convolution $*$. We refer to [7] for the continuity of $*$ in the weak, weak* and vague topologies.

LEMMA 1.1. *Let S be a locally compact semigroup. Then $\text{supp}(\mu * \nu) \subseteq (\text{supp}(\mu)\text{supp}(\nu))^-$ for $\mu, \nu \in M(S)$, and equality holds for $\mu, \nu \geq 0$, where $\text{supp}(\mu)$ denotes the support of μ .*

Proof. (Cf. 1.1, p. 686, [5]).

LEMMA 1.2. *Let $\alpha: X \rightarrow Y$ be a continuous map (resp. morphism) between locally compact spaces (resp. semigroups). Then $M(\alpha): M(X) \rightarrow M(Y)$ given by*

$$[M(\alpha)(\mu)][f] = \mu(f \circ \alpha), \quad f \in C^b(Y)$$

is a norm-decreasing linear morphism (resp. algebra morphism) continuous in the weak topology. Moreover, if α is proper, then $M(\alpha)$ is also continuous in both weak* and vague topologies.

Proof. Straightforward.

LEMMA 1.3. Let Y be a closed subspace of a locally compact space X . Then every $f \in K(Y)$ (resp. $f \in C_0(Y)$) has an extension $F \in K(X)$ (resp. $F \in C_0(X)$).

Proof. This follows from (7.40, p. 99, [1]) and the following commutative diagram:

$$\begin{array}{ccc}
 X & \longrightarrow & X \cup \{\infty\} \\
 \uparrow & & \uparrow \\
 Y & \longrightarrow & Y \cup \{\infty\} \longrightarrow C \\
 \parallel & & \parallel \\
 Y & \longrightarrow & C \quad (f(\infty) = 0).
 \end{array}$$

PROPOSITION 1.4. Let S be a locally compact semigroup and $e^2 = e \in S$. Then $\delta_e * M(S) * \delta_e = M(eSe)$ is a Banach subalgebra of $M(S)$. In fact, if $i: eSe \rightarrow S$ is the inclusion map, then $M(i): M(eSe) \rightarrow M(S)$ is an embedding. (Note that, unless mentioned otherwise, our statements are to apply to each of the topologies mentioned before.)

Proof. We first observe from Lemma 1.1 that $\delta_e * M(S) * \delta_e \subseteq M(eSe)$ and that δ_e is the identity for $M(eSe)$, whence $M(eSe) = \delta_e * M(eSe) * \delta_e \subseteq \delta_e * M(S) * \delta_e$ and thus $\delta_e * M(S) * \delta_e = M(eSe)$. Since $\mu \mapsto \delta_e * \mu * \delta_e$ is a Banach space linear retraction, $M(eSe)$ is a linear closed norm retract of $M(S)$. As to the others, we will show the weak embedding only. Let $M(i)(\mu_\alpha) \xrightarrow{w} M(i)(\mu)$ in $M(S)$ and $f \in C^b(eSe)$; then f has an extension $F \in C^b(S)$ given by $F(s) = f(ese)$ and thus $\mu_\alpha(f) = [M(i)(\mu_\alpha)](F) \rightarrow [M(i)(\mu)](F) = \mu(f)$. Hence $M(i)$ is an embedding.

For the purpose of this paper it is therefore no loss of generality to assume that S is a monoid with the identity e .

PROPOSITION 1.5. Suppose that S acts on the left on a locally compact space X . If $\mu \in M(X)$ and $f \in C^b(X)$, then $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx)\mu(dx)$.

Proof. Let $\epsilon > 0$ be given. By the regularity of $|\mu|$, there exists a compact subset $K \subseteq X$ so that $|\mu|(X \setminus K) < \epsilon$. For this K and a given $s \in S$, let

$$\varphi(t) = \sup\{|f(tx) - f(sx)| : x \in K\}.$$

Then $\varphi(t) \rightarrow 0$ as $t \rightarrow s$; otherwise, there exist nets $t_\alpha \rightarrow s$, and $x_\alpha \rightarrow x_0$ in K so that $|f(t_\alpha x_\alpha) - f(sx_\alpha)| > \epsilon$ which contradicts to the continuity of f at sx_0 . Hence

$$\begin{aligned} |f_\mu(t) - f_\mu(s)| &\leq \int_K \varphi(t) |\mu|(dx) + \int_{X \setminus K} 2\|f\| |\mu|(dx) \\ &\leq \varphi(t) |\mu|(K) + 2\|f\| \epsilon \leq 3\|f\| \epsilon \end{aligned}$$

whenever t is close enough to s . Hence $f_\mu \in C^b(S)$.

2. H -invariant measures. Let H be any compact group acting on the left on a locally compact space X . A $\mu \in M(X)$ is called H -invariant if $\int f(hx)\mu(dx) = \int f(x)\mu(dx)$ for all $f \in C^b(X)$, $h \in H$.

For convenience, we will denote by $M_H(X)$ the Banach subspace of all H -invariant measures in $M(X)$. We now assume that S acts on the left on X and H is a compact subgroup of units in S . Suppose now that $f \in C^b(X)$ and $\mu \in M_H(X)$. By Proposition 1.4, $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx)\mu(dx)$. If we set $(fs)(x) = f(sx)$, then we note that $f_\mu(sh) = \int (fs)(hx)\mu(dx) = \mu(fs) = \int f(sx)\mu(dx) = f_\mu(s)$ for all $h \in H$. Hence f_μ is constant on left cosets sH in S . If $S/H = \{sH : s \in S\}$ and $p : S \rightarrow S/H$ is given by $p(s) = sH$, then $F \mapsto F \circ p : C^b(S/H) \rightarrow C^b(S)$ is an isometry onto $C^b_H(S)$ of all functions which are constant on orbits sH . Hence there is a unique function $\tilde{f}_\mu \in C^b(S/H)$ such that $\tilde{f}_\mu \circ p = f_\mu$. If now $\mu \in M_H(S/H)$ and $\nu \in M_H(X)$, then we define

$$\mu * \nu(f) = \mu(\tilde{f}_\nu)$$

on $C^b(X)$, which we will write

$$\mu * \nu(f) = \int f(sx)\mu(ds)\nu(dx), \quad \dot{s} = p(s).$$

As $(\widetilde{fh})_\nu = (\tilde{f}_\nu)h$, we have $\mu * \nu(fh) = \mu((\widetilde{fh})_\nu) = \mu(\tilde{f}_\nu h) = \mu(\tilde{f}_\nu) = \mu * \nu(f)$, whence $\mu * \nu \in M_H(X)$. In particular, if $\mu, \nu \in M_H(S/H)$, then $\mu * \nu \in M_H(S/H)$.

LEMMA 2.1. $M(p): M(S) \rightarrow M(S/H)$ is a norm-decreasing continuous linear morphism mapping $w_H * M(S)$ into $M_H(S/H)$ where w_H is the normalized Haar measure on H .

Proof. We observe first that $w_H * M(S) \subseteq M_H(S)$ by invariance of w_H , and that $M(p)$ maps $M_H(S)$ into $M_H(S/H)$. And since $M(p)$ is continuous in various topologies, then so is any restriction and corestriction of $M(p)$.

LEMMA 2.2. $M(p)$ induces norm-preserving bijections $M(S) * w_H \rightarrow M(S/H)$ and $w_H * M(S) * w_H \rightarrow M_H(S/H)$.

Proof. It suffices to show bijections only (cf. 2.45, p. 20, [6]).

(1) Surjectivity: Let $f \in C^b(S)$ and set $f_H = \int f(sh)w_H(dh)$. Then $f_H \in C^b_H(S)$ and hence defines a unique $\widetilde{f}_H \in C^b(S/H)$ such that $\widetilde{f}_H \circ p = f_H$. If now $\nu' \in M(S/H)$, then $f \mapsto \nu'(f_H)$ is a bounded linear functional. Hence there is a $\nu \in M(S)$ with $\nu(f) = \nu'(\widetilde{f}_H)$. Now $\nu * w_H(f) = \nu(f_H) = \nu'(\widetilde{f}_H) = \nu'(\widetilde{f}_H)_H = \nu'(f)$. Thus $\nu * w_H = \nu$, i.e. $\nu \in M(S) * w_H$. Now suppose that even $\nu' \in M_H(S/H)$. Then

$$\begin{aligned} w_H * \nu(f) &= \int f(hx)w_H(dh)\nu(dx) = \int \nu(fh)w_H(dh) \\ &= \int \nu'(\widetilde{f}_H)_H w_H(dh) = \int \nu'(\widetilde{f}_H h)w_H(dh) = \nu'(\widetilde{f}_H) \end{aligned}$$

since $\nu' \in M_H(S/H)$. The last term equals $\nu(f_H) = \nu(f)$. Thus $w_H * \nu = \nu$, i.e. $\nu \in w_H * M(S) * w_H$. Now, for $f \in C^b(S/H)$, $[M(p)(\nu)](f) = \nu(f \circ p) = \nu'(\widetilde{f}_H)$. But $(f \circ p)_H \circ p = (f \circ p)_H = f \circ p$, whence $f = (f \circ p)_H$; thus $\nu'(\widetilde{f}_H) = \nu'(f)$. This shows $M(p)(\nu) = \nu'$ in both cases, i.e. $M(S/H)$ is in the image of $M(S) * w_H$ and $M_H(S/H)$ is in the image of $w_H * M(S) * w_H$ under $M(p)$. (2) Injectivity: For $\mu, \nu \in M(S) * w_H$, we note that $M(p)(\mu) = M(p)(\nu)$ implies $\mu(f) = [M(p)(\mu)](f_H) = [M(p)(\nu)](f_H) = \nu(f)$ for $f \in C^b(S)$, hence $\mu = \nu$.

LEMMA 2.3. $M(p): w_H * M(S) * w_H \rightarrow M_H(S/H)$ is an algebra morphism.

Proof. First of all, we observe the following facts: (1) For $\mu \in w_H * M(S) * w_H$ and $f \in C^b(S)$, $\mu(f) = [M(p)(\mu)](f_H)$. (2) For $\nu \in w_H * M(S) * w_H$ and $f \in C^b(S/H)$, $f_\nu \in C^b_H(S)$ is well defined by

$$\begin{aligned} f_{\dot{\nu}}(x) &= \int f(xy)[M(p)(\nu)](dy) = \int f(xy)\dot{\nu}(dy) \\ &= \int f \circ p(xy)\nu(dy), \text{ with } \dot{\nu} = M(p)(\nu). \end{aligned}$$

Then, if $\mu, \nu \in w_H * M(S) * w_H$ and $f \in C^b(S/H)$, we have

$$\begin{aligned} [M(p)(\mu * \nu)](f) &= \mu * \nu(f \circ p) = \int f \circ p(xy)\mu(dx)\nu(dy) \\ &= \int f(xy)\mu(dx)[M(p)(\nu)](dy) \\ &= \mu(f_{\dot{\nu}}) = [M(p)(\mu)](\widetilde{(f_{\dot{\nu}})_H}) \\ &= [M(p)(\mu)](\widetilde{f_{\dot{\nu}}}) = [M(p)(\mu) * M(p)(\nu)](f). \end{aligned}$$

PROPOSITION 2.4. $M(p): w_H * M(S) * w_H \rightarrow M_H(S/H)$ is a norm-preserving algebra isomorphism.

Proof. It remains to show that $M(p)|_{w_H * M(S) * w_H}$ is open which follows from the facts that $\mu(f) = [M(p)(\mu)](f_H)$ for all $\mu \in w_H * M(S) * w_H$, and that $f \in K(S)$ (resp. $f \in C_0(S)$) implies $f_H \in K(S)$ (resp. $f_H \in C_0(S)$) and thus $\widetilde{f_H} \in K(S/H)$ (resp. $\widetilde{f_H} \in C_0(S/H)$).

COROLLARY 2.5. Let H be normal in S (2.1, p. 17, [3]). Then $M(p): M(S) \rightarrow M(S/H)$ is a continuous algebra morphism mapping $w_H * M(S) * w_H$ isomorphically onto $M_H(S/H)$.

COROLLARY 2.6. Let $P_H(S/H)$ denote the totality of H -invariant probability measures in $P(S/H)$. Then $M(p): w_H * P(S) * w_H \rightarrow P_H(S/H)$ is an isomorphism.

In the remainder, we assume that w is an idempotent probability measure with compact support on S ; then $w = \mu_E * w_H * \mu_F$ [4].

LEMMA 2.7. The maps $w * M(S) * w \xrightleftharpoons[\beta]{\alpha} w_H * M(S) * w_H$ defined via $\alpha(\mu) = w_H * \mu * w_H$ and $\beta(\nu) = w * \nu * w$ are mutually inverse norm-preserving continuous algebra morphisms so that $\alpha(w) = w_H$ and $\beta(w_H) = w$.

Proof. The proof in (3.1–2, [8]) yields this.

PROPOSITION 2.8.

$$w * M(S) * w \cong w_H * M(S) * w_H \cong M_H(S/H)$$

algebraically and topologically.

ACKNOWLEDGEMENT. The author wishes to thank the referee for many helpful suggestions.

REFERENCES

1. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag (1971).
2. H. Heyer, *Infinitely divisible probability measures on compact groups*, Lecture Notes in Mathematics 247, Springer-Verlag (1972), 55–247.
3. K. H. Hofmann and P. S. Mostert, *Elements of Compact Semigroups*, Charles E. Merrill Books, Columbus, Ohio (1966).
4. J. S. Pym, *Idempotent measures on semigroups*, Pacific J. Math., **12** (1962), 685–698.
5. ———, *Idempotent probability measures on compact semi-topological semigroups*, Proc. Amer. Math. Soc., **21** (1969), 499–501.
6. J. L. Taylor, *Measure Algebras*, Regional Conference Series in Mathematics, Number 16 (1972).
7. J. Yuan, *On the continuity of convolution*, Semigroup Forum **10** (1975), 367–372.
8. ———, *On the groups of units in semigroups of probability measures*, Pacific J. Math., (to appear).

Received May 12, 1975 and in revised form October 7, 1975.

NATIONAL TSING HUA UNIVERSITY, TAIWAN

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1976 Pacific Journal of Mathematics
All Rights Reserved

Allan Russell Adler and Catarina Isabel Kiefe, <i>Pseudofinite fields, pro-cyclic fields and model-completion</i>	305
Christopher Allday, <i>The stratification of compact connected Lie group actions by subtori</i>	311
Martin Bartelt, <i>Commutants of multipliers and translation operators</i>	329
Herbert Stanley Bear, Jr., <i>Ordered Gleason parts</i>	337
James Robert Boone, <i>On irreducible spaces. II</i>	351
James Robert Boone, <i>On the cardinality relationships between discrete collections and open covers</i>	359
L. S. Dube, <i>On finite Hankel transformation of generalized functions</i>	365
Michael Freedman, <i>Uniqueness theorems for taut submanifolds</i>	379
Shmuel Friedland and Raphael Loewy, <i>Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue</i>	389
Theodore William Gamelin, <i>Uniform algebras spanned by Hartogs series</i>	401
James Guyker, <i>On partial isometries with no isometric part</i>	419
Shigeru Hasegawa and Ryōtarō Satō, <i>A general ratio ergodic theorem for semigroups</i>	435
Nigel Kalton and G. V. Wood, <i>Homomorphisms of group algebras with norm less than $\sqrt{2}$</i>	439
Thomas Laffey, <i>On the structure of algebraic algebras</i>	461
Will Y. K. Lee, <i>On a correctness class of the Bessel type differential operator S_μ</i>	473
Robert D. Little, <i>Complex vector fields and divisible Chern classes</i>	483
Kenneth Loudon, <i>Maximal quotient rings of ring extensions</i>	489
Dieter Lutz, <i>Scalar spectral operators, ordered l^p-direct sums, and the counterexample of Kakutani-McCarthy</i>	497
Ralph Tyrrell Rockafellar and Roger Jean-Baptiste Robert Wets, <i>Stochastic convex programming: singular multipliers and extended duality singular multipliers and duality</i>	507
Edward Barry Saff and Richard Steven Varga, <i>Geometric overconvergence of rational functions in unbounded domains</i>	523
Joel Linn Schiff, <i>Isomorphisms between harmonic and P-harmonic Hardy spaces on Riemann surfaces</i>	551
Virinda Mohan Sehgal and S. P. Singh, <i>On a fixed point theorem of Krasnoselskii for locally convex spaces</i>	561
Lewis Shilane, <i>Filtered spaces admitting spectral sequence operations</i>	569
Michel Smith, <i>Generating large indecomposable continua</i>	587
John Yuan, <i>On the convolution algebras of H-invariant measures</i>	595