KOROVKIN APPROXIMATIONS IN $L_p$-SPACES

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The main result is a characterization of finite Korovkin sets for positive operators in $l_p$. It follows that a finite set containing a positive function is a Korovkin set in $l_p$ if and only if it is a Korovkin set in $c_0$. The methods also show:

**Proposition.** Let $X$ be a compact subset of $\mathbb{R}^n$. Let $K$ be a subspace of $C(X)$ containing the constants. If $K$ is a Korovkin set in $C(X)$, then $K$ is Korovkin set in $L_p(X)$.

Several related results are also given. For example a question of G. G. Lorentz about the restrictions of Korovkin set in $C(X)$ to a subset $Y \subseteq X$ is answered.

Let $\mathcal{L}$ be a class of operators on a Banach space $E$. A subset $K \subseteq E$ is an $\mathcal{L}$-Korovkin set if whenever

(i) $\{L_i\}$ is a bounded sequence in $\mathcal{L}$, and
(ii) $L_i k \to k$ for each $k \in K$;

we have

(iii) $L_i f \to f$ for each $f$ in $E$.

Let $\mathcal{L}^1$ be the class of norm one operators on $E$. If $E$ is also a lattice, let $\mathcal{L}^+$ denote the positive operators on $E$; and, $\mathcal{L}^{1,+} = \mathcal{L}^1 \cap \mathcal{L}^+$.

After Korovkin showed that $\{1, x, x^2\}$ is an $\mathcal{L}^+$-Korovkin set in $C[0, 1]$, interest in this field has been in characterizing the Korovkin subsets of the classic Banach spaces.

Papers by Berens and Lorentz [3], Franchetti [8, 9], Krasnosilskii and Lifsic [13], Lorentz [14], Saskin [18], Scheffeld [19], and Wulbert [22] identified the various types of Korovkin sets in $C(X)$ spaces. Berens and Lorentz [3] have essentially characterized the $\mathcal{L}^{1,+}$-Korovkin subsets of $L_1$ spaces (see §3 of this article, also see [Lorentz, 14] and [Wulbert, 22]), and Dzjadyk [7] has shown that $\{1, \sin x, \cos x\}$ is an $\mathcal{L}^+$-Korovkin set in $L_p[0, 2\pi]$. (See also [James, 11], and [Zaricka, 24].)

The results here are related to identifying $\mathcal{L}^+$-Korovkin subsets of $L_p$-spaces. A sufficient condition is presented that encompasses the known (and the suspected) $\mathcal{L}^+$-Korovkin sets. For example each $\mathcal{L}^+$-Korovkin set in $C[a, b]$ that contains constants is also an $\mathcal{L}^+$-Korovkin set in $L_p[a, b]$. The main result given is a characterization of finite $\mathcal{L}^+$-Korovkin sets in $l_p$. A consequence of this characterization is that the $L_p$ spaces have the same finite $\mathcal{L}^+$-Korovkin sets. That is, if $K$ is a finite subset of both $l_r$ and $l_s$, and $K$ contains a
positive sequence, then $K$ is $L^+\text{-Korovkin}$ in $l_p$ if and only if $K$ is $L^+\text{-Korovkin}$ in $l_r$.

We use the last two sections of the paper to give short direct generalizations of some related Korovkin theorems. For example, a recent result by Bernau and Lacey [5] enables the removal of the last conditions from the characterization of $L^{1,+}\text{-Korovkin}$ subsets of $L_p$-spaces with an easy argument.

G. G. Lorentz [14] proved that if $X$ is a compact metric space, and $K$ is $\text{"Korovkin}$ set in $C(X)$ containing a constant, then for each closed subset $Y \subseteq X$, $K|_Y$ is an $L^+\text{-Korovkin}$ set in $C(Y)$. Lorentz asked if the property was true for any compact Hausdorff space $X$. A counterexample is given in section two.

**Notation.** If $X$ is a compact Hausdorff space $C(X)$ is the space of continuous real functions on $X$. For $x \in X$, $\xi(x)$ is the linear functional on $C(X)$ given by $\xi(x)(f) = f(x)$. If $K$ is a linear subspace of $C(X)$, we say $x \in cb K$, the Choquet boundary of $K$, if the only positive linear functional on $C(X)$ that agrees with $\xi(x)$ on $K$ is $\xi(x)$ itself. If $F$ is a subset of a set $Y$, $\psi_F$ is the characteristic function of $F$. We use $f|_F$ to denote the restriction of a function $f$ to the domain $F$, and for a set of functions $K$, $K|_F = \{k|_F: k \in K\}$. The dual of a normed space $E$ is written $E^*$.

As usual, $c$ denotes the space of convergent sequences with the sup norm,

$$c_0 = \{x(\ell) \in c: \lim x(\ell) = 0\}, \quad \text{and}$$

$$l_p = \{x(\ell) \in c_0: \|x\|_p = \sqrt[p]{\sum |x(\ell)|^p} < \infty\}.$$  

The norm on $l_p$ is assumed to be $\|\cdot\|_p$ as given above. We will frequently view these sequence spaces as spaces of continuous functions on the one point compactification of the integers.

Let $L$ be a class of linear operators on a normed space $E$. Let $K$ be a subset of $E$. A member $f \in E$ is in the $L$-shadow of $K$ if $L_n f \to f$ for each bound sequence $\{L_n\} \subseteq L$ such that $L_n k \to k$ for each $k \in K$. Hence $K$ is an $L$-Korovkin set if the $L$-shadow of $K$ is $E$. Since the $L$-shadow of $K$ is the same as the $L$-shadow of the span of $K$ we will often assume that $K$ is already a linear subspace of $E$.

1. $L^+\text{-Korovkin}$ sets in $L_p$-spaces. The main result of this section is the characterization of finite $L^+\text{-Korovkin}$ subsets of $l_r$-spaces. The condition is sufficient in general, and provides an accessible class of $L^+\text{-Korovkin}$ sets in $l_p$-spaces.

We also show that an $L^+\text{-Korovkin}$ set of an $L_p$-space contains
three functions. The interest in this fact comes from the surprising observation that that \( \{1, x\} \) is \( L^+\)-Korovkin in \( L_p[0, 1] \) (see §3).

Let \( K \) be a linear subspace of a normed linear lattice \( E \). Let \( f \in E \). Two sets of vectors \( \{u_i\}_{i=1}^n \) is an \( \varepsilon \)-trap for \( f \) if there is a vector \( e \) such that:

1. \(-e + \bigvee_{i=1}^m l_i \leq f \leq e + \bigwedge_{i=1}^n u_i,\)
2. \( \bigwedge_{i=1}^n u_i - \bigvee_{i=1}^m l_i + 2\varepsilon \| < \varepsilon, \) and
3. \( \|e\| < \varepsilon.\)

**DEFINITION.** \( K \) traps \( f \) if for each \( \varepsilon > 0, K \) contains an \( \varepsilon \)-trap for \( f.\)

**PROPOSITION 1.1.** If \( K \) traps \( f, \) then \( f \) is in the \( L^+ \)-shadow of \( K.\)

**Proof.** Let \( L_i \) be a sequence of positive operators such that \( L_k \to k \) for all \( k \) in \( K \) and \( \|L_i\| < B.\) Then for \( k \) sufficiently large,

\[
\left\| \bigwedge_{i=1}^n L_k(u_i) - \bigwedge_{i=1}^n u_i \right\| < \varepsilon, \text{ and } \left\| \bigvee_{i=1}^m L_k(l_i) - \bigvee_{i=1}^m l_i \right\| < \varepsilon.
\]

We also have,

\[
-L_k(e) + \bigvee_{i=1}^m L_k(l_i) \leq -L_k(e) + L_k \left( \bigvee_{i=1}^m l_i \right)
\]

\[
\leq L_k(f)
\]

\[
\leq L_k(e) + L_k \left( \bigwedge_{i=1}^n u_i \right)
\]

\[
\leq L_k(e) + \bigwedge_{i=1}^n L_k(u_i).
\]

Since,

\[
\left\| \bigwedge_{i=1}^n L_k(u_i) - \bigvee_{i=1}^m L_k(l_i) + 2L_k(e) \right\| \leq \varepsilon B,
\]

we have,

\[
\left\| L_k f - f \right\| \leq \left\| L_k f - L_k(e) - \bigwedge_{i=1}^n L_k(u_i) \right\|
\]

\[
+ \left\| L_k e \right\| + \left\| \bigwedge_{i=1}^n L_k(u_i) - \bigwedge_{i=1}^n u_i \right\|
\]

\[
+ \left\| \bigwedge_{i=1}^n u_i - f \right\|
\]

\[
\leq 2\varepsilon (B + 1).
\]

We need the following known result. [Alfsen, 1, Cor. 1.5.10].

Let \( X \) be a compact Hausdorff space. Let \( K \) be a linear subspace of \( C(X) \) that contains the constants and separates the points of \( X.\)
LEMMA 1.2. If \( f \in C(X) \) and \( x \in \text{cb}K \) then
\[
f(x) = \inf \{k(x) \mid k \in K, k \geq f\}.
\]

COROLLARY 1.3. Let \( X \) and \( K \) be as above. Let \( \mu \) be a positive finite, regular Borel measure on \( X \). If the support of \( \mu \) is contained in \( \text{cb} K \), then \( K \) is an \( L^+ \)-Korovkin set in \( L_p(X, \mu), 1 \leq p < \infty \).

Proof. From the lemma and Dini's theorem \( K \) traps every continuous function. Since the \( L^+ \)-shadow of \( K \) is closed, and the continuous functions are dense in \( L_p(X, \mu) \), the corollary is proved.

COROLLARY 1.4. Let \( X, K, \) and \( \mu \) be as above. If \( \text{cb} K = X \) then \( K \) is an \( L^+ \)-Korovkin set in \( L_p(X, \mu) \). In particular if \( X \) is metrizable and \( K \) is \( L^+ \)-Korovkin in \( C(X) \), then \( K \) is \( L^+ \)-Korovkin in \( L_p(X, \mu) \).

Proof. If \( X \) is metrizable the Choquet boundary of an \( L^+ \)-Korovkin set is \( X \) [14]. (Also see §2.)

EXAMPLE 1.5. (a) (Dzjadyk) \( \{1, \sin x, \cos x\} \) is an \( L^+ \)-Korovkin set in \( L_p[0, 2\pi] \).
(b) \( \{1, x, x^2\} \) is an \( L^+ \)-Korovkin set in \( L_p[0, 1] \).
(c) \( \{1, x, y, x^2, y^2\} \) is an \( L^+ \)-Korovkin set in \( L_p([0, 1] \times [0, 1]) \).

In the above corollaries the \( \varepsilon \)-traps constructed are exact in the sense that \( \varepsilon = 0 \). Unfortunately such \( \varepsilon \)-traps cannot generally be constructed.

PROPOSITION 1.6. If \( K \) is a finite dimensional subspace of an infinite dimensional \( L_p \) space, then there is an \( f \in L_p \) which cannot be bounded above by any \( k \in K \).

Proof. Let \( k_1, \ldots, k_n \) be a basis for \( K \), and let \( w = \Sigma^{|k|} \).

If \( k \geq f \) then there is a multiple of \( w \) which also bounds \( f \).

If \( w \) has a finite range a.e., then the infinite dimensionality of \( L_p \) can be used to construct an \( f \in L_p \) which cannot be bounded by \( w \). Otherwise looking at level sets we can find a countable family of disjoint measurable sets \( A(n) \) such that
\[
0 < \int_{A(n)} w^p \leq \left( \frac{1}{n^3} \right)^p.
\]

Let
\[
f(x) = \begin{cases} nw(x) & \text{on } A(n) \\ 0 & \text{otherwise} \end{cases}
\]
then \( f \in L_p \) and cannot be bounded by \( w \).

**DEFINITION.** For the remainder of this section let \( V \) be either \( c_0 \) or \( l_p \) for some \( 1 \leq p < \infty \).

With a series of lemmas we will prove a characterization theorem for finite dimensional \( \mathcal{L}^+ \)-Korovkin sets in \( V \).

**DEFINITION.** \( K \subseteq V \) contains essentially positive members if for every \( \varepsilon > 0 \), and every integer \( x \) there is a \( k \in K \) for which

\[
\begin{align*}
(1) & \quad k(x) \geq 1, \text{ and} \\
(2) & \quad \| k \wedge 0 \| < \varepsilon .
\end{align*}
\]

(for example—if \( K \) contains a strictly positive function, \( K \) contains essentially positive members.)

**THEOREM 1.7.** Let \( K \) be a finite dimensional subspace of \( V \) then:

1. \( K \) is an \( \mathcal{L}^+ \)-Korovkin set, and
2. \( K \) contains essentially positive members

if and only if

3. \( K \) traps every member of \( V \).

Proposition 1.1 proved that (3) implies (1), and it is trivial that (3) implies (2).

Let \( K \) be a linear subspace of \( V \).

Let

\[
T = \{ f \in V : K \text{ traps } f \} .
\]

**LEMMA 1.8.** \( T \) is a closed linear space.

**Proof.** Clearly \( K \) traps \( f \), implies \( K \) traps \( \alpha f \), for all \( \alpha \in \mathbb{R} \).

Suppose \( k \) traps \( f \) and \( g \).

Since it is always true that

\[
x \wedge y + z = (x + z) \wedge (y + z) ,
\]

it follows that

\[
\bigwedge_{i=1}^n \bigwedge_{j=1}^s (u_i + v_j) = \bigwedge_{i=1}^n u_i + \bigwedge_{j=1}^s v_j .
\]

Therefore if \( \{ u_i \}_i \) and \( \{ v_j \}_j \) are \( \varepsilon \)-traps for \( f \) and \( g \), then

\[
\{ u_i + v_j : i = 1, \ldots, n, j = 1, \ldots, s \}
\]

is a \( 2\varepsilon \)-trap for \( f + g \).
It is also easy to see that $T$ is closed.

**Lemma 1.9.** Let $K$ be an $\mathcal{L}^+\text{-Korovkin subspace of } V$. If $p \in V^*$ is nonnegative and $p(k) = (i)$ for some integer $i$ and all $k$ in $K$ then $p = \xi(i)$.

**Proof.** Suppose $p$ is as above. Let

$$(Pf)(j) = \begin{cases} f(j) & j \neq i \\ p(f) & j = i \end{cases}.$$ 

Then $P$ carries $k$ onto $k$ for all $k \in K$. Hence $P$ is the identity and $p = \xi(i)$.

In particular $K$ separates the integers.

**Lemma 1.10.** Let $K$ be a subspace of $V$ for which $\text{cb} K = \{1, 2, 3, \ldots \}$. For each integer $i$ there is a $k \in K$ for which $k(i) < k(j)$ for all $j \neq i$, and $k(i) < 0$.

**Proof.** Let $K'$ be the span of $K$ and 1 in $(c)$. From Lemma 1.2 there is an $\alpha \in \mathbb{R}$ and a $k$ in $K$ such that

1. $k(j) + \alpha \geq 0$ for $j \neq i$
2. $k(i) + \alpha < -1$.

Since $\lim_{j \to \infty} k(j) = 0$, $\alpha \geq 0$. Hence this $k$ has the desired properties.

**Lemma 1.11.** Let $K$ be a finite dimensional $\mathcal{L}^+\text{-Korovkin set in } V$. Let $w(i)$ be a strictly positive sequence such that $w_k \in (c)_0$ for all $k$ in $K$. Then each integer $i$ is in $\text{cb}(wK)$.

**Proof.** Let $p$ be a nonnegative sequence in $l_1$, such that $p(wk) = w(i)k(i)$ for each $k \in K$. Let $g \in V$. Using Caratheodory’s theorem, the Hahn-Banach theorem, and the characterization of the extreme points of the unit ball of $(c)^*$, there is a finite set of integers $\{x_j\}_{j=1}^n$ and nonnegative numbers $\{\lambda_j\}_{j=0}^n$ such that,

$$p(f) = \lambda_0 f(\infty) + \sum_{j=1}^n \lambda_j f(x_j) \text{ for all } f \in wK \oplus g \oplus 1$$

where $\infty$ denotes the point at infinity.

Let

$$q(t) = \begin{cases} \lambda_j w(x_j)/w(x_i) & \text{for } t = x_j, j = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$
Now Lemma 1.9 applies to $q$, and $p(g) = q(g) = g(i)$. Since $g$ was arbitrary the lemma is proved.

**Lemma 1.12.** Let $K$ be a finite dimensional subspace of $V$. There is a sequence $p$ such that

\[
\begin{align*}
(1) & \quad p > 0, \\
(2) & \quad pK \subseteq c_0, \text{ and } \\
(3) & \quad \frac{1}{p} \epsilon V.
\end{align*}
\]

**Proof.** Let $k_1, \ldots, k_n$ be a basis for $K$. Let

\[
w(x) = \sum_{i=1}^{n} |k_i(x)|.
\]

It suffices to consider the case in which $w$ has no zeros. It follows that $k(x)/w(x)$ is bounded for each $k \in K$. Thus if there is a $q \in c_0$ such that $w/q \in V$, then

\[
p = q \left/ \sum_{i=1}^{n} |k_i| \right.
\]

is the desired function.

To find such a $q$ when $V$ is an $l_p$ space, let $N(\epsilon)$ be the smallest integer such that

\[
\sum_{j > N(\epsilon)} w(j)^p \leq \epsilon, \quad \text{and let}
\]

\[
q(j) = \left( \frac{1}{n} \right)^{1/p} \quad \text{for} \quad N\left( \frac{1}{n^2} \right) \leq j < N\left( \frac{1}{(n+1)^2} \right).
\]

If $V = c_0$, let $N(\epsilon)$ be the smallest integer such that

\[
\sup_{j > N(\epsilon)} |w(j)| < \epsilon,
\]

then let

\[
q(j) = \frac{1}{n} \quad \text{for} \quad N\left( \frac{1}{n^2} \right) \leq j < N\left( \frac{1}{(n+1)^2} \right).
\]

**Lemma 1.13.** Let $K$ be a finite dimensional $L^\alpha$-Korovkin subspace of $V$.

(a) For each integer $i$ and each $\epsilon > 0$ there is a $k \in K$ such that

\[
\begin{align*}
(1) & \quad k(i) = -1, \quad \text{and} \\
(2) & \quad ||k \wedge 0|| < 1 + \epsilon.
\end{align*}
\]

(b) If in addition each member of $K$ is also in $l_q$ then the norm in (2) can be taken to be the $l_q$ norm.

**Proof.** For Lemma 1.12 there is a positive sequence $p$ such that
$pK \subseteq c_0$ and $1/p \in V$ ($1/p \in l_q$, resp.). We may also assume that $\|1/p\| = 1$ ($\|1/p\|_q = 1$ resp.). Let

$$w(j) = \begin{cases} \frac{p(j)}{\varepsilon} & j \neq i \\ 1 & j = i \end{cases}$$

By Lemma 1.10 and Lemma 1.11 there is a $k \in K$ such that

$$-1 = (wk)(i) < (wk)(j) \quad (j \neq i).$$

Thus

$$k(i) = -1, \text{ and } k(j) \geq 1/w(j).$$

**Lemma 1.14.** Let $K$ be a subspace of $V$ that contains essentially positive function and which satisfies the conclusion of Lemma 13(a), then for each $i$, $K$ traps $\psi_{\{i\}}$.

**Proof.** Let $0 < \varepsilon < 1/2$. The lower sequence $\{l_i\}$ for the definition of an $\varepsilon$-trap for $\psi_{\{i\}}$ is guaranteed by hypothesis.

Since $K$ contains essentially positive functions for each integer $j$ there is a $k_j \in K$ such that

1. $k_j(i) = 1$, and
2. $\|k_j \wedge 0\| < \varepsilon/2^{j+1}$.

Let $m_j \in K$ be a function (guaranteed by hypothesis) such that

3. $m_j(j) = -k_j(j) \wedge 0$, and
4. $\|m_j \wedge 0\| < (\varepsilon/2^{j+1} - m_j(j))$.

For $j \neq i$ let,

$$u_j = (k_j + m_j)/[(k_j + m_j)(i)],$$

then there is an $n$ for which $\{u_j\}_{j=1, j \neq i}^n$ forms the upper sequence in the definition of an $\varepsilon$-trap for $\psi_{\{i\}}$.

**Proof of Theorem 1.7.** The theorem is now immediate from Lemma 1.14, Lemma 1.13 and Lemma 1.8.

**Theorem 1.15** Let $K$ be a finite dimensional subspace of $l_p$ that contains a strictly positive function. Then $K$ is $L^+_\text{-Korovkin}$ if and only if it is an $L^+_\text{-Korovkin}$ subspace of $c_0$.

**Proof.** The necessity is immediate from Theorem 1.7. The sufficiency follows from Lemma 1.13(b), Lemma 1.14 and Lemma 1.8.
EXAMPLE 1.16. Let $X = \{1/i\}_{i=1}^{\infty} \cup \{0\}$, and let $K'$ be a finite dimensional subspace of $C(X)$ that contains the constants and such that $\{1/i\}_{i=1}^{\infty} \subseteq cbK$. Let $w \in l_p$.

For $k \in K'$ let

$$(Tk)(i) = w(i)k \left(\frac{1}{i}\right).$$

Then $Tk \in l_p$. Let $K = \{Tk: k \in K'\}$. Then in view of Lemma 1.2, $K$ satisfies the conclusion of Lemma 1.13(a) (even with $\varepsilon = 0$). Hence Lemma 1.14 implies that $K$ is an $\mathcal{L}^+$-Korovkin set in $l_p$. For example, this shows that $K = \{1/i^2, 1/i^3, 1/i^4\}$ is $\mathcal{L}^+$-Korovkin in each $l_p$, by letting $w(i) = i^2$ and $K' = \{1, x, x^2\}$.

PROPOSITION 1.17. If $L_p(X, \Sigma, \mu)$ contains a two-dimensional $\mathcal{L}^+$-Korovkin set, then $L_p(X, \Sigma, \mu)$ is two dimensional.

Proof. We again use several lemmas. For these let $K$ be a two-dimensional subspace of $L_p = L_p(X, \Sigma, \mu)$.

LEMMA 1.18. If there exists positive functionals $\phi_1$ and $\phi_2$ on $L_p$ and a set $Y$ of positive measure such that:

1. if $k \in K$, $\phi_1(k) \geq 0$, and $\phi_2(k) \geq 0$ then $k \geq 0$ on $Y$
2. for each pair of real numbers $r_1, r_2$ there is a $k \in K$ such that $\phi_1(k) = r_1$ and $\phi_2(k) = r_2$ and
3. $\dim L_p|_Y \geq 3$,

then $K$ is not $\mathcal{L}^+$-Korovkin.

Proof. For $f$ in $L_p$, let $Lf$ be the unique member $k$ of $K$ such that

$$\phi_i(f) = \phi_i(k) \quad i = 1, 2.$$

Now simply let

$$Pf(x) = \begin{cases} f(x) & x \in Y \\ (Lf)(x) & x \in Y \end{cases}.$$

Then $P$ is a nontrivial positive operator which acts as the identity on $K$.

LEMMA 1.19. Let $g$ be a measurable positive function that is bounded and bounded away from zero. Let

$$K' = \{gk: k \in K\}$$

then $K$ is $\mathcal{L}^+$-Korovkin if and only if $K'$ is $\mathcal{L}^+$-Korovkin.
Proof. If suffices to show that if $K$ is $\mathcal{L}^+$-Korovkin then $K'$ is also. Let $L_n$ be a bounded sequence of positive operators, such that

$$L_n(k') \rightarrow k' \quad \text{for each} \quad k' \in K'.$$

Let

$$P_nf = g^{-1}L_n(gf).$$

Since

$$P_nk \rightarrow k \quad \text{for all} \quad k \in K,$$

$$P_n(g^{-1}f) \rightarrow g^{-1}f \quad \text{for all} \quad f \in L_p.$$

Hence

$$L_nf \rightarrow f \quad \text{for all} \quad f \in L_p.$$

**Lemma 1.20.** Let $F \subseteq X$ be a set of positive measure which is not an atom. If $K$ is $\mathcal{L}^+$-Korovkin then $\dim K|_F = 2$.

**Proof.** Again one easily constructs a nontrivial positive operator that is the identity on $K$.

**Lemma 1.21.** A two-dimensional subspace $H$ of $\mathbb{R}^3$ that does not contain a positive vector, has a nonnegative annihilator.

**Proof.** Let $a = (a_1, a_2, a_3)$ be an annihilator of $H$. If $H$ does not have a nonnegative annihilator we may assume that $a_1 > 0 > a_2$. Let $h = (h_1, h_2, h_3)$ be a member in $H$ such that $h_3 = 0$. Then $a(h) = 0$ implies $\text{sgn} h_1 = \text{sgn} h_2$. Since $H$ also contains some vector whose third coordinate is positive, $H$ contains a vector with all positive coordinates.

**Lemma 1.22.** If $K$ is $\mathcal{L}^+$-Korovkin then there is an $F \subseteq X$ and a $k \in K$ such that

1. $\dim L_p|_F \geq 3$, and
2. $k$ is bounded, positive and bounded away from zero on $F$.

**Proof.** If $X$ is not purely atomic the lemma follows from Lemma 1.20. If $X$ is purely atomic the lemma follows from Lemmas 1.20 and 1.21, since if $P$ is a nonnegative annihilator of $K$, $Pf = f + p(f)\psi F$ is a positive operator for any set $F$ of finite measure.

**Proof of the proposition.** Suppose $K$ is $\mathcal{L}^+$-Korovkin. From Lemmas 1.19 and 1.22 we may assume that there is a set $F \subseteq X$
such that \( \dim(L_p|_F) \geq 3 \), that \( K \) is spanned by functions \( k_1 \) and \( k_2 \), and that \( k_i \) is identically 1 on \( F \). From Lemma 1.20 we can find subsets \( F_1, F_2 \) and \( F_3 \) of positive finite measure such that

\[
\max k_2|_{F_1} < \min k_2|_{F_3} \leq \max k_2|_{F_3} < \min k_2|_{F_2} \text{ a.e.}
\]

Furthermore if \( F \) is not purely atomic we may assume that \( \dim L_p|_{F_3} \geq 3 \). Hence letting \( \phi_i f = \int_{F_i} f \) \( (i = 1, 2) \), and \( Y = F_3 \) contradicts Lemma 1.18. If \( F \) is purely atomic we may assume that each \( F_i \) is an atom, and then letting \( \phi_i f = f(F_i) \) and \( Y = \bigcup_{i=1}^n \{ F_i \} \) would also contradict Lemma 1.18.

2. Korovkin sets in \( C(X) \). Let \( X \) be metrizable, and let \( K \) be a subspace of \( C(X) \) that contains the constants. G. G. Lorentz [14] showed that \( K \) is \( \mathcal{L}^+ \)-Korovkin in \( C(X) \) if and only if \( \mathcal{L}^+ K = X \). It follows that if \( Y \) is a closed subset of \( X \) then \( K|_Y \) is \( \mathcal{L}^+ \)-Korovkin in \( C(Y) \). Answering a question by Lorentz, we will give examples of a compact Hausdorff space \( X \), and an \( \mathcal{L}^+ \)-Korovkin sets \( K \subseteq C(X) \) whose restrictions to closed subsets of \( X \) fail to be Korovkin. The examples also extend a result by E. Sheffield [19].

**Definition.** \( K \subseteq C(X) \) is \( \mathcal{L} \)-Korovkin for nets if every bounded net of operators in \( \mathcal{L} \) that converges strongly to the identity on \( K \), also converges strongly to the identity on \( C(X) \).

**Lemma 2.1.** Let \( X \) be a compact Hausdorff space, \( K \) is \( \mathcal{L}^+ \)-Korovkin for nets if and only if \( \mathcal{L}^+ K = X \).

**Proof.** This is a minor variant of known results. The sufficiency can be obtained from the method of proof of Lemma 1 in [Wulbert, 22]. The necessity follows from the following known construction [Lorentz, 14]. Let \( \{ U_a \} \) be a neighborhood base for a point \( x \in X \). Suppose \( \mu \) is a positive measure in \( C(X)^* \) such that

\[
k(x) = \int k d\mu \quad \text{for all} \quad k \in K.
\]

Let \( g_x \) be a continuous function that is 1 at \( x \) and vanishes off \( U_x \). Let

\[
L_a(f) = (1 - g_x)f + \left( \int f d\mu \right) g.
\]

Then

\[
L_a(k) \longrightarrow k \quad \text{for all} \quad k.
\]
but also
\[(L_n f)(x) \longrightarrow \int f \, d\mu.\]

The following is also a variant of the proof in [Wulbert, 22].

**Lemma 2.2.** Let \( \{L_n\} \) be a bounded sequence of positive operators on \( C(X) \) such that \( L_n k \to k \) for all \( k \in K \subseteq C(X) \). If \( Y \) is a countably compact subset of \( cbK \), then for each \( f \in C(X) \), \( L_n f \) converges uniformly to \( f \) on \( Y \).

**Corollary 2.3.** Let \( X \) be an open countably compact dense subset of a compact Hausdorff space \( Y \). Assume that \( Y - X \) contains two points, and let
\[K = \{ f \in C(Y) : f \text{ constant on } Y - X \} .\]
Then \( K \) is \( \mathcal{L}^+\)-Korovkin, but not \( \mathcal{L}^+\)-Korovkin for nets.

**Examples 2.4.** (1) Let \( X \) be locally compact and countably compact. Let \( Y = \beta X \) be the Stone-Čech compactification of \( X \). If \( Y - X \) contains two points then \( X \) and \( Y \) satisfy the conditions of the corollary.

(2) Let \( W \) be the space of ordinals less than the first uncountably ordinal. Let \( X = W \times W \), then \( X \) and \( Y = \beta X \) satisfy the properties of part (1) above.

(3) Let \( Y \) be an \( F \)-space. Let \( G \) be a finite subset of \( Y \) containing two points, and let \( X = Y - G \). Then \( X \) and \( Y \) satisfy the conditions of the corollary. (See [Gillman and Jerison, 10, p. 215].)

(4) In \( N \) denotes the integers then \( \beta N - N \) is an \( F \)-space.

**Example 2.5.** Let \( X, Y \) and \( K \) be as in the corollary then \( K \) is \( \mathcal{L}^+\)-Korovkin in \( C(Y) \), but \( k|_{Y - K} \) is not \( \mathcal{L}^+\)-Korovkin in \( C(Y - X) \).

**Remark 2.5.** Let \( X \) and \( Y \) be as in the corollary and let \( J \) be the ideal of continuous functions vanishing on \( Y - X \). Let \( y \in Y - X \).
Since the operator \( P \) given by
\[(Pf)(x) = f(x) + f(y)\]
is a positive mapping that acts as the identity on \( J \), \( J \) is not an \( \mathcal{L}^+\)-Korovkin set in \( C(Y) \). However it only requires minor modification to show that \( J \) is an \( \mathcal{L}^-\)-Korovkin set, although it is not \( \mathcal{L}^-\)-Korovkin for nets.

E. Sheffold [19] gave the first example of a set that was an
\(L^1\)-Korovkin set but not \(L^1\)-Korovkin for nets. Using a different method Sheffield showed that if \(Y\) is an \(F\)-space, and \(J\) is the ideal of all continuous functions vanishing at a single point, then \(J\) has the above properties.

R. M. Minkova [15] has proved a Korovkin type theorem involving convergence of the higher order derivatives for functions in \(C^r[0, 1]\). Indeed let \(X\) be an open-bounded subset of \(\mathbb{R}^n\). Let \(Y\) be the closure of \(X\) and let \(C^r(X)\) be the continuous real-valued functions on \(Y\), with \(r\) bounded, continuous (Frechet) derivatives on \(X\). Let the norm on \(C^r(X)\) be the sum of the uniform norms of the derivatives

\[
\|f\| = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^n\|_\infty.
\]

An operator \(T\) on \(C(X)\) is \(r\)-smooth if \(T(C^r(X)) \subseteq C^r(X)\) and \(T\) is continuous on \(C^r(X)\).

**Proposition 2.6.** Let \(K\) be a subspace of \(C(X)\) that contains the constants and for which \(cb K\) is dense in \(X\). Let \(\{T_i\}\) be a bounded sequence of positive \(r\)-smooth operators on \(C(X)\) such that

1. \(\{T_i\}\) is uniformly bounded as operators on \(C^r(X)\), and
2. \(T_i k \to k\) for all \(k \in K\),

then
3. \(T_i f^{(j)} \to f^{(j)}\) uniformly for each \(f \in C^r(X)\), and for each \(j = 0, 1, 2, \cdots, r-1\).

**Proof.** This easily follows by induction from Ascoli’s theorem since in this setting \((T_i f)(x) \to f(x)\) for all \(x \in cb K\) (Lemma 2.2).

Minkova used a delicate estimate of Landau to bound the derivative of a function with bounds for the function and its second derivative, and proved the case of the above proposition obtained when \(X\) is a compact interval of the line, and \(K\) is an \(L^1\)-Korovkin set.

3. \(L^{1,+}\)-Korovkin sets in \(L_p\). Let \((X, \Sigma, \mu)\) be a finite measure space, and let \(K\) be a subspace of \(L_1(X, \Sigma, \mu)\) that contains the constants. Let \(E\) be the closed linear sublattice generated by \(K\). Since the conditional expectation operator is a contractive projection of \(L_1\) onto \(E\), the \(L^{1,+}\)-shadow of \(K\) is contained in \(E\). Berens and Lorentz [3] have in fact shown that \(E\) is the \(L^{1,+}\)-shadow of \(K\). Bernau and Lacey [5] have announced that every closed sublattice of an \(L_p\)-space is the range of a contractive projection. Hence the restrictions in the Berens-Lorentz theorem can be removed.

**Theorem 3.1.** Let \(K\) be a subset of \(L_p\). The \(L^{1,+}\)-shadow of
$K$ is the closed linear sublattice of $L_p$ generated by $K$.

Proof. Let $S$ be the $\mathcal{L}^{1,+}$-shadow of $K$. It is obvious that $S$ is closed. To show $S$ is a lattice it suffices to show that $f \vee g \in S$ when both $f \in S$ and $g \in S$. Let $L_i$ be a sequence of positive contractive on $L_p$ such that $L_p k \to k$ for all $k \in K$. Since $f \vee g$ dominates both $f$ and $g$

$$L_i(f \vee g) \geq L_i(f) \vee L_i(g).$$

We also know that $\|f \vee g\| \geq \|L_i(f \vee g)\|$ and that

$$L_i(f) \vee L_i(g) \to f \vee g.$$  

Hence if $f \vee g \geq 0$, $\lim L_i(f \vee g) = f \vee g$. Indeed, if we are working in $L_i$, this limit is found by inspecting the integral $\|L_i(f \vee g) - f \vee g\|$. Otherwise the statement follows from the uniform convexity of $L_p$. Therefore if $f$ and $g$ are arbitrary members of $S$, $|f| \vee |g| \in S$, and

$$f \vee g + |f| \vee |g| = (f + |f| \vee |g|) \vee (g + |f| \vee |g|) \in S,$$

thus $f \vee g \in S$.

The $\mathcal{L}^{1,+}$-shadow of $K$, therefore, contains the closed lattice generated by $K$. The converse statement is immediate from the result of Bernau and Lacey mentioned before the theorem.

Remark 3.2. Let $X$ be a compact metric space, and let $K$ be a subspace of $C(X)$ containing the constants. The lattice characterization of the $\mathcal{L}^{1,+}$-shadow of $K$ does not apply. In particular the space spanned by 1 and $x$ is not an $\mathcal{L}^{1,+}$-Korovkin set. However, it does follow from the proof of Lemma 2.1, and Lemma 1.2. that if $K$ is a Korovkin set then, the closed sublattice generated by $K$ is all of $C(X)$.

References

KOROVKIN APPROXIMATIONS IN $L_p$-SPACES


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