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**MULTIPLIERS ON A BANACH ALGEBRA WITH A BOUNDED  
APPROXIMATE IDENTITY**

JOHN WAYNE DAVENPORT

## MULTIPLIERS ON A BANACH ALGEBRA WITH A BOUNDED APPROXIMATE IDENTITY

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**Let  $A$  be a Banach algebra with a bounded approximate identity  $\{e_\alpha \mid \alpha \in A\}$ , and  $M(A)$  the multiplier algebra on  $A$ . In this paper, we obtain a representation for  $M(A)$  such that each multiplier operator appears as a multiplicative operator. The proof makes use of the weak-\* compactness of the net  $\{Te_\alpha \mid \alpha \in A\}$  and the algebraic properties of a multiplier.**

1. **Introduction.** In 1951, J. G. Wendel showed that the left centralizers on  $L_1(G)$ ,  $G$  a locally compact group, was equivalent to  $C_0(G)^*$ , the space of regular Borel measures on  $G$ . Thus, if  $T$  is a centralizer and  $x$  is any element in  $L_1(G)$  then  $Tx = \xi * x$  for some Borel measure  $\xi$ . It is also well known that if  $A$  is a Banach algebra with an identity element then any multiplier on  $A$  is determined by its action on the identity element. In this paper, we show that if  $A$  is a Banach algebra with a bounded approximate identity then there exist a continuous isomorphism of  $A$  such that each multiplier defined on  $A$  is given by point-wise multiplication. In the case that the approximate identity is uniformly bounded by one, the representation is norm preserving. Thus we obtain an isometric isomorphism for all multipliers on  $L_1(G)$  and for all multipliers on any  $B^*$ -algebra such that the action of a multiplier is given by point-wise multiplication by a fixed element in  $A$ .

2. The representation space for  $M(A)$ .

**DEFINITION 2.1.** Let  $A$  be a Banach algebra and  $T$  a mapping from  $A$  into  $A$ . The map  $T$  is a multiplier provided

$$x(Ty) = (Tx)y \quad (x, y \in A).$$

Every multiplier turns out to be a continuous function and the set of all multipliers on  $A$  under pointwise operations is a commutative subalgebra of  $B(A)$ , the set of all bounded linear operators on  $A$  ([5]).

**NOTATION 2.2.** In this paper, a Banach algebra with a bounded approximate identity will be denoted by  $A$  and the multiplier algebra on  $A$  will be denoted by  $M(A)$ . For any Banach algebra  $X$ , we denote the weak-\* convergence of a net in  $X^*$ , the dual space of  $X$ , indexed by  $\alpha \in A$ , by " $\lim_\alpha^{w^*} (\cdot)$ ". Unless otherwise stated, we denote the bound on the approximate identity by  $M$ .

DEFINITION 2.3. Let  $X$  be a Banach algebra. The algebra  $X$  is said to have a bounded approximate identity provided there exists a net  $\{e_\alpha \mid \alpha \in A\}$  in  $X$  and a  $M > 0$  such that

$$\begin{aligned} 2.3.1 \quad & \|e_\alpha\| < M && (\alpha \in A) \\ 2.3.2 \quad & \lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x && (x \in X). \end{aligned}$$

DEFINITION 2.4. Let  $\{e_\alpha \mid \alpha \in A\}$  denote the approximate identity on  $A$ , and  $B_* = \{f \in A^* \mid f \cdot e_\alpha \rightarrow f\}$  where  $f \cdot a(x) = f(ax)$  for each  $a, x \in A$  and  $f \in A^*$ . The set  $B_*$  is a closed subspace of  $A^*$  and  $B_* = \{f \cdot a \mid f \in A^*, a \in A\}$  ([3]). By defining

$$\begin{aligned} 2.4.1 \quad & [G, f] = G(f \cdot a) && (a \in A, f \in B_*, G \in B_*^*) \\ 2.4.2 \quad & F \cdot G(f) = F[G, f] && (f \in B_*, F, G \in B_*^*), \end{aligned}$$

the dual space,  $B_*^*$ , becomes a Banach algebra. This follows since the above definitions are the restrictions to  $B_*$  of the Arens product on  $A^{**}$  which makes  $A^{**}$  into a Banach algebra such that if  $\pi$  is the canonical embedding of  $A$  into  $A^{**}$  then  $\pi$  is an isometric isomorphism ([5]).

LEMMA 2.5. *There exists a norm reducing isomorphism of  $A$  into  $B_*^*$ .*

*Proof.* We define  $\tau: A \rightarrow B_*^*$  by  $\tau a(f) = f(a) = \pi a|_{B_*}$ .

Clearly  $\tau$  is linear and since  $B_* = \{f \cdot a \mid f \in A^*, a \in A\}$ , it follows that  $\tau$  is one-to-one. From  $|\tau a(f)| = |f(a)| < \|f\| \cdot \|a\|$ , we see that  $\|\tau a\| < \|a\|$ , for all  $a \in A$ .

LEMMA 2.6. *Let  $\{F_\alpha \mid \alpha \in A\}$  be a net in  $B_*^*$ ;  $a \in A$ ; and  $F, G \in B_*^*$ , then the following properties are satisfied:*

$$\begin{aligned} 2.6.1 \quad & \text{if } \lim_\alpha^{w_{k^*}} F_\alpha = F \text{ then } \lim_\alpha^{w_{k^*}} F_\alpha \cdot G = F \cdot G \\ 2.6.2 \quad & \text{if } \lim_\alpha^{w_{k^*}} F_\alpha = F \text{ then } \lim_\alpha^{w_{k^*}} \tau a \cdot F_\alpha = \tau a \cdot F \\ 2.6.3 \quad & \text{if } F \cdot \tau a = 0 \text{ for all } a \in A \text{ or } \tau a \cdot F = 0 \text{ for all } a \in A \text{ then } F = 0. \end{aligned}$$

*Proof.* These properties follow from a straightforward application of the definitions of the operations involved.

LEMMA 2.7. *The Banach algebra  $B_*^*$  has an identity element which we denote by  $J$ .*

*Proof.* From  $\|\tau e_\alpha\| < \|e_\alpha\| < M$ , it follows that the net  $\{\tau e_\alpha\}$  has a weak- $*$  convergent subnet. Let  $J = \lim_\alpha^{w_{k^*}} \tau e_\alpha$ . Since

$$[J, f](x) = J(f \cdot x) = \lim_\alpha \tau e_\alpha(f \cdot x) = \lim_\alpha f(xe_\alpha) = f(x),$$

for all  $x \in A$ , we have that  $[J, f] = f$  for all  $f \in B_*$ . Thus  $F \cdot J = F$ , for all  $F \in B_*^*$ . Since  $\tau a \cdot F$  is weak-\* continuous in  $F$ , it also follows that  $J \cdot F(f) = \lim_{\alpha} \tau e_{\alpha} \cdot F(f) = \lim_{\alpha} F(f \cdot e_{\alpha}) = F(f)$  for all  $f \in B_*$  and  $F \in B_*^*$ . Thus  $J \cdot F = F$  for all  $F \in B_*^*$ .

**THEOREM 2.8.** *Let  $A$  be a Banach algebra with a bounded approximate identity  $\{e_{\alpha} \mid \alpha \in A\}$ . Then there exists a map  $\mu$  from  $M(A)$  into  $B_*^*$  such that  $\mu$  is a continuous, algebraic isomorphism of  $M(A)$  into  $B_*^*$ . Furthermore*

$$\tau(Ta) = (\mu T) \cdot \tau a = \tau a \cdot (\mu T) \quad (a \in A, T \in M(A)).$$

*Proof.* Let  $T \in M(A)$ . Since  $\|Te_{\alpha}\| < \|T\| \cdot M$ , the net  $\{\tau(Te_{\alpha}) \mid \alpha \in A\}$  has a weak-\* convergent subnet in  $B_*^*$ . If  $\{\tau(Te_{\beta}) \mid \beta \in \Gamma\}$  converges to  $G$  and  $\{\tau(Te_{\alpha}) \mid \alpha \in A\}$  converges to  $F$ , each in the weak-\* topology; then, for each  $f \in B_*$ , we have that

$$\begin{aligned} F(f) &= \lim_{\alpha} \tau(Te_{\alpha})(f) = \lim_{\alpha} \tau(Te_{\alpha}) \cdot J(f) \\ &= \lim_{\alpha} \lim_{\beta} \tau Te_{\alpha} \cdot \tau e_{\beta}(f) = \lim_{\alpha} \lim_{\beta} (\tau Te_{\alpha}(e_{\beta}))(f) \\ &= \lim_{\alpha} \lim_{\beta} \tau e_{\alpha} \cdot \tau Te_{\beta}(f) = \lim_{\alpha} \tau e_{\alpha} \cdot G(f) = G(f). \end{aligned}$$

Now we define the mapping  $\mu$  from  $M(A)$  to  $B_*^*$  by

$$\mu(T) = F = \lim_{\alpha}^{wk*} \tau(Te_{\alpha}) \quad (T \in M(A)).$$

The previous remarks show that  $\mu$  is well defined. We first observe that if  $F = \mu(T)$ , then

$$\tau a \cdot F(f) = \lim_{\alpha} \tau a \cdot \tau Te_{\alpha}(f) = \lim_{\alpha} \tau Ta \cdot \tau e_{\alpha}(f) = \tau(Ta)(f).$$

Thus

$$2.8.1. \quad \tau a \cdot \mu(T) = \tau(Ta) \quad (a \in A, T \in M(A)).$$

By Lemma 2.7, the identity element of  $B_*^*$  is the weak-\* limit of a subnet of  $\{\tau e_{\alpha} \mid \alpha \in A\}$ . Let  $\{\tau e_{\beta}\}$  denote this subnet. Hence we have

$$\begin{aligned} \mu(T) \cdot \tau a(f) &= \lim_{\beta} \tau e_{\beta} \cdot \mu(T) \cdot \tau a(f) = \lim_{\beta} \tau e_{\beta} \cdot \mu(T) \cdot \tau a(f) \\ &= \lim_{\beta} \tau e_{\beta} \cdot \tau Ta(f) = \tau Ta(f). \end{aligned}$$

Therefore,

$$2.8.2. \quad \mu T \cdot \tau a = \tau Ta \quad (a \in A, T \in M(A)).$$

Let  $x, y \in A$  and  $T \in M(A)$ . Then

$$\begin{aligned}\tau x \cdot \mu(TS) \cdot \tau y &= \tau(TSx)y = \tau Sx \cdot \tau Ty = \mu S \cdot \tau x \cdot \mu T \cdot \tau y \\ &= \tau x \cdot \mu S \cdot \mu T \cdot \tau y\end{aligned}$$

and thus by Lemma 2.6, it follows that  $\mu(TS) = \mu(S) \cdot \mu(T)$ . But C. N. Kellogg [4] proved that  $M(A)$  is a closed commutative sub-algebra of  $B(A)$ , the set of all bounded linear operators on  $A$ . Thus  $\mu(TS) = \mu(ST) = \mu(T) \cdot \mu(S)$  and therefore  $\mu$  is homomorphic.

If  $\mu(T) = \mu(S)$  for some  $T, S \in M(A)$  where  $\mu(T) = \lim_{\alpha}^{wk^*} \tau T e_{\alpha}$  and  $\mu(S) = \lim_{\beta}^{wk^*} \tau S e_{\beta}$  then for each  $f \in B_*$ , and  $a \in A$ , we have

$$\begin{aligned}\tau(Ta)(f) &= \lim_{\alpha} \tau(Ta) \cdot \tau e_{\alpha}(f) = \lim_{\alpha} \tau a \cdot \tau T e_{\alpha}(f) \\ &= \tau a \cdot \mu(T)(f) = \tau a \cdot \mu(S)(f) = \tau a \cdot \lim_{\beta} \tau(S e_{\beta})(f) \\ &= \lim_{\beta} \tau a \cdot \tau(S e_{\beta})(f) = \lim_{\beta} \tau(Sa) \cdot e_{\beta}(f) = \tau(Sa)(f).\end{aligned}$$

Since  $\tau$  is one-to-one, it follows that  $Ta = Sa$  for each  $a \in A$ . Thus  $\mu$  is one-to-one.

From  $\mu(T) = \lim_{\alpha} \tau T e_{\alpha}$  and  $\|\tau T e_{\alpha}\| < \|T e_{\alpha}\| < \|T\| \cdot \|e_{\alpha}\| < \|T\| \cdot M$ , it follows that  $\mu$  is continuous.

**COROLLARY 2.9.** *If  $M = 1$ , then  $M(A)$  is isometrically  $*$ -isomorphic to a subspace of  $B_*^*$ .*

*Proof.* This follows from Theorem 2.8 and the fact that  $\|\tau a\| = \|a\|$ .

For  $A = L_1(G)$ ,  $G$  a nondiscrete locally compact abelian group, the space  $B_*$  is the space of uniformly continuous bounded functions on  $G$  and  $B_*^*$  is the space  $M(G)$  of bounded measures of the maximal ideal space of  $B_*$ . If  $G$  is compact then  $M(A) = M(G)$ . In the case that  $A$  is a  $B^*$ -algebra, we have the following result.

**COROLLARY 2.10.** *If  $A$  is a  $B^*$ -algebra then  $M(A)$  is isometrically  $*$ -isomorphic to a subspace of  $A^{**}$ . Furthermore, if  $\mu(T) = F$  for  $T \in M(A)$  and  $F \in A^{**}$ , then*

$$\pi a \cdot F = F \cdot \pi a = \pi Ta \quad (a \in A)$$

where the above operation is the Arens product on  $A^{**}$ .

*Proof.* D. C. Taylor [7] has shown that  $A^* = \{f \cdot a \mid f \in A^*, a \in A\} = \{a \cdot f \mid f \in A^*, a \in A\}$ . Thus  $B_* = A^*$  and  $B_*^* = A^{**}$ . In this case the product operation on  $B_*^* = A^{**}$  becomes the Arens product and the involution on  $A^{**}$  is given by  $F^*(f) = \overline{F(f)}$  where  $\overline{f(x^*)}$  [2]. Since a  $B^*$ -algebra possesses an approximate identity uniformly bounded by one, the result follows from Corollary 2.9.

COROLLARY 2.11. *Let  $A$  be a  $B^*$ -algebra. Then  $F \in A^{**}$  belongs to  $\mu(M(A))$  if and only if the operator  $F$  commutes with  $\pi A$  and  $F \cdot \pi a$  is continuous in the weak-\* topology on  $A^*$  for each  $a \in A$ .*

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#### REFERENCES

1. R. Arens, *Operations induced in function classes*, *Monat. fur Math.*, **55** (1951), 1-19.
2. F. F. Bonsall, and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Notes, Series II, London, 1971.
3. P. Curtis and Figa-Talamanca, *Factorization theorems for Banach algebras* in *Function Algebras*, edited by F. T. Birtel, Scott, Foresman and Co., Chicago, Ill., 1966, 169-185.
4. C. N. Kellogg, *Centralizers and  $H^*$ -algebras*, *Pacific J. Math.*, **17** (1966), 121-129.
5. R. Larson, *An Introduction to the Theory of Multipliers*, Springer, New York, 1971.
6. C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, Princeton, 1960.
7. D. C. Taylor, *The strict topology for double centralizer algebras*, *Trans. Amer. Soc.*, **150** (1970), 633-643.
8. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, *Pacific J. Math.*, **2** (1952), 251-261.

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