SUBSTITUTION IN NASH FUNCTIONS

GUSTAVE ADAM EFROYMSON
SUBSTITUTION IN NASH FUNCTIONS

GUSTAVE EFROYMSON

Let $D$ be a domain in $\mathbb{R}^n$. In this paper $D$ is assumed to be defined by a finite number of strict polynomial inequalities. A Nash function on $D$ is a real valued analytic function $f(x)$ such that there exists a polynomial $p(z, x_1, \ldots, x_n)$ in $\mathbb{R}[z, x_1, \ldots, x_n]$ such that $p(f(x), x) = 0$ for all $x$ in $D$. Let $\mathcal{A}_D$ be the ring of such functions on $D$. For any real closed field $\mathbb{L}$ containing $\mathbb{R}$, use the Tarski-Seidenberg theorem to extend $f$ to a function from a domain $D_\mathbb{L}$ (defined by the same inequalities as $D$), $D_\mathbb{L} \subseteq \mathbb{L}^n$, to $\mathbb{L}$. Now let $\varphi: \mathcal{A}_D \rightarrow \mathbb{L}$ be a homomorphism. Since $\mathbb{R}[x_1, \ldots, x_n] \subset \mathcal{A}_D$, $\varphi x = (\varphi x_1, \ldots, \varphi x_n)$ is a well defined point in $L^{(\mathbb{R})}$ and is in $D_\mathbb{L}$. So $f(\varphi x)$ is defined for any $f$ in $\mathcal{A}_D$. In this paper it is shown that $f(\varphi x) = \varphi f$. From this result one can deduce Mostowski's version of the Hilbert Nullstellensatz for $\mathcal{A}_D$.

As for the Nullstellensatz, since D. Dubois [2], and J. J. Risler [8], independently proved the real Nullstellensatz for polynomial rings, there have been various successful attempts to extend the result to other types of rings, for example, [4], [9]. In [5], a partial result was obtained for Nash rings and then, in [7], T. Mostowski proved the Nullstellensatz for Nash rings. There is still a question as to whether the result holds for Nash rings on more general domains than those considered here.

1. Mostowski's theorem. We first recall some definitions.

**Definition 1.** A set $C$ contained in $\mathbb{R}^n$ is said to be semi-algebraic if it is defined by Boolean operations (finite union, finite intersection, complement) on sets of the form \{ $a \in \mathbb{R}^n$ | $p(a) > 0$, for $p(x)$ in $\mathbb{R}[x_1, \ldots, x_n]$\}. That is, $C$ is defined by a finite number of polynomial inequalities.

**Definition 2.** Let $D$ be a set defined by a finite intersection of sets of the form \{ $a \in \mathbb{R}^n$ | $p(a) > 0$, for $p(x)$ in $\mathbb{R}[x_1, \ldots, x_n]$\}. Then $\mathcal{A}_D = \{ f: D \rightarrow \mathbb{R} | f$ is analytic on $D$ and there exists a polynomial $p(z, x)$ in $\mathbb{R}[z, x_1, \ldots, x_n]$ such that for all $x$ in $D$, $p(f(x), x) = 0$\}. This ring is called the ring of Nash functions on $D$.

**Definition 3.** We wish to define certain subrings of $\mathcal{A}_D = \mathbb{A}$. Namely, let $B_0 = \mathbb{R}[x_1, \ldots, x_n] \cap \mathcal{A}_D$. Let $B_i = \vee B_i(\sqrt{f})$ for $f$ in $B_i$ and $f > 0$ on $D$. Let $B_2 = \vee B_2(\sqrt{f})$ for $f$ in $B_1$ and $f > 0$ on $D$. 

137
MOSTOWSKI'S THEOREM. Let \( D \) be as above and let \( C_1 \) and \( C_2 \) be two disjoint closed semi-algebraic sets contained in \( D \). Then there exists a function \( g \) in \( B_2 \) such that \( g(C_1) > 0 \) and \( g(C_2) < 0 \).

We will give a proof of this result in this section which is similar to Mostowski's proof, but, by proving a stronger version of Thom's lemma (the Separation Lemma below), we are able to simplify the finish of the proof of Mostowski's theorem.

SEPARATION LEMMA. We start with a finite number of polynomials \( f_1(x_1, \ldots, x_n), \ldots, f_s(x_1, \ldots, x_n) \) in \( \mathbb{R}[x_1, \ldots, x_n] \). Then the roots of the \( f_i \) divide up \( \mathbb{R}^n \) into a union of semi-algebraic sets. By Theorem 2.1 in [5], we can further divide up the sets so \( \mathbb{R}^n = \bigcup T_i \), a finite disjoint union of connected semi-algebraic sets bounded by the zeros of the \( f_i \)’s. We now claim we can find:

(a) a further finite subdivision of each \( T_i = \bigcup T_{ij} \) a disjoint union of semi-algebraic sets,

(b) a finite number of polynomials \( f_1, \ldots, f_s, f_{s+1}, \ldots, f_m \) derivable from the original polynomials, so that

1. \( \text{Sign} f_i(T_{ij}) \) is constant where \( \text{Sign} (f) = +, -, \) or 0.
2. Given \( i_1, i_2 \) so that \( T_{i_1} \cap T_{i_2} = \emptyset \), then for all \( j_1, j_2 \) there exists some \( f_k \) with either

\[
-f_k(T_{i_1j_1}) \geq 0 \quad \text{and} \quad f_k(T_{i_2j_2}) < 0,
\]
or

\[
-f_k(T_{i_1j_1}) \geq 0 \quad \text{and} \quad -f_k(T_{i_2j_2}) < 0.
\]

Proof. We consider the polynomials \( f_1, \ldots, f_s \) as polynomials in \( x_n \) with coefficients in \( \mathbb{R}[x_1, \ldots, x_{n-1}] \). We can divide up \( \mathbb{R}^{n-1} \) into a disjoint union \( \bigcup S_j \) of a finite number of connected semi-algebraic sets so that above each \( S_j \), the polynomials \( \partial^k f_i / \partial x_j^k \) have a constant number of real roots and none intersect. We let the \( T_j \)'s be the regions above the \( S_j \) either defined as a root of one of the \( \partial^k f_i / \partial x_j^k \) or a connected region bounded by adjacent roots. We first check Thom’s lemma which asserts that regions above a fixed \( S_j \) are separated by the partials of the \( f_j \). It is clearly enough to check this for one \( f_j = f \).

Note, if the regions have a simple root of \( f \) between them, then \( f \) itself will separate unless there is more than one root of \( f \) in between, in which case \( \partial f / \partial x_n \) will have a root in between and induction on degree of \( f \) will handle it. Similarly, a multiple root of \( f \) will also be a root of \( \partial f / \partial x_n \). If the regions are (1) a simple root of \( f \) and (2) a root of one of its derivatives, and they are adjacent,
then $f$ separates one way and the corresponding $\partial^k f/\partial x^k$ the other way, etc.

The $S_i$ are semi-algebraic sets in $\mathbb{R}^{n-1}$ and so they can be separated by induction on $n$ ($n = 1$ is trivial). So if $T_i$ and $T_j$ have projections $S_i$ and $S_j$ which have disjoint closures, the polynomials separating $S_i$ and $S_j$ will also separate $T_i$ and $T_j$. So the only case left to handle is where $\overline{S_i} \cap \overline{S_j} = \emptyset$ but $\overline{T_i} \cap \overline{T_j} = \emptyset$. Since we can assume $f = \sum_{i=0}^d a_i x^n_i$ and $\text{Sign } a_d$ constant on $S_i$ and on $S_j$, we may as well assume $\text{Sign } a_d \neq 0$ on $S_i$. We have to consider various cases.

Case 1. $S_j \subseteq \overline{S_i}$. Let $\pi: \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$. Let $f_Q$ denote the polynomial $\sum_{i=0}^d a_i(Q)x^n_i$ for $Q$ in $\mathbb{R}^{n-1}$. Choose $P$ in $T_j$ and we wish to find a polynomial which will be $\neq 0$ at $P$ and of opposite sign or 0 on $T_i$. We have several sub-cases.

(i) Assume $f_{\pi(P)}$ is not the zero polynomial and that $x_a(P)$ is not a root of $f_{\pi(P)}$. Then, since $T_i$ either is, or is bounded by, a root $x_a = \alpha_a(x_1, \ldots, x_{n-1})$ above $S_i$, one sees that respectively, either $f$ itself separates $P$ from $T_j$, or else there is a root of $f$ above $S_i$ between $P$ and $T_i$. In the second case, if this is a simple root of $f$ and there are not others in between, then $f$ separates. If the root is a multiple root, or if there is more than one in between, then $\partial f/\partial x_a$ will have a root between $P$ and $T_i$ and induction will work.

(ii) Assume $f_{\pi(P)}$ is not the zero polynomial and that $x_a(P)$ is a root of $f_{\pi(P)}$. Note that either $f_{\pi(P)}$ or $(\partial f/\partial x_a)_{\pi(P)}$ changes sign in an interval $(x_a(P) - \delta, x_a(P) + \delta)$ about $x_a(P)$. Thus, for $Q$ near $\pi(P)$ ($Q$ in $S_i$), $f_Q$ or $(\partial f/\partial x_a)_{Q}$ will also change sign on this interval. In the first instance, $f$ will have a root $\alpha_a(x_1, \ldots, x_{n-1})$ above $S_i$, which will have $P$ in its closure. There will then be a root of $\partial f/\partial x_a$ which will be in between these roots and so separate $T_i$ from $P$. If $\partial f/\partial x_a(\pi(P))$ changes sign on the interval about $x_a(P)$, then looking at the graph of $y = f_{\pi(P)}(x_a)$ and the graph of $y = f_Q(x_a)$ for $Q$ near $\pi(P)$, one sees that $\partial f/\partial x_a$ again has a root between $T_i$ and $P$. Proceed by induction.

(iii) Assume that $f_{\pi(P)}$ is the zero polynomial, i.e., that all
\[ a_i(\pi(P)) = 0. \] Then if some irreducible factor of \( a_d \) divides all \( a_i \) and vanishes at \( \pi(P) \), one can divide this factor out. Otherwise, \( \pi(P) \) lies in the intersection of the zeros of two relatively prime polynomials in \( R[x, \ldots, x_{n-1}] \) derivable from the \( a_i \)'s. For, either two \( a_i \)'s have relatively prime factors which vanish at \( P \), or else one of the \( a_i \) has an irreducible factor which vanishes at \( P \) and is not real. Not real means that the factor does not generate the ideal of its real zeros. This follows from the real nullstellensatz for polynomials (see Theorem 2.1, in [4]). In the factor of \( a_i \) which we also call \( a_i \) is not real, then its real locus is contained in its singular locus (see Theorem 2.1, in [3]), and so is in the zero set of all the \( \partial a_i/\partial x_j \). In this way eventually one will obtain two relatively prime polynomials \( u(x, \ldots, x_{n-1}) \) and \( v(x, \ldots, x_{n-1}) \) which vanish at \( P \). By elimination theory, one can assume that \( u \) doesn't involve \( x_{n-1} \) and \( v \) doesn't involve \( x_{n-2} \). Then, using \( u, v, u - v \) and \( u + v \), one can define new regions subdividing \( S_i \) so that in these regions, which we also call \( S_i \), we have \( \pi(P) = \lim Q \) for \( Q \) in \( S_i \) and \( x_i(Q) - x_i(P) \) \( i = 1, 2, \ldots, n - 2 \). Now let \( \beta(x_1, \ldots, x_{n-2}) = \min \alpha(x_1, \ldots, x_{n-1}) \) for \( (x_1, \ldots, x_{n-1}) \) is \( S_i \) and where \( \alpha \) is a boundary of \( T_i \). And do the same thing for \( \max \alpha \). Then, since \( f(x_1, \ldots, x_{n-1}, \alpha(x_1, \ldots, x_{n-1})) = 0 \), we derive

\[
\frac{\partial f}{\partial x_{n-1}}(x_1, \ldots, x_{n-1}, \alpha(x_1, \ldots, x_{n-1})) + \frac{\partial f}{\partial x_n}(x_1, \ldots, x_{n-1}, \alpha(x_1, \ldots, x_{n-1})) \frac{\partial \alpha}{\partial x_{n-1}} = 0 .
\]

So either \( \beta(x_1, \ldots, x_{n-1}) \) is a root of the polynomial obtained from \( \partial f/\partial x_{n-1} \) and \( f \) by eliminating \( x_{n-1} \) or else the minimum occurs on the boundary, in which case one can use elimination theory on \( f \) and one of the \( u, v, u + v, \) or \( u - v \) to get a polynomial with root \( \beta \). In any case, we get a new polynomial involving one less variable than \( f \) which will have a root \( \beta \) between \( T_i \) and \( P \), at least near \( P \), in \( S_i \). By subdividing \( S_i \) again, one can assume that the new polynomial does have a root between \( T_i \) and \( P \) above \( S_i \). Then the induction can proceed.

Case 2. \( S_j \not\subseteq \bar{S}_i \). One must consider the various \( S_k \subseteq \bar{S}_i \cap \bar{S}_j \) and see what happens to the roots of \( f \) above \( S_k \). If \( a_d = 0 \) on \( S_k \), then one can use the minimizing techniques of Case 1, (iii) to reduce to the two variable case where \( a_d \) or one of its irreducible factors will separate \( T_i \) and \( T_j \). When \( a_d \neq 0 \) on \( S_k \), let \( T_i' = T_i \cap \pi^{-1}(S_k) \) and \( T_j' = T_j \cap \pi^{-1}(S_k) \). As long as \( T_i' \) and \( T_j' \) are not adjacent roots, one can find some \( \partial^i f/\partial x_i \), which is \( > 0 \) on \( T_i' \) and \( < 0 \) on \( T_j' \) and by continuity the same holds on \( T_i \) and \( T_j \). If these are adjacent
roots, then one chooses the polynomial so it vanishes on $T_t$ and is $< 0$ on $T_{t'}$, etc.

One must subdivide the regions $T_t$ so that the new polynomials have constant sign on these pieces.

**Mostowski's Theorem.** Let $D$ be defined by $p_i(x_1, \ldots, x_n) > 0,$ $i = 1, \ldots, s$. Then, if $C_1$ and $C_2$ are semi-algebraic closed disjoint subsets of $D$, there exists $g$ in $B_2$ with $g(C_1) > 0$ and $g(C_2) < 0$.

**Proof.** First a remark. Mostowski's proof would show that one could choose $g$ in $B_1$, but this seems to have no advantage in the applications.

Let $f_t, \ldots, f_t$ be the polynomials defining $C_1$ and $C_2$. By the Separation Lemma, we can find more polynomials including the $f$'s and $p$'s, say, $f_t, \ldots, f_n$ and a subdivision $R^n = \bigcup_{i,j} T_{tij}$, $T_t = \bigcup_i T_{ti}$ so that $C_1 = \bigcup_i T_{ti}$ for $i$ in $I$, and $C_2 = \bigcup_i T_{ti}$ for $i$ in $I$. Moreover, Sign $f_k(T_{tij})$ is constant and for all $\bar{T}_{t1} \cap \bar{T}_{t2} = \emptyset$, and all $j_1, j_2$, there exists $f_k$ with $\pm f_k(T_{tij}) \geq 0$ and $\pm f_k(T_{tij}) < 0$.

So choose $T_{t1} \subseteq C_1$ and choose each $\pm f_k$ with $\pm f_k(T_{t1}) \geq 0$. We consider the chosen $\pm f_k$'s as our $f_k$'s. Then, for each $T_{tij} \subseteq C_2$, there exists $f_k$ with $f_k(T_{tij}) < 0$.

Let $h = \sum_k (|f_k| - f_k)$. Then $h = 0$ on $T_{t1}$ and $h > 0$ on $C_2$. Let

$$\varepsilon(x) = \prod_{i=1}^s p_i(x)/(2 + ||x||^L)$$

for $L = \sum_{i=1}^s \deg p_i + 1$ and if no $p_i$, then let the numerator $=1$. Since $h > 0$ on $C_2$, we can let $\gamma(r) = \min\{h(x) | \varepsilon(x) = r, x \in C_3\}$. Then $\gamma(r) > 0$ for $0 < r < 1$ and $\gamma(r)$ is an algebraic function. Thus there exists $N$ so that $\gamma(r) > r^N$ for all $r$ for which $\gamma(r)$ is defined. It follows that $h(x) > \varepsilon(x)^N$ on $C_2$. We let

$$g_{ij}(x) = \sum_k (\sqrt{f_k^i + \varepsilon(x)^N/(t + 1)^2} - f_k) > h(x).$$

Moreover, $g_{ij}(x) < h(x) + \varepsilon(x)^N$, so on $T_{t1}$, we have $g_{ij}(x) < \varepsilon(x)^N$. Thus $g_{ij}(x) - \varepsilon(x)^N$ is $< 0$ on $T_{t1}$ and $> 0$ on $C_2$.

In a similar way, one can find some $g_{ij}$ for each $T_{tij} \subseteq C_i$ so that $g_{ij} > 0$ on $C_2$ and $g_{ij} < 0$ on $T_{tij}$. Each $g_{ij} \in B_n$. Now note that $\sum (|g_{ij} - g_{ij}|) = 0$ on $C_2$ and $> 0$ on $C_i$. Then as above by modifying this function one can obtain

$$g = \sum (\sqrt{g_{ij}^2 + \varepsilon(x)^N/M^2} - g_{ij}) - \varepsilon(x)^N$$

for some large integer $M$ which will have the desired properties.
2. Substitution in Nash functions. Recall the situation. We have $D$ as in §1 to be \{ $a$ in $\mathbb{R}^n$ with $p_i(a) > 0$, $i = 1, \ldots, s$ \}. And $A = \{ f : D \to \mathbb{R}$ with $f$ algebraic and analytic \}. Let $\varphi : A \to L$ be a homomorphism of $A$ into a real closed field. Since $A \supset \mathbb{R}[x_1, \ldots, x_n]$, $\varphi x_i$ is defined for $i = 1, \ldots, n$. In [5], §2, it was shown that $\varphi f(x) = z$ is equivalent to some elementary statement $A(x, z)$, so one can define a new function $f_L : D_L \to L$, where $D_L = \{ (a_1, \ldots, a_n) \text{ in } L^n \text{ with all } p_i(a_1, \ldots, a_n) > 0 \}$ by setting $f_L(\alpha) = \delta$ if and only if $A(\alpha, \delta)$.

THEOREM 2.1. With the notation as above, $f_L(\varphi x_1, \ldots, \varphi x_n) = \varphi f$.

Proof. This goes in several steps and occupies most of this section.

LEMMA 2.2. We can assume that if $p_f = \sum_{i=0}^d a_i(x)z^i$ is the irreducible polynomial for $f$ over $\mathbb{R}(x_1, \ldots, x_n)$; then $a_d(\varphi x) \neq 0$, and $\partial p_f/\partial z(\varphi f, \varphi x) \neq 0$.

Proof. Let $a$ be any point of $D$. For our original $f$,

$$R[x, f]_{(a, f(a))} = (R[x, z]/(p_f))_{(a, f(a))}$$

is a local ring and is etale over $\mathbb{R}[x]_{(a)}$. But by [6], Corollary 7.5, p. 11, this implies that there exists $g$ in $R[x, f]_{(a, f(a))}$ with $p_f(z, x)$ irreducible and $\partial p_f/\partial z(g(a), x) \neq 0$. Let $\alpha(x)$ be the leading coefficient of $p_f(z)$. So $f(x) = q_1(g, x)/q_2(g, x)$ with $g_2(g(a), a) \neq 0$. Let

$$h_a = (\partial p_f/\partial z(g(x), x)q_2(g(x), x)a(x))^2.$$ 

Then $h_a \neq 0$ near $a$, and we can construct such an $h_a$ for every $a$ in $D$. Let $V_a = V(h_a)$ = zero set of $h_a$ in $D$. It is clear that $\bigcap V_a = \emptyset$ taking the intersection over all $a$ in $D$. We claim that there exists a finite number of $V_a$ whose intersection is empty. To prove this we argue as in [5], Lemma 3.1. First choose some $h_{a_1}$ and let $W$ be a connected non-singular piece of $V(h_{a_1})$. Let $a_2 \in W$ and then $h_{a_2}$ can vanish only on a smaller dimensional piece of $W$. By continuing this process one gets the result.

Let $h$ be the sum of these $h_a$'s. Then $h - \varepsilon(x)^N > 0$ on $D$ for $N$ large enough. So $\sqrt{h - \varepsilon(x)^N} \in A$ and so $\varphi(\sqrt{h - \varepsilon(x)^N}) = \varphi(h - \varepsilon(x)^N) > 0$. This implies that $\varphi h_a > 0$ for some $a$. Take the corresponding $g$ for this $h_a$. Then $\partial p_f/\partial z(\varphi g, \varphi x) \neq 0$, $q_2(\varphi g, \varphi x) \neq 0$, and $\alpha(\varphi x) \neq 0$.

So suppose we have proved our theorem for $g$. Then since $f = q_1(g, x)/q_2(g, x)$, we have
\[ \Psi_f = q_i(g_{\phi}, \phi x)/q_i(g_{\phi}, \phi x) = q_i(g(\phi x), \phi x)/q_i(g(\phi x), \phi x). \]

But \( f(b) = q_i(g(b), b)/q_i(g(b), b) \) for all \( b \) in \( R^* \) with \( q_i(g(b)) \neq 0 \) so, by the Tarski-Seidenberg principle: [1] or [5], Theorem 1.8, the same holds for all \( b \) in \( L^* \). In particular \( f(\phi x) = q_i(g(\phi x), \phi x)/q_i(g(\phi x), \phi x) \) and this implies that \( \Psi_f = f(\phi x) \).

So we now assume that \( \partial \phi_f/\partial z(\phi_f, \phi x) \neq 0 \). Consider \( R[x, z]/(\phi_f) \) and normalize this ring. Let \( t_i(z, x), \cdots, t_s(z, x) \) generate the normalization (considered mod \( p_f \)). So, as usual, \( R[x, z]/(\phi_f, \cdots) \rightarrow R[x, z]/(\phi_f) \) induces \( \pi: C^{n+s+1} \rightarrow C^{s+1} \), with the branches of \( V(\phi_f) \) separated in \( C^{n+s+1} \). Of course \( C = \) complex numbers.

Note that \( t_i(\phi_f, \phi x) \) is defined since \( \partial \phi_f/\partial z(\phi_f, \phi x) \neq 0 \).

Let

\[ C_1 = \{ (x, f(x), t_i(f(x), x), \cdots, t_s(f(x), x)) \mid x \in D \} \]

and

\[ C_2 = \{ (x, z, t_i(z, x), \cdots, t_s(z, x)) \mid p_f(z, x) = 0 \}, \quad x \in D, \ z \neq f(x). \]

Then \( C_1 \) and \( C_2 \) are closed disjoint semi-algebraic sets in \( D \times R^{n+1} \), so by Mostowski's theorem, there exists \( g(x, z, t) \) in \( B_{2, D \times R^{n+1}} \) with \( g(C_1) > 0 \) and \( g(C_2) < 0 \).

Now let \( h(x) = g(x, f(x), t_i(f(x), x), \cdots, t_s(f(x), x)) \). We have to show that \( h(x) \in A \) and that \( \phi h = g(\phi x, \phi f, t_i(\phi f, \phi x), \cdots, t_s(\phi f, \phi x)) \). But since each \( t_i(f(x), x) \) is integral over \( R[x], \forall a \in D \) and analytic on \( D \) except for a thin set, \( t_i(f(x), x) \) is in fact analytic on \( D \). The rest follows from

**Lemma 2.3.** Let \( g \in B_2 \) and \( h_1, \cdots, h_r \in A_D \) so that \( g(h_1, \cdots, h_r) \) is defined and in \( A_D \). Then \( \phi g = g(\phi h_1, \cdots, \phi h_r) \).

**Proof.** It is enough to show this for \( g \) in \( B_1 \), as a repeat of the same argument will finish the proof. So let \( g = a(x) + b(x)\sqrt{f(x)} \) where \( f, a, b \in B_2 \) and \( f > 0 \) on \( D \). Now \( f(h_1, \cdots, h_r) \) has

\[ \phi(f(h_1, \cdots, h_r)) = f(\phi h_1, \cdots, \phi h_r) \quad \text{as} \quad f \in B_2. \]

Since \( \phi(\sqrt{f(h_1, \cdots, h_r)}) = f(h_1, \cdots, h_r) \), it follows that

\[ \phi(\sqrt{f(h_1, \cdots, h_r)}) = \pm \sqrt{\phi f(h_1, \cdots, h_r)}. \]

But

\[ \phi(\sqrt{f(h_1, \cdots, h_r)}) = \phi \sqrt{f(h_1, \cdots, h_r)} > 0 \]

and so Lemma 2.3 follows.
To finish the proof of Theorem 2.1, note that

\[(*)\] Given \((x, z)\) in \(D \times \mathbb{R}\) if \(p_f(z, x) = 0\) and \(g(x, z, t, (x, x), \ldots, t, (x, x))\) is defined and \(\geq 0\); then \(f(x) = z\).

By Tarski-Seidenberg, this statement also holds in \(D \times L\) and since we have \(p_f(\varphi f, \varphi x) = 0\) and \(g(\varphi x, \varphi f, t, (\varphi f, \varphi(x)), \ldots, t, (\varphi f, \varphi(x)) > 0\), it follows that \(f(\varphi x) = \varphi f\).

We now show that the Nullstellensatz proved by Mostowski is an easy corollary of Theorem 2.1.

**Theorem 2.4 (Mostowski).** Let \(\mathcal{I}\) be an ideal of \(A\). Then \(I(V_{D}(\mathcal{I})) = \mathcal{I}\) iff \(\mathcal{I}\) is a real ideal (i.e. \(\sum \lambda_i \in \mathcal{I}\) implies each \(\lambda_i \in \mathcal{I}\)).

**Proof.** First note that \(A\) is Noetherian by [5], Theorem 3.4, and so \(\mathcal{I} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r\), where each \(\mathcal{P}_i\) is a real prime. It will be sufficient then to show that for each \(i\), \(I(V_{D}(\mathcal{P}_i)) = \mathcal{P}_i\). So consider \(\mathcal{P}_i\) a real prime in \(A\) and by the Noetherian property of \(A\), we have \(\mathcal{P}_i = (f_1, \ldots, f_i)\) for some \(f_1, \ldots, f_i\) in \(A\). Let \(L\) be a real closure of the quotient field of \(A/\mathcal{P}_i\). Then we have \(\varphi : A \twoheadrightarrow A/\mathcal{P}_i \twoheadrightarrow L\) where \(\varphi\) is the total map.

Now \(g \in I(V_{D}(\mathcal{P}_i))\) iff; \((*)\) For all \(x \in D\), \(f_i(x) = 0, \ldots, f_i(x) = 0\); implies \(g(x) = 0\). By Tarski-Seidenberg, \((*)\) holds for \(L\). But \(\varphi f_i = 0\) for all \(i\) so by Theorem 2.1, \(f_i(\varphi x) = 0\). This implies, by \((*)\) that \(g(\varphi x) = 0\). Again applying Theorem 2.1, we see that \(\varphi g = 0\). So \(g \in \mathcal{I}\).

**Theorem 2.5 (Mostowski).** Let \(f \in A\), \(f \geq 0\) on \(D\). Then \(f\) is a sum of squares in \(K\), the quotient field of \(A\).

**Proof.** If \(f\) is not a sum of squares, order \(K\) so that \(-f > 0\). Then, if \(L\) is a real closure of \(K\), one has \(\varphi : A \hookrightarrow L\). Since for all \(x \in \mathbb{R}^n\), \(f(x) \geq 0\); by Tarski-Seidenberg, the same holds for \(x \in L^*\). Thus \(f(\varphi x) \geq 0\). By Theorem 2.1, \(\varphi f = f(\varphi x) \geq 0\), but as \(\varphi f\) is the image of \(f\) in \(L\) and \(L\) is ordered so \(f < 0\), we have a contradiction.

**References**


Received April 12, 1975.

*University of New Mexico*
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ralph Artino, Gevrey classes and hypoelliptic boundary value problems</td>
<td>1</td>
</tr>
<tr>
<td>B. Aupetit, Caractérisation spectrale des algèbres de Banach commutatives</td>
<td>23</td>
</tr>
<tr>
<td>Leon Bernstein, Fundamental units and cycles in the period of real quadratic number fields. I</td>
<td>37</td>
</tr>
<tr>
<td>Leon Bernstein, Fundamental units and cycles in the period of real quadratic number fields. II</td>
<td>63</td>
</tr>
<tr>
<td>Robert F. Brown, Fixed points of automorphisms of compact Lie groups</td>
<td>79</td>
</tr>
<tr>
<td>Thomas Ashland Chapman, Concordances of noncompact Hilbert cube manifolds</td>
<td>89</td>
</tr>
<tr>
<td>William C. Connett, V and Alan Schwartz, Weak type multipliers for Hankel transforms</td>
<td>125</td>
</tr>
<tr>
<td>John Wayne Davenport, Multipliers on a Banach algebra with a bounded approximate identity</td>
<td>131</td>
</tr>
<tr>
<td>Gustave Adam Efroymson, Substitution in Nash functions</td>
<td>137</td>
</tr>
<tr>
<td>John Sollon Hsia, Representations by spinor genera</td>
<td>147</td>
</tr>
<tr>
<td>William George Kitto and Daniel Eliot Wulbert, Korovkin approximations in $L_p$-spaces</td>
<td>153</td>
</tr>
<tr>
<td>Eric P. Kronstadt, Interpolating sequences for functions satisfying a Lipschitz condition</td>
<td>169</td>
</tr>
<tr>
<td>Gary Douglas Jones and Samuel Murray Rankin, III, Oscillation properties of certain self-adjoint differential equations of the fourth order</td>
<td>179</td>
</tr>
<tr>
<td>Takaši Kusano and Hiroshi Onose, Nonoscillation theorems for differential equations with deviating argument</td>
<td>185</td>
</tr>
<tr>
<td>David C. Lantz, Preservation of local properties and chain conditions in commutative group rings</td>
<td>193</td>
</tr>
<tr>
<td>Charles W. Neville, Banach spaces with a restricted Hahn-Banach extension property</td>
<td>201</td>
</tr>
<tr>
<td>Norman Oler, Spaces of discrete subsets of a locally compact group</td>
<td>213</td>
</tr>
<tr>
<td>Robert Olin, Functional relationships between a subnormal operator and its minimal normal extension</td>
<td>221</td>
</tr>
<tr>
<td>Thomas Thornton Read, Bounds and quantitative comparison theorems for nonoscillatory second order differential equations</td>
<td>231</td>
</tr>
<tr>
<td>Robert Horace Redfield, Archimedean and basic elements in completely distributive lattice-ordered groups</td>
<td>247</td>
</tr>
<tr>
<td>Jeffery William Sanders, Weighted Sidon sets</td>
<td>255</td>
</tr>
<tr>
<td>Aaron R. Todd, Continuous linear images of pseudo-complete linear topological spaces</td>
<td>281</td>
</tr>
<tr>
<td>J. Jerry Uhl, Jr., Norm attaining operators on $L^1[0, 1]$ and the Radon-Nikodým property</td>
<td>293</td>
</tr>
<tr>
<td>William Jennings Wickless, Abelian groups in which every endomorphism is a left multiplication</td>
<td>301</td>
</tr>
</tbody>
</table>