MULTIPLIERS ON A BANACH ALGEBRA WITH A BOUNDED APPROXIMATE IDENTITY

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Let $A$ be a Banach algebra with a bounded approximate identity $\{e_\alpha \mid \alpha \in \Lambda\}$, and $M(A)$ the multiplier algebra on $A$. In this paper, we obtain a representation for $M(A)$ such that each multiplier operator appears as a multiplicative operator. The proof makes use of the weak-* compactness of the net $\{T e_\alpha \mid \alpha \in \Lambda\}$ and the algebraic properties of a multiplier.

1. Introduction. In 1951, J. G. Wendel showed that the left centralizers on $L_1(G)$, $G$ a locally compact group, was equivalent to $C_0(G)^*$, the space of regular Borel measures on $G$. Thus, if $T$ is a centralizer and $x$ is any element in $L_1(G)$ then $T x = \xi * x$ for some Borel measure $\xi$. It is also well known that if $A$ is a Banach algebra with an identity element then any multiplier on $A$ is determined by its action on the identity element. In this paper, we show that if $A$ is a Banach algebra with a bounded approximate identity then there exist a continuous isomorphism of $A$ such that each multiplier defined on $A$ is given by point-wise multiplication. In the case that the approximate identity is uniformly bounded by one, the representation is norm preserving. Thus we obtain an isometric isomorphism for all multipliers on $L_1(G)$ and for all multipliers on any $B^*$-algebra such that the action of a multiplier is given by point-wise multiplication by a fixed element in $A$.

2. The representation space for $M(A)$.

**Definition 2.1.** Let $A$ be a Banach algebra and $T$ a mapping from $A$ into $A$. The map $T$ is a multiplier provided

$$x(Ty) = (Tx)y \quad (x, y \in A).$$

Every multiplier turns out to be a continuous function and the set of all multipliers on $A$ under pointwise operations is a commutative subalgebra of $B(A)$, the set of all bounded linear operators on $A([5])$.

**Notation 2.2.** In this paper, a Banach algebra with a bounded approximate identity will be denoted by $A$ and the multiplier algebra on $A$ will be denoted by $M(A)$. For any Banach algebra $X$, we denote the weak-* convergence of a net in $X^*$, the dual space of $X$, indexed by $\alpha \in \Lambda$, by $\lim_{\alpha \to \infty}^w (\cdot)$. Unless otherwise stated, we denote the bound on the approximate identity by $M$. 

131
DEFINITION 2.3. Let $X$ be a Banach algebra. The algebra $X$ is said to have a bounded approximate identity provided there exists a net $\{e_\alpha \mid \alpha \in \Lambda\}$ in $X$ and a $M > 0$ such that

\[ \| e_\alpha \| < M \quad (\alpha \in \Lambda) \]

\[ \lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x \quad (x \in X). \]

DEFINITION 2.4. Let $\{e_\alpha \mid \alpha \in \Lambda\}$ denote the approximate identity on $A$, and $B_\ast = \{f \in A^\ast \mid f \cdot e_\alpha \to f\}$ where $f \cdot a(x) = f(ax)$ for each $a, x \in A$ and $f \in A^\ast$. The set $B_\ast$ is a closed subspace of $A^\ast$ and $B_\ast = \{f \cdot a \mid f \in A^\ast, a \in A\}$ ([3]). By defining

2.4.1 \[ [G, f] = G(f \cdot a) \quad (a \in A, f \in B_\ast, G \in B_\ast^\ast) \]

2.4.2 \[ F \cdot G(f) = F[G, f] \quad (f \in B_\ast, F, G \in B_\ast^\ast), \]

the dual space, $B_\ast^\ast$, becomes a Banach algebra. This follows since the above definitions are the restrictions to $B_\ast$ of the Arens product on $A^{**}$ which makes $A^{**}$ into a Banach algebra such that if $\pi$ is the canonical embedding of $A$ into $A^{**}$ then $\pi$ is an isometric isomorphism ([5]).

LEMMA 2.5. There exists a norm reducing isomorphism of $A$ into $B_\ast^\ast$.

Proof. We define $\tau : A \to B_\ast^\ast$ by $\tau(a)(f) = f(a) = \pi a |_{B_\ast}$.

Clearly $\tau$ is linear and since $B_\ast = \{f \cdot a \mid f \in A^\ast, a \in A\}$, it follows that $\tau$ is one-to-one. From $|\tau(a)(f)| = |f(a)| < \| f \| \cdot \| a \|$, we see that $\| \tau(a) \| < \| a \|$, for all $a \in A$.

LEMMA 2.6. Let $\{F_a \mid a \in \Lambda\}$ be a net in $B_\ast^\ast$; $a \in A$; and $F, G \in B_\ast^\ast$, then the following properties are satisfied:

2.6.1 if $\lim_a^{w^*} F_a = F$ then $\lim_a^{w^*} F_a \cdot G = F \cdot G$

2.6.2 if $\lim_a^{w^*} F_a = F$ then $\lim_a^{w^*} \tau a \cdot F_a = \tau a \cdot F$

2.6.3 if $F \cdot \tau a = 0$ for all $a \in A$ or $\tau a \cdot F = 0$ for all $a \in A$ then $F = 0$.

Proof. These properties follow from a straightforward application of the definitions of the operations involved.

LEMMA 2.7. The Banach algebra $B_\ast^\ast$ has an identity element which we denote by $J$.

Proof. From $\| \tau e_\alpha \| < \| e_\alpha \| < M$, it follows that the net $\{\tau e_\alpha\}$ has a weak-* convergent subnet. Let $J = \lim_a^{w^*} \tau e_\alpha$. Since

\[ [J, f](x) = J(f \cdot x) = \lim_\alpha \tau e_\alpha(f \cdot x) = \lim_\alpha f(x e_\alpha) = f(x), \]
for all \( x \in A \), we have that \([J, f] = f\) for all \( f \in B_*\). Thus \( F \cdot J = F\), for all \( F \in B_*^*\). Since \( \tau a \cdot F\) is weak-* continuous in \( F\), it also follows that \( J \cdot F(f) = \lim_{a} \tau e_a \cdot F(f) = \lim_{a} F(f \cdot e_a) = F(f)\) for all \( f \in B_*\) and \( F \in B_*^*\). Thus \( J \cdot F = F\) for all \( F \in B_*^*\).

**Theorem 2.8.** Let \( A \) be a Banach algebra with a bounded approximate identity \( \{ e_\alpha \mid \alpha \in \Lambda\} \). Then there exists a map \( \mu \) from \( M(A) \) into \( B_*^* \) such that \( \mu \) is a continuous, algebraic isomorphism of \( M(A) \) into \( B_*^* \). Furthermore

\[
\tau(Ta) = (\mu T) \cdot \tau a = \tau a \cdot (\mu T) \quad (a \in A, \ T \in M(A)).
\]

**Proof.** Let \( T \in M(A) \). Since \( \|T e_\alpha\| < \|T\| \cdot M \), the net \( \{\tau(T e_\alpha) \mid \alpha \in \Lambda\} \) has a weak-* convergent subnet in \( B_*^* \). If \( \{\tau(T e_\alpha) \mid \beta \in \Gamma\} \) converges to \( G \) and \( \{\tau(T e_\alpha) \mid \alpha \in \Lambda\} \) converges to \( F \), each in the weak-* topology; then, for each \( f \in B_* \), we have that

\[
F(f) = \lim_{a} \tau(T e_\alpha)(f) = \lim_{a} \tau(T e_\alpha) \cdot J(f)
= \lim_{a} \lim_{\beta} \tau e_\alpha \cdot \tau e_\beta(f) = \lim_{a} \lim_{\beta} (\tau e_\alpha \cdot \tau e_\beta)(f)
= \lim_{a} \lim_{\beta} \tau e_\alpha \cdot \tau e_\beta(f) = \lim_{a} \tau e_\alpha \cdot G(f) = G(f).
\]

Now we define the mapping \( \mu \) from \( M(A) \) to \( B_*^* \) by

\[
\mu(T) = F = \lim_{a} \tau(T e_\alpha) \quad (T \in M(A)).
\]

The previous remarks show that \( \mu \) is well defined. We first observe that if \( F = \mu(T) \), then

\[
\tau a \cdot F(f) = \lim_{a} \tau a \cdot \tau e_\alpha(f) = \lim_{a} \tau T a \cdot \tau e_a(f) = \tau(Ta)(f).
\]

Thus

2.8.1. \( \tau a \cdot \mu(T) = \tau(Ta) \quad (a \in A, \ T \in M(A)) \).

By Lemma 2.7, the identity element of \( B_*^* \) is the weak-* limit of a subnet of \( \{\tau e_\alpha \mid \alpha \in \Lambda\} \). Let \( \{\tau e_\beta\} \) denote this subnet. Hence we have

\[
\mu(T) \cdot \tau a(f) = \lim_{\beta} \tau e_\beta \cdot \mu(T) \cdot \tau a(f) = \lim_{\beta} \tau e_\beta \cdot \mu(T) \cdot \tau a(f)
= \lim_{\beta} \tau e_\beta \cdot \tau T a(f) = \tau(Ta)(f).
\]

Therefore,

2.8.2. \( \mu T \cdot \tau a = \tau(Ta) \quad (a \in A, \ T \in M(A)) \).

Let \( x, y \in A \) and \( T \in M(A) \). Then
\[\tau x \cdot \mu(TS) \cdot \tau y = \tau(TSx)y = \tau Sx \cdot \tau Ty = \mu S \cdot \tau x \cdot \mu T \cdot \tau y = \tau x \cdot \mu S \cdot \mu T \cdot \tau y\]

and thus by Lemma 2.6, it follows that \(\mu(TS) = \mu(S) \cdot \mu(T)\). But C. N. Kellogg [4] proved that \(M(A)\) is a closed commutative subalgebra of \(B(A)\), the set of all bounded linear operators on \(A\). Thus \(\mu(TS) = \mu(ST) = \mu(T) \cdot \mu(S)\) and therefore \(\mu\) is homomorphic.

If \(\mu(T) = \mu(S)\) for some \(T, S \in M(A)\) where \(\mu(T) = \lim_{\alpha} \tau Te_{\alpha}\) and \(\mu(S) = \lim_{\beta} \tau Se_{\beta}\) then for each \(f \in B_*\), and \(a \in A\), we have

\[\tau(Ta)(f) = \lim_{\alpha} \tau(Ta) \cdot \tau e_{\alpha}(f) = \lim_{\alpha} \tau a \cdot \tau Te_{\alpha}(f) = \tau a \cdot \mu(T)(f) = \tau a \cdot \mu(S)(f) = \tau a \cdot \lim_{\beta} \tau(Se_{\beta})(f) = \lim_{\beta} \tau a \cdot \tau(Se_{\beta})(f) = \lim_{\beta} \tau(Sa) \cdot e_{\beta}(f) = \tau(Sa)(f)\]

Since \(\tau\) is one-to-one, it follows that \(Ta = Sa\) for each \(a \in A\). Thus \(\mu\) is one-to-one.

From \(\mu(T) = \lim_{\alpha} \tau Te_{\alpha}\) and \(\|\tau Te_{\alpha}\| < \|Te_{\alpha}\| < \|T\| \cdot \|e_{\alpha}\| < \|T\| \cdot M\), it follows that \(\mu\) is continuous.

**Corollary 2.9.** If \(M = 1\), then \(M(A)\) is isometrically \(*\)-isomorphic to a subspace of \(B_*\).

**Proof.** This follows from Theorem 2.8 and the fact that \(\|\tau a\| = \|a\|\).

For \(A = L_1(G)\), \(G\) a nondiscrete locally compact abelian group, the space \(B_*\) is the space of uniformly continuous bounded functions on \(G\) and \(B_*\) is the space \(M(G)\) of bounded measures of the maximal ideal space of \(B_*\). If \(G\) is compact then \(M(A) = M(G)\). In the case that \(A\) is a \(B^*\)-algebra, we have the following result.

**Corollary 2.10.** If \(A\) is a \(B^*\)-algebra then \(M(A)\) is isometrically \(*\)-isomorphic to a subspace of \(A^{**}\). Furthermore, if \(\mu(T) = F\) for \(T \in M(A)\) and \(F \in A^{**}\), then

\[\pi a \cdot F = F \cdot \pi a = \pi Ta \quad (a \in A)\]

where the above operation is the Arens product on \(A^{**}\).

**Proof.** D. C. Taylor [7] has shown that \(A^* = \{f \cdot a \mid f \in A^*, a \in A\} = \{a \cdot f \mid f \in A^*, a \in A\}\). Thus \(B_* = A^*\) and \(B_*^{**} = A^{**}\). In this case the product operation on \(B_*\) becomes the Arens product and the involution on \(A^{**}\) is given by \(F^*(f) = \overline{F(f)}\) where \(\overline{f(x^*)}\) [2]. Since a \(B^*\)-algebra possesses an approximate identity uniformly bounded by one, the result follows from Corollary 2.9.
COROLLARY 2.11. Let $A$ be a $B^*$-algebra. Then $F \in A^{**}$ belongs to $\mu(M(A))$ if and only if the operator $F$ commutes with $\pi A$ and $F \cdot \pi a$ is continuous in the weak-* topology on $A^*$ for each $a \in A$.

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REFERENCES


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<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ralph Artino</td>
<td>Gevrey classes and hypoelliptic boundary value problems</td>
<td>1</td>
</tr>
<tr>
<td>B. Aupetit</td>
<td>Caractérisation spectrale des algèbres de Banach commutatives</td>
<td>23</td>
</tr>
<tr>
<td>Leon Bernstein</td>
<td>Fundamental units and cycles in the period of real quadratic number fields. I</td>
<td>37</td>
</tr>
<tr>
<td>Leon Bernstein</td>
<td>Fundamental units and cycles in the period of real quadratic number fields. II</td>
<td>63</td>
</tr>
<tr>
<td>Robert F. Brown</td>
<td>Fixed points of automorphisms of compact Lie groups</td>
<td>79</td>
</tr>
<tr>
<td>Thomas Ashland Chapman</td>
<td>Concordances of noncompact Hilbert cube manifolds</td>
<td>89</td>
</tr>
<tr>
<td>William C. Connett, V and Alan Schwartz</td>
<td>Weak type multipliers for Hankel transforms</td>
<td>125</td>
</tr>
<tr>
<td>John Wayne Davenport</td>
<td>Multipliers on a Banach algebra with a bounded approximate identity</td>
<td>131</td>
</tr>
<tr>
<td>Gustave Adam Efroymson</td>
<td>Substitution in Nash functions</td>
<td>137</td>
</tr>
<tr>
<td>John Sollion Hsia</td>
<td>Representations by spinor genera</td>
<td>147</td>
</tr>
<tr>
<td>William George Kitto and Daniel Eliot Wulbert</td>
<td>Korovkin approximations in ( L_p )-spaces</td>
<td>153</td>
</tr>
<tr>
<td>Eric P. Kronstadt</td>
<td>Interpolating sequences for functions satisfying a Lipschitz condition</td>
<td>169</td>
</tr>
<tr>
<td>Gary Douglas Jones and Samuel Murray Rankin, III</td>
<td>Oscillation properties of certain self-adjoint differential equations of the fourth order</td>
<td>179</td>
</tr>
<tr>
<td>Takaši Kusano and Hiroshi Onose</td>
<td>Nonoscillation theorems for differential equations with deviating argument</td>
<td>185</td>
</tr>
<tr>
<td>David C. Lantz</td>
<td>Preservation of local properties and chain conditions in commutative group rings</td>
<td>193</td>
</tr>
<tr>
<td>Charles W. Neville</td>
<td>Banach spaces with a restricted Hahn-Banach extension property</td>
<td>201</td>
</tr>
<tr>
<td>Norman Oler</td>
<td>Spaces of discrete subsets of a locally compact group</td>
<td>213</td>
</tr>
<tr>
<td>Robert Olin</td>
<td>Functional relationships between a subnormal operator and its minimal normal extension</td>
<td>221</td>
</tr>
<tr>
<td>Thomas Thornton Read</td>
<td>Bounds and quantitative comparison theorems for nonoscillatory second order differential equations</td>
<td>231</td>
</tr>
<tr>
<td>Robert Horace Redfield</td>
<td>Archimedean and basic elements in completely distributive lattice-ordered groups</td>
<td>247</td>
</tr>
<tr>
<td>Jeffery William Sanders</td>
<td>Weighted Sidon sets</td>
<td>255</td>
</tr>
<tr>
<td>Aaron R. Todd</td>
<td>Continuous linear images of pseudo-complete linear topological spaces</td>
<td>281</td>
</tr>
<tr>
<td>J. Jerry Uhl, Jr.</td>
<td>Norm attaining operators on ( L^1[0, 1] ) and the Radon-Nikodým property</td>
<td>293</td>
</tr>
<tr>
<td>William Jennings Wickless</td>
<td>Abelian groups in which every endomorphism is a left multiplication</td>
<td>301</td>
</tr>
</tbody>
</table>