OSCILLATION PROPERTIES OF CERTAIN SELF-ADJOINT DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

GARY DOUGLAS JONES AND SAMUEL MURRAY RANKIN, III
Assuming oscillation, a connection between the decreasing and increasing solutions of

\[(ry'')'' = py\]

is established. With this result, it is shown that if \(r \equiv 1\) and \(p\) positive and monotone the decreasing solution of (1) is essentially unique. It is also shown that if \(p > 0\) and \(r \equiv 1\) the decreasing solution tends to zero.

It will also be assumed that \(p\) and \(r\) are positive and continuous and at times continuously differentiable on \([a, +\infty)\). By an oscillatory solution of (1) will be meant a solution \(y(x)\) such that there is a sequence \(\{x_n\}_{n=1}^{\infty}\) diverging to \(+\infty\) such that \(y(x_n) = 0\) for every \(n\). Equation (1) will be called oscillatory if it has an oscillatory solution.

Equation (1) has been studied previously by Ahmad [1], Hastings and Lazer [3], Leighton and Nehari [8] and Keener [7].

Hastings and Lazer [3] have shown that if \(p > 0\), \(r = 1\) and \(p' \geq 0\) then (1) has two linearly independent oscillatory solutions which are bounded on \([a, +\infty)\). They further show that if \(\lim_{t\to\infty} p(t) = +\infty\) then all oscillatory solutions tend to zero. Our result will show that there is a nonoscillatory solution which goes to zero "faster" than the oscillatory ones.

Keener [7] shows the existence of a solution \(y\) of (1) such that \(\text{sgn } y = \text{sgn } y'' = \text{sgn } y' = \text{sgn } (ry'')'\). Under the additional hypothesis that \(\liminf p(t) \neq 0\) he shows that \(y(t) \to 0\) as \(t \to \infty\). We will give a condition for \(y(t) \to 0\) where \(\liminf p(t)\) can be zero.

Ahmad [1] shows that if (1) is nonoscillatory then every solution \(z\) of (1) with the properties of \(y\) above satisfy \(z = cy\) for some constant \(c\).

The following lemmas due to Leighton and Nehari [8] will be basic in our investigation.

**Lemma 1.** If \(y\) is a solution of (1) with \(y(c) \geq 0\), \(y'(c) \geq 0\), \(y''(c) \geq 0\) and \((r(c)y''(c))' \geq 0\) but not all zero for \(c \geq a\) then \(y(x)\), \(y'(x)\), \(y''(x)\) and \((r(x)y''(x))'\) are positive for \(x > c\).

**Lemma 2.** If \(y\) is a solution of (1) with \(y(c) \geq 0\), \(y''(c) \geq 0\), \(y'(c) \leq 0\) and \((r(c)y''(c))' \leq 0\) but not all zero for \(c \geq a\) then \(y(x) > 0\), \(y''(x) > 0\), \(y'(x) < 0\) and \((r(x)y''(x))' < 0\) for \(x \in [a, c)\).
We will also use the following theorem of Keener [7].

**THEOREM 1.** There exists a solution \( w(x) \) of (1) which has the following property:

\[
\begin{align*}
w(x)w'(x)w''(x)[r(x)w''(x)]' & \neq 0; \\
(P) \quad \text{sgn} \ w(x) = \text{sgn} \ w''(x) \neq \text{sgn} \ w'(x) = \text{sgn} \ [r(x)w''(x)]' \\
\end{align*}
\]

for \( a \leq x \).

We will first show a connection between the decreasing solution of (1) given by Theorem 1 and the solution that tends to \( \infty \) given by Lemma 1. We will use the fact that if \( y_1, y_2 \) and \( y_3 \) are solutions of (1) then \( r(x)W(y_i, y_2, y_3; x) = r(x)\det(y_i^j(x)) \) \((i, j = 1, 2, 3, 4)\) is a solution of (1). Further we have

**LEMMA 3.** If \( y_1, y_2, y_3, y_4 \) is a basis for the solution space of (1) then \( W_{123} = rW(y_1, y_2, y_3), W_{124} = rW(y_1, y_2, y_4), W_{134} = rW(y_1, y_3, y_4) \) and \( W_{234} = rW(y_2, y_3, y_4) \) is a basis for the solution space of (1).

**Proof.** Let

\[
A = \begin{vmatrix}
y_1 & y_2 & y_3 & y_4 \\
y'_1 & y'_2 & y'_3 & y'_4 \\
y''_1 & y''_2 & y''_3 & y''_4 \\
(ry''_1)' & (ry''_2)' & (ry''_3)' & (ry''_4)'
\end{vmatrix}
\]

Then

\[
\text{adj} \ A = \begin{vmatrix}
(rW''_{123})' & -rW''_{124} & W'_{134} & -W'_{234} \\
-(rW''_{132})' & rW''_{124} & -W'_{134} & W'_{234} \\
(rW''_{124})' & -rW''_{134} & W'_{124} & -W'_{123} \\
-(rW''_{134})' & rW''_{123} & -W'_{124} & W'_{123}
\end{vmatrix}
\]

Thus since \( \det A \neq 0 \), \( \det \text{adj} A \neq 0 \). Consequently \( W_{123}, W_{124}, W_{134} \) and \( W_{234} \) is a basis for the solution space of (1).

**LEMMA 4.** Let \( y_1, y_2, y_3, y_4 \) be a basis for the solution space of (1). Then there is a basis for the solution space of (1), \( z_1, z_2, z_3, z_4 \) such that \( W_{123} = rW(z_1, z_2, z_3) = k_1y_1, W_{124} = rW(z_1, z_2, z_4) = k_2y_2, W_{134} = \)
rW(z_i, z_3, z_4) = k_3y_i and W_{234} = rW(z_2, z_3, z_4) = k_4y_i, where k_i \neq 0, i = 1, 2, 3, 4 is a constant.

Proof. Let u_1, u_2, u_3, u_4 be a basis for the solution space of (1). Then rW(u_1, u_2, u_3), rW(u_1, u_2, u_4), rW(u_2, u_3, u_4) is also a basis for the solution space of (1) by Lemma 3. Thus each y_i is a linear combination of the rW's. Suppose

y = c_1rW(u_1, u_2, u_3) + c_2rW(u_1, u_2, u_4) + c_3rW(u_2, u_3, u_4) + c_4rW(u_2, u_3, u_4) where c_i \neq 0. Letting

v_1 = c_1u_1 + c_4u_4, v_2 = c_2u_2 - c_4u_4, v_3 = c_1u_3 + c_2u_4 and v_4 = u_4, we have

W(v_1, v_2, v_3) = c_1[c_1W(u_1, u_2, u_3) + c_2W(u_1, u_2, u_4) + c_3W(u_2, u_3, u_4) + c_4W(u_2, u_3, u_4)], W(v_1, v_2, v_4) = c_1W(u_1, u_2, u_4), W(v_2, v_3, v_4) = c_1W(u_2, u_3, u_4), W(v_2, v_3, v_4) = c_1W(u_2, u_3, u_4). Repeating the argument three times gives the desired result.

LEMMA 5. Let z be a nonoscillatory solution of (1). Then the solution space of

\( (2) \quad z(y')' + rz'y'' + z''ry' - (rz'')'y = 0 \)

is a three dimensional subspace of (1). Further, if z satisfies the conditions of Lemma 1 or Theorem 1 then (2) is oscillatory if and only if (1) is oscillatory.

Proof. Using Lemma 4, choose solutions y_1, y_2, y_3 of (1) such that kz = rW(y_1, y_2, y_3), where k \neq 0. Then

\[
\begin{vmatrix}
  y_1 & y_2 & y_3 & y \\
  y_1' & y_2' & y_3' & y' \\
  ry_1'' & ry_2'' & ry_3'' & ry'' \\
  (ry_1'')' & (ry_2'')' & (ry_3'')' & (ry'')' \\
\end{vmatrix} = 0
\]

is equivalent to (2). Thus, the first part of the lemma follows. It follows from Lemma 1 that if z satisfies the conclusion of the lemma and if y is a solution of (2) such that y(d) = y'(d) = 0, r(d)y''(d) = 1 where d > c, then y(x) > 0 for x > d, or using the definition of Hanan [2], (2) is \( C_{11} \). In the same way it follows from Lemma 2 that if y is a solution of (2) where z satisfies (P) such that y(d) = y'(d) = 0, r(d)y''(d) = 1 then y(x) > 0 for x \in [a, d], i.e. (2) is \( C_{1} \) [2]. Writing (2) is the form

\( (3) \quad (ry''/z) + rz''y'/z^2 - (rz'')'y/z^2 = 0 \),

we have by [4, Theorem 3, p. 338] that (3) is \( C_{1}(C_{11}) \) if and only if

\( (4) \quad [(ry''/z) + rz''y'/z^2]' = -(rz'')'y/z^2 \).
is $C_I(C_I)$. It then follows, using the methods of Hanan [2] that (3) is oscillatory if and only if (4) is oscillatory. Since $z$ satisfies (2), choose a basis for the solution space of (2) of the form $z, u_1, u_2$. Then $zu'_1 - u_1z'$ and $zu'_2 - u_2z'$ satisfy (4) and

\[(5) \quad (ry'/z) + [2rz''/z^2]y = 0.\]

But Leighton and Nehari [8, p. 335, 3.4] show that (5) is oscillatory if and only if (1) is oscillatory. Thus the result follows.

**Theorem 2.** Suppose (1) is oscillatory. If there exist two linearly independent solutions $n_1$ and $n_2$ of (1) which satisfy (P), then there is a $c \geq a$ and an oscillatory solution $u$ of (1) such that $u + N$ is oscillatory, where $N$ is the solution defined by $N(c) = N'(c) = N''(c) = 0, (r(c)N''(c))' = 1$.

**Proof.** Consider the equation

\[(6i) \quad n_i(ry'')' - n_i'ry'' + n_i''ry' - (rn''_i)y = 0, \quad i = 1, 2.\]

By Lemma 5, each of the equations (6) are oscillatory and $C_I$. Since $n_1$ and $n_2$ are linearly independent, we can choose $c \geq a$ such that $n'_1(c)n_2(c) - n_1''(c)n_2(c) \neq 0$. Let $u_i$ be the solution of (6i) defined by $u_i(c) = u_i'(c) = 0, r(c)u_i'' = 1$ for $i = 1, 2$. Since (6i) is $C_I$ and $u_i(c) = 0$, it follows that $u_i$ and $u_2$ are oscillatory solutions of (1). But $u_i(c) - u_2(c) = u_i'(c) - u_2'(c) = u_i''(c) - u_2''(c) = 0, (r(c)u_i'(c))' - (r(c)u_2'(c))' = n'_1(c)n_1(c) - n_2'(c)/n_2(c) \neq 0$. Thus $u_1 - u_2$ is a multiple of $N$ and the result follows.

**Theorem 3.** Suppose (1) is oscillatory. If there is a $c \geq a$ and an oscillatory solution $u$ of (1) such that $u + N$ is oscillatory, where $N$ is the solution of (1) defined by $N(c) = N'(c) = N''(c) = 0, (r(c)N''(c))' = 1$ then (1) has a basis for the solution space with all oscillatory elements.

**Proof.** Let $z$ be a solution of (1) that satisfies (P). Then (2) is $C_I$ and oscillatory. Thus there is a basis for the solution space of (2), say $\{u_1, u_2, u_3\}$, with all oscillatory elements [5]. Since $N$ does not satisfy (2), there is a constant $0 < k < 1$ such that $u + kN$ is not in the solution space of (2). Since $u + N$ is oscillatory, $u + kN$ is oscillatory. Thus $\{u + kN, u_1, u_2, u_3\}$ is a basis for the solution space of (1).

**Theorem 4.** Suppose (1) has a basis for its solution space with all oscillatory elements. Then there are two linearly independent
solutions $n_1$ and $n_2$ of (1) which satisfy (P).

Proof. Suppose $\{y_1, y_2, y_3, y_4\}$ is a basis for the solution space of (1) with all oscillatory elements. By Lemma 4 there is a basis $\{u_1, u_2, u_3, u_4\}$ of (1) such that $W_{1i} = k_i y_i$, $W_{2i} = k_i y_2$, $W_{3i} = k_i y_3$, $W_{4i} = k_i y_4$ where $k_i \neq 0$ for $i = 1, 2, 3, 4$. Since $y_i$ is oscillatory, there is a sequence $\{x_i\} \to \infty$ such that $y_i(x_i) = 0$ for every $i$. Since $W_{1i} = k_i y_i$, for every $x_i$ there are constants $c_{i_j}$ for $j = 1, 2, 3$ such that $c_{i_1} + c_{i_2} + c_{i_3} = 1$ and

$$u_i = c_{i_1}u_1 + c_{i_2}u_2 + c_{i_3}u_3$$

has a triple zero at $x_i$. Since $\{c_{i_j}\}_{i=1}^n$ are bounded for $i = 1, 2, 3$, we can assume without loss of generality that

$$\lim_{i \to \infty} c_{i_j} = c_j \text{ for } j = 1, 2, 3.$$

Hence using Lemma 2 and an argument such as in [7, p. 281]

$$W_1 = c_1 z_1 + c_2 z_2 + c_3 z_3$$

satisfies (P). In the same way there are constants $d_{i_j}$, $i = 2, 3, 4$; $j = 1, 2, 3$, such that

$$W_2 = d_2 z_1 + d_2 z_2 + d_3 z_4$$
$$W_3 = d_3 z_1 + d_3 z_2 + d_3 z_4$$
$$W_4 = d_4 z_2 + d_4 z_3 + d_4 z_4$$

satisfy the (P). Clearly at least two of $W_1, W_2, W_3, W_4$ are linearly independent.

We will now use the above theorems to prove the following results for

(6) \hspace{1cm} y'' = p(x)y.

**Theorem 5.** Suppose (6) is oscillatory, $p \in C'[a, +\infty)$ and $p$ is monotone. Then there is a unique solution of (6) (up to constant multiples) which satisfies (P). Further, a basis for the solution space of (1) has at most three oscillatory elements.

Proof. Suppose there are two solutions of (6) that satisfy (P) and are linearly independent. Then by Theorem 1, there is an $c \geq a$ and an oscillatory solution $u$ of (6) such that $u + N$ is oscillatory, where $N$ is the solution defined by $N(c) = N'(c) = N''(c) = 0$, $N'''(c) = 1$. By Lemma 1, $N(x)$, $N'(x)$, $N''(x)$, and $N'''(x)$ are positive for $x > c \geq a$. Thus $N, N'$ and $N''$ are unbounded. Multiplying (6) by $y'$ where $y$
is a solution of (6) and integrating from \( a \) to \( x \), we obtain

\[
G[y(x)] = y''''(x) - 2y'(x)y'''(x) + p(x)y''(x) = G[y(a)] + \int_a^x p'(t)y''(t)dt.
\]

Assuming that \( p'(x) \leq 0 \), \( G[y(x)] \) is bounded. Let \( \{x_n\}_{n=1}^{\infty} \) be the sequence of maximum points of \( u''(x) \). Then \( u''''(x_n) \leq u''''(x_n) + p(x_n)u''(x_n) = G[u(x_n)] \). But since \( u + N \) is oscillatory and \( N'' \) is unbounded, \( u'''' \) is unbounded, contradicting the boundedness of \( G[y(x)] \). The second part of the conclusion follows from Theorem 4.

If \( p'(x) \geq 0 \), Lazer and Hastings [3] have shown that all oscillatory solutions are bounded. The results then follow from the above theorems.

Whether or not the conclusion of Theorem 5 is true without the monotone condition on \( p \) is an open question.

We conclude with the following observation.

**THEOREM 6.** If \( n(x) \) is a solution of (6) satisfying the conditions of Theorem 1 where (6) is oscillatory, then \( \lim_{x \to \infty} n(x) = 0 \)

**Proof.** Equation (6) is oscillatory if and only if

(7) \( (y'/n^2)' + (2n''/n^3)y = 0 \)

is oscillatory. But, as in [6] it can be shown that \( \lim_{x \to \infty} x^n n''(x) = 0 \). Thus if \( \lim_{x \to \infty} n(x) = c > 0 \) (7) is nonoscillatory.

**REFERENCES**


Received November 13, 1975

MURRAY STATE UNIVERSITY
Pacific Journal of Mathematics
Vol. 63, No. 1 March, 1976

Ralph Artino, Gevrey classes and hypoelliptic boundary value problems .......... 1
B. Aupetit, Caractérisation spectrale des algèbres de Banach commutatives ...... 23
Leon Bernstein, Fundamental units and cycles in the period of real quadratic number fields. I ............................................................. 37
Leon Bernstein, Fundamental units and cycles in the period of real quadratic number fields. II ...................................................... 63
Robert F. Brown, Fixed points of automorphisms of compact Lie groups ....... 79
Thomas Ashland Chapman, Concordances of noncompact Hilbert cube manifolds ................................................................. 89
William C. Connett, V and Alan Schwartz, Weak type multipliers for Hankel transforms ......................................................... 125
John Wayne Davenport, Multipliers on a Banach algebra with a bounded approximate identity .................................................. 131
Gustave Adam Efroymson, Substitution in Nash functions .......................... 137
John Sollion Hsia, Representations by spinor genera .................................. 147
William George Kitto and Daniel Eliot Wulbert, Korovkin approximations in $L_p$-spaces ......................................................... 153
Eric P. Kronstadt, Interpolating sequences for functions satisfying a Lipschitz condition ............................................................... 169
Gary Douglas Jones and Samuel Murray Rankin, III, Oscillation properties of certain self-adjoint differential equations of the fourth order .......... 179
Takashi Kusano and Hiroshi Onose, Nonoscillation theorems for differential equations with deviating argument .................................. 185
David C. Lantz, Preservation of local properties and chain conditions in commutative group rings ........................................... 193
Charles W. Neville, Banach spaces with a restricted Hahn-Banach extension property ............................................................... 201
Norman Oler, Spaces of discrete subsets of a locally compact group ............ 213
Robert Olin, Functional relationships between a subnormal operator and its minimal normal extension ........................................... 221
Thomas Thornton Read, Bounds and quantitative comparison theorems for nonoscillatory second order differential equations .................... 231
Robert Horace Redfield, Archimedean and basic elements in completely distributive lattice-ordered groups ..................................... 247
Jeffery William Sanders, Weighted Sidon sets ............................................ 255
Aaron R. Todd, Continuous linear images of pseudo-complete linear topological spaces ......................................................... 281
J. Jerry Uhl, Jr., Norm attaining operators on $L^1[0, 1]$ and the Radon-Nikodým property ................................................................. 293
William Jennings Wickless, Abelian groups in which every endomorphism is a left multiplication .............................................. 301