BANACH SPACES WITH A RESTRICTED HAHN-BANACH
EXTENSION PROPERTY

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We shall study the class of real Banach spaces \( B \) with the following restricted Hahn-Banach extension property: For each Banach space \( C \) with a dense set of cardinality \( \leq \) some fixed cardinal \( \mathfrak{R} \), and for each subspace \( A \) of \( C \) and bounded linear map \( T_0: A \to B \), there exists an extension \( T: C \to B \) such that \( \| T \| = \| T_0 \| \). Suprisingly, there exist Banach spaces in this class which are not isometrically isomorphic to \( C(X) \) for a compact Hausdorff \( X \).

The combined results of Goodner, Hasumi, Kelley and Nachbin show that those Banach spaces with the Hahn-Banach extension property, that is, those Banach spaces which are injective in the category \( \mathscr{B} \) of Banach spaces and linear maps of norm \( \leq 1 \), are precisely the Banach spaces of the form \( C(X) \), where \( X \) is compact Hausdorff and extremally disconnected \([5],[6],[7],[11]\). In this paper, we wish to study those Banach spaces which enjoy a restricted Hahn-Banach extension property, where the existence of an extension is only required for spaces which are relatively small.

To be more precise, let \( \mathfrak{R} \) be an infinite cardinal. We shall say that a Banach space \( C \) is \( \mathfrak{R} \)-separable if \( C \) has a dense subset of cardinality \( \mathfrak{R} \). As usual, the word "separable" standing alone means \( \mathfrak{R}_0 \)-separable. We shall call a Banach space \( B \) \( \mathfrak{R} \)-injective if \( B \) has the following restricted Hahn-Banach extension property: Let \( C \) be an \( \mathfrak{R} \)-separable Banach space, let \( A \) be a subspace of \( C \), let \( i: A \hookrightarrow C \) be the inclusion map, and let \( T_0: A \to B \) be a bounded linear map. Then there exists a bounded linear map \( T \) with \( \| T \| = \| T_0 \| \), making the following diagram commute:

\[
\begin{array}{ccc}
A & \hookrightarrow & C \\
\downarrow \scriptstyle T_0 & & \downarrow \scriptstyle T \\
B & & \\
\end{array}
\]

We shall study the \( \mathfrak{R} \)-injective Banach spaces in this paper. We shall only consider real Banach spaces here. We shall characterize the Banach spaces of type \( C(X) \) which are \( \mathfrak{R} \)-injective. We shall also show that there are a good many other \( \mathfrak{R} \)-injective Banach spaces! Finally, we shall show that if an \( \mathfrak{R} \)-injective Banach space also happens to be \( \mathfrak{R} \)-separable, then it is in fact injective in the full category \( \mathscr{B} \). This contrasts rather sharply with the situation.
in the category $\mathcal{B}$ of Banach spaces and bounded linear maps. Sobczyk showed that $c_0$ satisfies diagram 1, with $c_0 = B$ and $R = R_0$, except that the extension $T$ may have a larger norm than $T_*$. On the other hand, Phillips showed that there is no continuous linear projection of $m$ onto $c_0$, so $c_0$ is not injective in the full category $\mathcal{B}$ (cf. [3, p. 25], [13]).

\textbf{\(\mathcal{R}\)-Injectives of Type \(C(X)\).}

First, let us prove a theorem which will enable us to characterize the \(\mathcal{R}\)-injective spaces of type \(C(X)\). To motivate this theorem, the reader should recall the Stone-Nakano theorem, which says, among other things, that a compact Hausdorff space $X$ is extremally disconnected if and only if $C(X)$ is a boundedly complete vector lattice under the usual ordering [12]. Thus, the Goodner-Hasumi-Kelley-Nachbin theorem may be rephrased to assert that the injectives in $\mathcal{B}$ are exactly the $C(X)$'s which are also boundedly complete vector lattices. It is thus not surprising that a property similar to lattice completeness would play a role in the study of $\mathcal{R}$-injectives. We shall say that an ordered normed linear space $B$ satisfies condition $a_n$ if the following is true:

\hspace{1cm} (a_n) For each $\mathcal{R}$-separable subspace $V$ of $B$, the following is true: Given a subset $\mathcal{F}$ of $V$ which is bounded above in norm by $m$ and of cardinality $\leq \mathcal{R}$, there exists at least one $b \in B$ such that $\| b \| \leq m$, $f \leq b$ for all $f \in \mathcal{F}$, and if $v \in V$ and $f \leq v$ for all $f \in \mathcal{F}$, then $b \leq v$.

\textbf{Theorem 1.} Let $X$ be a compact Hausdorff space and let $B$ be a closed subspace of $C(X)$. Then $B$ is $\mathcal{R}$-injective if $B$ satisfies property $a_n$.

Conversely, suppose $B$ is $\mathcal{R}$-injective, and suppose $B$ contains a subset $Q$ with the following properties. $Q$ consists of nonnegative functions none of which are identically 0, $Q$ contains a dense set of cardinality $\leq \mathcal{R}$, and the set of points at which the function in $Q$ attain their suprema is dense in $X$. Then $B$ satisfies condition $a_n$.

\textbf{Proof.} First suppose that $B$ satisfies condition $a_n$. The proof that $B$ is $\mathcal{R}$-injective follows Goodner's idea of replacing real valued sublinear functionals with $C(X)$ valued sublinear functionals in Banach's original proof of the Hahn-Banach theorem [3, pp. 135-137], [5]. Let $A$ be a subspace of $C$, let $C$ be $\mathcal{R}$-separable, and let $T_i: A \to B$ be a bounded linear map. Let $\phi: C \to B$ be defined as follows: Let $\mathcal{H}$ be a dense subset of cardinality $\leq \mathcal{R}$ of the unit ball of $A$. We know that the set $T_i(\mathcal{H})$ has cardinality $\leq \mathcal{R}$ and
is bounded above in norm by $\|T_0\|$. Clearly $T_0(A)$ is $\mathcal{R}$-separable. By condition $a_\mathcal{R}$, there exists $u \in B$ which bounds $T_0(\mathcal{K}) \cup \{0\}$ from above, and which satisfies $\|u\| \leq \|T_0\|$. Because $\mathcal{K}$ is dense in the unit ball of $A$, we have $T_0(c) \leq \|c\| \|u\|$ for all $c \in C$. Let $p(c) = \|c\| \|u\|$. Then $p$ is a sublinear map, and $T_0$ is dominated by $p$. Furthermore, if $S$ is any linear map from a subspace of $C$ to $B$ which is dominated by $p$, then $S$ is continuous. In fact, $S(c) \leq \|c\| \|u\|$ and $-S(c) = S(-c) \leq \|c\| \|u\|$, so $\|S\| \leq \|T_0\|$.

Now suppose $A'$ is a proper subspace of $C$ containing $A$, and suppose $T'$ is an extension of $T_0$ to $A'$ which is dominated by $p$. Let $\mathcal{H}'$ be a dense subset of the unit ball of $A'$ of cardinality $\leq \mathcal{R}$. Let $z \in C \sim A'$. As in Banach's proof of the Hahn-Banach theorem, we obtain for each $(x, y) \in A' \times A'$, $-p(-y - z) - T'(y) \leq p(x + z) - T'(x)$. Let $V = \text{the linear hull of } \{-p(-y - z) - T'(y); y \in A'\} \cup \{p(x + z) - T'(x); x \in A'\}$. The continuity of $p$ and $T'$ together with the $\mathcal{R}$-separability of $A'$ implies that $V$ is also $\mathcal{R}$-separable. We would like to apply condition $a_\mathcal{R}$ to a set $\mathcal{F} = \{-p(-y - z) - T'(y); y \in \text{some dense set in } A'\}$ to obtain the existence of $c \in B$, such that $-p(-y - z) - T'(y) \leq c \leq p(x + z) - T'(x)$ for all $(x, y) \in A' \times A'$. But such a set $\mathcal{F}$ would not be bounded in norm, so we shall consider a sequence of sets $\mathcal{F}_n = \{-p(-y - z) - T'(y); y \in n\mathcal{K}, n = 1, 2, \cdots\}$. Since $\mathcal{H}$ is a subset of the unit ball of $A'$, each set $\mathcal{F}_n$ is bounded in norm. By condition $a_\mathcal{R}$ applied to $\mathcal{F}_n$ and $V$, there exists $c_n \in B$ such that $w \leq c_n \leq p(x + z) - T'(x)$ for each $w \in \mathcal{F}_n$ and $x \in A'$. Since $\mathcal{H}$ is dense in $U$, the unit ball of $A'$, we have $-p(-y - z) - T'(y) \leq c_n \leq p(x + z) - T'(x)$ for each $x \in A'$ and $y \in n U$. Let $W = \text{the linear hull of } V$ and $\{c_n; n = 1, 2, \cdots\}$. Pick $x_0 \in A'$ and $y_0 \in U$. Since $U \subseteq n U$, $-p(-y_0 - z) - T'(y_0) \leq c_n \leq p(x_0 + z) - T'(x_n)$ for $n = 1, 2, \cdots$, so the set $\mathcal{C} = \{c_n; n = 1, 2, \cdots\}$ is bounded in norm. Clearly $\mathcal{C}$ is $\mathcal{R}$-separable, so by condition $a_\mathcal{R}$ applied to $\mathcal{C}$ and $W$, there exists $c \in B$ such that $c_n \leq c \leq p(x + z) - T'(x)$ for each $n$ and each $x \in A'$. Hence $-p(-y - z) - T'(y) \leq c \leq p(x + z) - T'(x)$ for all $(x, y) \in A' \times A'$. The rest of the proof now follows from Zorn’s lemma or transfinite induction exactly as in Banach’s original proof (cf. [3, p. 10]).

Conversely, suppose that $B$ is $\mathcal{R}$-injective, and that $Q \subseteq B$ contains a dense subset of cardinality $\leq \mathcal{R}$ and consists of nonnegative elements. Let $Y \subseteq X$ be the set of points at which the elements of $Q$ attain their suprema, and suppose $Y$ is dense in $X$. We wish to show that $B$ satisfies condition $a_\mathcal{R}$. Let $V$ be an $\mathcal{R}$-separable subspace of $B$, and let $\mathcal{F}$ be a subset of $V$ of cardinality $\leq \mathcal{R}$ which is bounded above by $m$ in norm. Let $f \in l^\infty(X)$ be the pointwise sup-
remum of the set $\mathcal{F}$. Then $\|f\|_\infty \leq m$. Let $A$ be the closed linear hull of $V \cup Q$, let $C$ be the linear hull of $A$ and $\{f\}$ in $l^\infty(X)$, and consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i} & C \\
\downarrow{I} & & \downarrow{P} \\
B & \xrightarrow{\bar{P}} & \end{array}
\]

Here, $I$ is the inclusion map and $P$ is the map guaranteed by $\mathcal{R}$-injectivity. We assert that $b = P(f)$ is the element whose existence is required by condition $a_\mathcal{R}$.

Clearly it suffices to show that the map $P$ is positive. For then, if $v \in \mathcal{F}$, we have $v \leq f$, which implies that $v = P(v) \leq P(f)$. Also, if $v \in V$ and $v$ is an upper bound for $\mathcal{F}$, then $f \leq v$, which implies $P(f) \leq P(v) = v$. Finally $\|P(f)\| \leq \|f\|_\infty \leq m$. But it is easy to show that $P$ is positive. Let $c \in C$ be $\geq 0$. Let $y \in Y$. There exists at least one $q \in Q$ such that $q \geq 0$, and $q(y) = \|q\| > 0$. Let $\lambda > 0$ be such that $\|\lambda q\| = \|c\|$. Then $\|c\| \geq \|\lambda q - c\| \geq \|P(\lambda q - c)\| = \|\lambda q - P(c)\| \geq \lambda q(y) - P(c)(y) = \|c\| - P(c)(y)$. Thus $P(c) \geq 0$ on $Y$. Since $P(c) \in C(X)$ and $Y$ is dense in $X$, we have $P(c) \geq 0$ on $X$. This concludes the proof of Theorem 1.

If we pick $B = C(X)$ and $Q = \{1\}$ in Theorem 1, we immediately have

**Corollary 1.** $C(X)$ is $\mathcal{R}$-injective if and only if $C(X)$ satisfies condition $a_\mathcal{R}$.

Clearly, $C(X)$ satisfies condition $a_\mathcal{R}$ for every cardinal $\mathcal{R}$ if and only if $C(X)$ is a boundedly complete vector lattice. Thus we have another proof of the following result of Goodner and Nachbin [5], [11]:

**Corollary 2.** $C(X)$ is injective in the category $\mathcal{B}_i$ if and only if $C(X)$ is a boundedly complete vector lattice. Equivalently, using the Stone-Nakano theorem, $C(X)$ is injective in the category $\mathcal{B}$, if and only if the compact Hausdorff space $X$ is extremally disconnected.

Of course, if $C(X)$ is a boundedly $N$-complete vector lattice, that is, if every set in $C(X)$ of cardinality $\leq \mathcal{R}$ which is bounded above has a least upper bound, then $C(X)$ clearly satisfies condition $a_\mathcal{R}$. From the Stone-Nakano theorem again, we know that $C(X)$ is boundedly $\mathcal{R}$-complete if and only if $X$ is both totally disconnected and
\( R \)-disconnected. Here, \( R \)-disconnected means the closure of a union of at most \( R \) clopen sets is again clopen, and the if part is true because we assume \( X \) is compact Hausdorff [11], [12]. Thus we have

**Corollary 3.** \( C(X) \) is \( R \)-injective if it is a boundedly \( R \)-complete vector lattice. Equivalently, \( C(X) \) is \( R \)-injective if the compact Hausdorff space \( X \) is both totally disconnected and \( R \)-disconnected.

We do not know whether there exist spaces \( C(X) \) which satisfy condition \( a_\alpha \), but which are not boundedly \( R \)-complete vector lattices. We conjecture that there are such spaces.

Other examples of \( R \)-injectives. Kelley showed that every injective in the category \( \mathcal{R} \) is of type \( C(X) \) [7]. This is not true for \( R \)-injectives! We with to give a general example of a proper subspace of \( C(X) \) which is \( N \)-injective, but which is not of type \( C(Z) \) for any compact Hausdorff \( Z \). To do this, we shall need an example of a compact Hausdorff space \( X \) with special properties. We shall call a point \( p \) of a topological space \( Y \) an \( R \, \! - \! \! - \text{P} \) point if the intersection of \( R \)-neighborhoods of \( p \) is again a neighborhood of \( p \). The standard name for an \( R \, \! - \! \! - \text{P} \) point is just \( P \)-point (cf. [4, Chapter 4]).

First, let us manufacture an example of a compact Hausdorff space \( X \) which contains a nonisolated \( R \, \! - \! \! - \text{P} \) point \( x_0 \), and which is totally disconnected and \( R \)-disconnected, but not extremally disconnected. For \( R = R_\alpha \), such examples are fairly ubiquitous in point set topology, and may be manufactured from the Stone space of suitable Boolean algebras, or by other means (cf. [2] for a recent and interesting example). The example we shall give is taken from Gillman and Jerison [4, Exercises 4N, 6M], who unfortunately only consider the case \( R = R_\alpha \). We shall consider the case of general \( R \), which introduces slight additional difficulties. We want to thank A. Hager for suggesting this and other examples of nonisolated \( R \, \! - \! \! - \text{P} \) points.

Let \( Y \) be a set of cardinality \( > R \). Let \( x_0 \in Y \). Topologize \( Y \) as follows: A set \( U \subseteq Y \) is open if \( x_0 \notin U \), or if \( x_0 \in U \) and \( Y \sim U \) has cardinality \( \leq R \). This amounts to giving \( Y \sim \{x_0\} \) the discrete topology, and letting neighborhoods of \( x_0 \) be complements of sets of cardinality \( \leq R \). Clearly a set \( F \) is closed in \( Y \) if \( x_0 \notin F \), or if \( x_0 \in F \) and the cardinality of \( F \) is \( \leq R \). From this, we see immediately that \( Y \) is normal. Clearly \( x_0 \) is a nonisolated \( R \, \! - \! \! - \text{P} \) point of \( Y \). To see that \( Y \) is not extremally disconnected, let \( U \) be a subset of \( Y \) such that \( x_0 \notin U \) and both \( U \) and \( Y \sim U \) have cardinality \( > R \). Then \( U \) is open, and the smallest closed set containing \( U \) is \( U \cup \{x_0\} \). But the closure of \( U, U \cup \{x_0\}, \) is not open, so \( Y \) is not extremally discon-
nected. By the Stone-Nakano theorem, $C_b(Y)$, the bounded continuous functions on $Y$, do not form a boundedly complete vector lattice.

On the other hand, it is easy to show directly that $C_b(Y)$ is a boundedly $\mathcal{R}$-complete vector lattice. Clearly, $f: Y \to \mathbb{R}$ is continuous if and only if, given $\varepsilon > 0$, there exists a set $F_\varepsilon$ of cardinality $\leq \mathcal{R}$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in F_\varepsilon$. Suppose $\mathcal{F} \subseteq C_b(Y)$ has cardinality $\leq \mathcal{R}$. Let $f$ be the pointwise supremum of $\mathcal{F}$. Let $\varepsilon > 0$. For each $h \in \mathcal{F}$, let $F_{h,\varepsilon}$ be a set of cardinality $\leq \mathcal{R}$ such that $|h(x) - h(x_0)| < \varepsilon$ for $x \in F_{h,\varepsilon}$. Let $F_\varepsilon = \bigcup (F_{h,\varepsilon}: h \in \mathcal{F})$. The $F_\varepsilon$ has cardinality $\leq \mathcal{R}$, and if $x \in F_\varepsilon$, then $|h(x) - h(x_0)| \leq \varepsilon$ for all $h \in \mathcal{F}$, so $|f(x) - f(x_0)| \leq \varepsilon$ as well. Thus $f$ is continuous, and so $C_b(Y)$ is boundedly $\mathcal{R}$-complete.

Let $X$ be the Stone-Cech compactification of $Y$. Then $C(X)$ is isometrically isomorphic, and isomorphic as an ordered Banach space, to $C_b(Y)$. Thus $C(X)$ is not boundedly complete, so $X$ is not extremally disconnected. On the other hand, $C(X)$ is boundedly $\mathcal{R}$-complete, so $X$ is totally disconnected and $\mathcal{R}$-disconnected. Finally $Y$ is dense in $X$, so $x_0$, being an $\mathcal{R} - P$ point of $Y$, is also an $\mathcal{R} - P$ point of $X$. Clearly, $x_0$ is not an isolated point of $X$.

By taking finite disjoint unions of copies of $X$, we may construct compact Hausdorff spaces with at least $n$ nonisolated $\mathcal{R} - P$ points, which are totally disconnected and $\mathcal{R}$-disconnected, but not extremally disconnected. Incidentally, $C(X)$ is a good example of a Banach space which is $\mathcal{R}$-injective, but not injective in $\mathcal{B}$. So is $C_0(X) = \{f \in C(X): f(x_0) = 0\}$! Because $x_0$ is an $\mathcal{R} - P$ point, $C_0(X)$ is a boundedly $\mathcal{R}$-complete vector sublattice of $C(X)$, and so satisfies condition $a_\mathcal{R}$. Hence, by Theorem 1, it is $\mathcal{R}$-injective. However, because $x_0$ is not isolated, $C_b(X)$ is not isometrically isomorphic to any $C(Z)$, and thus by the Goodner-Hasumi-Kelley-Nachbin theorem is not injective in $\mathcal{B}$. We shall not go into greater detail, because this example will be subsumed under the promised general example, which we shall now give in the form of a theorem:

**Theorem 2.** Let $X$ be a compact Hausdorff space which is totally disconnected and $\mathcal{R}$-disconnected, and which contains $n$ nonisolated $\mathcal{R} - P$ points, $x_1, \ldots, x_n$. Let $x_0 \in X$, let $c_1, \ldots, c_n \in [-1, 1]$, and let $B = \{f \in C(X): f(x_i) = c_i, f(x_0), i = 1, \ldots, n\}$. Assume $n \geq 1$ and $x_0, x_i, x_n$ are all distinct. Then $B$ is $\mathcal{R}$-injective. However if at least one $c_i \neq -1$, then $B$ is not isometrically isomorphic to $C(Z)$ for any compact Hausdorff space $Z$.

**Proof.** The key to the proof is the fact that for any $f \in C(X)$ and $c \in \mathbb{R}$, $\{x: f(x) = c\}$ is a $G_\delta$. Thus $f$ is constant in a neighborhood
of any $P$ point of $X$. Let $C$ be $\mathfrak{R}$-separable and let $A$ be a subspace of $C$. Let $T_0: A \to B$ be a bounded linear map. Let $V$ be the closure of $T_0(A)$, and let $\mathcal{N}$ be a dense subset of $V$ of cardinality $\leq \mathfrak{R}$. For $k \in \mathcal{N}$, let $G_{i,k}$ be a neighborhood of the $\mathfrak{R} - P$ point $x_i$ in which $k$ is constant. Since there are at most $\mathfrak{R} G_{i,k}$'s, the set $\bigcap \{G_{i,k}: k \in \mathcal{N}\}$ is also a neighborhood of $x_i$. For $i = 1, 2, \ldots, n$, let $G_i$ be a clopen neighborhood of $x_i$ contained in $\bigcap \{G_{i,k}: k \in \mathcal{N}\}$, and assume $x_0 \notin G_i$ and $G_i \cap G_j = \emptyset$ for $i \neq j$. Since $\mathcal{N}$ is dense in $V$, not only is each $G_i$ constant on $G_i$, but so is each $f \in \mathcal{N}$ constant on $G_i$, but so is each $f \in V$.

Let $Y = X \sim \bigcup_{i=1}^{n} G_i$ and let $r: C(X) \to C(Y)$ be the restriction map. Clearly $Y$ is not only compact, but also open in $X$. Thus $Y$ is both totally disconnected and $\mathfrak{R}$-disconnected, and hence $C(Y)$ is boundedly $\mathfrak{R}$-complete and hence is $\mathfrak{R}$-injective. Consequently, $r T_0: A \to C(Y)$ has an extension $S: C \to C(Y)$ with the same norm. Define $T: C \to B$ as follows: $T(c)(y) = S(c)(y)$ on $Y$, and $T(c)(x) = c S(c)(x_0)$ on $G_i$. Since each $G_i$ is clopen, $T(c) \in C(X)$. Clearly $T(C) \subseteq B$, $T$ is linear, and $\|T(c)\| = \|S(c)\|$, so $\|T\| = \|T_0\|$. Finally, if $a \in A$, then $T(a) = T_0(a)$, since $T_0(a)$ is constant on each $G_i$. Thus $B$ is $\mathfrak{R}$-injective.

We still must show that $B$ is not isomorphic to any $C(Z)$ if some $c_i \neq -1$. Let $\mathcal{E}$ be the set of extreme points of the unit ball of $B^*$, the dual of $B$, and endow $\mathcal{E}$ with the weak * topology. Be renumbering the $x_i$ if necessary, we may assume $\{x_i: c_i = -1\} = \emptyset$ or $\{x_{p+1}, \ldots, x_n\}$. In the former case, set $p = n$. Then $\mathcal{E}$ is homeomorphic to the union of two disjoint copies of $X \sim \{x_i, \ldots, x_p\}$ (corresponding to ± point evaluations) with $x_i$ in one copy identified with $\{x_{p+1}, \ldots, x_n\}$ in the other copy and vice-versa, if $p < n$. Since $x_i, \ldots, x_p$ are not isolated points of $X$, $\mathcal{E}$ is not compact. Therefore, $B$ cannot be isometrically isomorphic to a $C(Z)$ [3, p. 113].

The reader should note that if some of the $c_i$'s are $< 0$, then $B$ is not even a sublattice of $C(X)$. Actually, we can say even more. If $Z$ is a compact Hausdorff space and $\sigma: Z \to Z$ is a homeomorphism such that $\sigma^2$ is the identity map, then $C_\sigma(Z) = \{f \in C(Z): f \circ \sigma = -f\}$. If for some $c_{i_0}, c_{i_0} \neq 0$ and $-1 < c_i < 1$, then $B$ is not even isometrically isomorphic to any $C_\sigma(Z)$! For the set $S$ of extreme points of the dual unit ball of $B$ which are in minimal facets of the dual unit ball is clearly all of $\mathcal{E}$, and point evaluation at $x_{i_0}$ clearly lies in the weak * closure of $\mathcal{E}$. But for all $b \in B$, we have $b(x_i) < \|b\|$. Thus $B$ cannot be isomorphic to any $C_\sigma(Z)$ by a theorem of Jerison's [3, p. 121].

Each $\mathfrak{R}$-Injective is Almost of Type $C(X)$.

Despite the example we have just given, an $\mathfrak{R}$-injective Banach space is not too far removed from a space of type $C(X)$. First, the
spaces of type $C(X)$ share with the space of the example we have just given the property that their duals are isometrically isomorphic to a space of type $L'(\mu)$. (A long list of spaces which are preduals of spaces of type $L'(\mu)$ is given in [8, pp. 180–181].) I would like to thank Y. Benjamini and the referee for bringing the class of preduals of spaces of type $L'(\mu)$ to my attention.) A well known result of Lindenstrauss's states that each Banach space enjoying a finite dimensional extension property (which is much weaker than the extension property of $\mathcal{N}$-injectivity) is the predual of an $L'(\mu)$ space [9, Theorem 6.1]. Hence the $\mathcal{N}$-injective Banach space and the spaces of type $C(X)$ all belong to the rather large family of preduals of spaces of type $L'(\mu)$. But an $\mathcal{N}$-injective space is more closely related to the spaces of type $C(X)$ than this. In fact, it turns out that if $B$ is $\mathcal{N}$-injective, then $B$ is the direct limit of its $\mathcal{N}$-separable subspaces of type $C(X)$.

We may prove this, and more, essentially by means of a slight modification of Kelley's original proof that a Banach space which is injective in the category $\mathcal{S}$ is of type $C(X)$ [7]. In what follows, if $E$ is a Banach space, then $E^*$ shall denote its dual, and $U_E$ shall denote the closed unit ball of $E$. If $K$ is a convex subset of $E$, then $\text{ext } K$ shall denote the set of extreme points of $K$. If $Y \subseteq E$, then $\text{Cl } Y$ shall denote the closure of $Y$. The topology with respect to which the closure is taken will be specified whenever it is not clear from context. Finally, if $Y$ is a compact Hausdorff space and $y \in Y$, then $e_y \in C(Y)^*$ shall denote evaluation at the point $y$.

**Lemma 1.** Let $M$ and $N$ be Banach spaces, and let $S: M \to N$ be a linear map of norm $\leq 1$. Let $p$ be an extreme point of $U_N$, and let $L = S^{-1}(p) \cap U_M$. Then either $L = \emptyset$ or $L$ is a support of $U_M$.

Lemma 1 is a standard fact (cf. [7]).

**Theorem 3.** Let $B$ be an $\mathcal{N}$-injective Banach space. Let $A$ be an $\mathcal{N}$-separable subspace of $B$, and let $i: A \to B$ be the inclusion map. Let $W$ be a weak * relatively open subset of $\text{Cl ext } U_A$, such that $W \cap (-W) = \emptyset$ and $\text{Cl } (W \cup (-W)) = \text{Cl ext } U_A$. Let $Y = \text{Cl } W$. Here, the closures are taken with respect to the weak * topology. Endow $Y$ with the weak * topology, and let $j: A \to C(Y)$ be the natural isometric injection. Then there exists an isometric injection $p: C(Y) \to B$ such that $p \circ j = i$.

Before proving Theorem 3, three comments are in order. First, as Kelley observed, it is easy to produce such sets $W$: Simply apply Zorn's lemma to produce a set $W$ which is maximal with respect to...
the two properties \( W \cap (-W) = \emptyset \), and \( W \) is weak * open in \( \text{Cl ext } U_\alpha \). Second, from the Krein-Milman theorem, we know that if \( a \in A \), then \( \sup \{ y(a) : y \in U_\alpha \} \) is actually attained at some \( y \in \text{Cl ext } U_\alpha \). Since \( W \cap (-W) \) is dense in \( \text{Cl ext } U_\alpha \), it follows that \( j \) is an isometry. Finally, \( Y \) is clearly compact by the Alaoglu theorem.

**Proof.** First, we must show that \( C(Y) \) is \( \mathcal{A} \)-separable. Observe that \( j(A) \) is \( \mathcal{A} \)-separable, so the subalgebra in \( \mathcal{A} \) in \( C(Y) \) generated by \( j(A) \) and the function \( \equiv 1 \) is also \( \mathcal{A} \)-separable. But \( \mathcal{A} \) separates points of \( Y \) because \( A \) does. By the Stone-Weierstrass theorem, \( \mathcal{A} \) is dense in \( C(Y) \), so \( C(Y) \) is also \( \mathcal{A} \)-separable. From the \( \mathcal{A} \)-injectivity of \( B \), we conclude that there exists a linear map \( p : C(Y) \to B \) of norm 1 such that \( p \circ j = i \).

We will show that \( p \) is 1–1 by showing that its adjoint \( p^* : B^* \to C(Y)^* \) is onto. We assert that it suffices to show that \( \{ e_y : y \in W \cap \text{ext } U_\alpha \} \subseteq p^*(U_B) \). To see why, suppose this inclusion holds. We know \( p^* \) is weak * continuous and \( U_B \) is weak * compact, so \( p^*(U_B) \) is weak * compact and hence weak * closed in \( C(Y)^* \). Thus \( \text{Cl } \{ e_y : y \in W \cap \text{ext } U_\alpha \} \subseteq p^*(U_B) \). But the map \( y \to e_y \), \( y \in Y \), is a homeomorphism from \( Y \) onto the set \( \{ e_y : y \in Y \} \) endowed with the weak * topology. Furthermore, because \( W \) is an open subset of \( \text{Cl ext } U_\alpha \), we know that \( \text{Cl } (W \cap \text{ext } U_\alpha) = \text{Cl } W = Y \). Thus \( \{ e_y \in Y \} \subseteq p^*(U_B) \). From this, we conclude that \( \text{ext } U_{C(Y)} = \{ \pm e_y : y \in Y \} \), as well as the closed convex hull of \( \text{ext } U_{C(Y)} \), are contained in \( p^*(U_B) \). By the Krein-Milman theorem, \( U_{C(Y)} \subseteq p^*(U_B) \). Thus \( p^* \) is onto.

In fact, from this last inclusion, we may conclude that \( p \) is not only 1–1, but is actually an isometry. Suppose \( p \) were not an isometry. We know that \( \| p \| = 1 \), so there exists \( f \in C(Y) \) such that \( \| f \| = 1 \) and \( \| p(f) \| < 1 \). Let \( \mu \in C(Y)^* \) be a linear functional of norm 1 such that \( \mu(f) = 1 \). If \( \lambda \in U_B \), then \( |p^*(\lambda)(f)| = |\lambda(p(f))| \leq \| p(f) \| < 1 \), so \( \mu \in p^*(U_B) \). This is a contradiction.

We thus need only show that \( \{ e_y : y \in W \cap \text{ext } U_\alpha \} \subseteq p^*(U_B) \) in order to complete the proof of Lemma 2. We will do this by chasing the following commutative diagram of adjoint maps:

\[
\begin{array}{ccc}
A^* & \xleftarrow{j^*} & B^* \\
\Downarrow j^* & & \Downarrow p^* \\
C(Y)^* & \xrightarrow{1^*} & C(Y)^*
\end{array}
\]

Note that all of the maps involved have norm \( \leq 1 \). Let \( y \in Y \). Clearly \( j^*(e_y) = y \) for each \( y \in Y \). We assert that if \( y \in W \cap \text{ext } U_\alpha \),
then \( j^{*-1}(y) \cap U_{CA} = \{e_y\} \). Let \( L = j^{*-1}(y) \cap U_{CA} \). By Lemma 1, \( L \) is a support of \( U_{CA} \). \( L \) is closed in \( U_{CA} \) and hence is compact. By the Krein-Milman theorem, \( L \) has extreme points. Because \( L \) is a support of \( U_{CA} \), each extreme point \( w \) of \( L \) is also an extreme point of \( U_{CA} \), and hence is of the form \( \pm e_z \) for some \( z \in Y \). If \( w = -e_z \), then for each \( a \in A \) we have \( y(a) = j^*(-e_z(a)) = -e_z(j(a)) = -z(a) \), so \( y = -z \). Hence \( y \in W \cap (-C1 W) \). But \( W \cap (-C1 W) = \emptyset \), since \( W \cap (-W) = \emptyset \) and \( W \) is open. Thus \( w \) is not \( -e_z \) for any \( z \in Y \). If \( w = e_z \) for some \( z \in Y \), it is immediate that \( z = y \). Hence the only extreme point of \( L \) is \( e_y \). By the Krein-Milman theorem, \( L = \{e_y\} \).

Now let \( y \in W \cap \text{ext } U_{CA} \), and observe that \( i^{*-1}(y) \cap U_{B} \neq \emptyset \) by the Hahn-Banach theorem. Let \( z \in i^{*-1}(y) \cap U_{B} \). By the commutativity of diagram 2, \( j^*p^*(z) = i^*(z) = y \), so \( p^*(z) \in j^{*-1}(y) \cap U_{B} \), and hence \( p^*(z) = e_y \). Consequently, \( \{e_y : y \in W \cap \text{ext } U_{CA} \} \subseteq p^*(U_{B}) \), which completes the proof of Theorem 3.

**COROLLARY 1.** Suppose \( B \) is not only \( \mathcal{R} \)-injective, but also \( \mathcal{R} \)-separable. Then \( B \) is isometrically isomorphic to \( C(X) \), for some extremally disconnected compact Hausdorff space \( X \). Hence \( B \) is injective in the category \( \mathcal{B}_i \).

**Proof.** In the statement of Theorem 3, choose \( A = B \) and \( X = Y \). Then \( B \) is isometrically isomorphic to \( C(X) \). By Theorem 1, Corollary 1, \( C(X) \) satisfies condition \( a_\mathcal{R} \). But \( C(X) \) is \( \mathcal{R} \)-separable, so \( C(X) \) satisfies condition \( a_\mathcal{R} \) for every infinite cardinal \( \mathcal{R} \), and thus \( C(X) \) is a boundedly complete vector lattice. The remainder of the corollary now follows from Theorem 1, Corollary 2, and the Stone-Nakano theorem.

Corollary 1 may be rephrased to assert that if \( B \) is injective in the full subcategory of \( \mathcal{B}_i \) whose objects are all \( \mathcal{R} \)-separable spaces, then \( B \) is actually injective in \( \mathcal{B}_i \). As we mentioned in the introduction, Corollary 1 is interesting because the situation is dramatically different in the category \( \mathcal{B}_i \). It would be interesting to know if there are any nontrivial full subcategories of \( \mathcal{B}_i \) in which new injectives can arise. As we shall see in Corollary 5, the full subcategory \( \mathcal{C}_i \) of \( \mathcal{B}_i \), whose objects are all weakly compactly generated spaces, is not such a subcategory.

**COROLLARY 2.** \( B \) is injective in the category \( \mathcal{B}_i \) if and only if \( B \) is isometrically isomorphic to \( C(X) \), for some extremally disconnected compact Hausdorff space \( X \).
Proof. One way is Theorem 1, Corollary 2. Conversely, suppose $B$ is injective in $\mathcal{B}_i$. $B$ is $\mathcal{N}$-separable for some $\mathcal{N}$, and also $\mathcal{N}$-injective. Apply Corollary 1.

Corollary 2 is the full Goodner-Kelley-Nachbin characterization of injectives in $\mathcal{B}_i$ [5], [7], [11]. Of course, Corollary 2 is a bit of a cheat, because the proof of Theorem 3 is essentially Kelley's proof (slightly modified) that each injective in $\mathcal{B}_i$ is of type $C(X)$. However, the following corollary is somewhat more interesting because it may provide the first step toward a complete characterization of $\mathcal{N}$-injectives.

**Corollary 3.** Let $B$ be an $\mathcal{N}$-injective Banach space. Then $B$ is the direct limit of its $\mathcal{N}$-separable subspaces of type $C(Y)$.

Proof. It suffices to prove that $B$ is the union of such subspaces, and that if $A$ and $C$ are two such $\mathcal{N}$-separable subspaces, then there is an $\mathcal{N}$-separable subspace $D$ of type $C(Y)$ such that $A \cup C \subseteq D$. Both assertion follow immediately from Theorem 3.

I am indebted to Y. Benjamini for pointing out that Corollary 1 is actually strong enough to imply the following important result:

**Corollary 4.** There are no $\mathcal{N}$-injective, infinite dimensional, weakly compactly generated spaces.

Proof. Suppose $B$ is an $\mathcal{N}$-injective space which is also infinite dimensional and weakly compactly generated. Let $E$ be an infinite dimensional separable subspace of $B$. By a fundamental result of Lindenstrauss's, there exists a separable subspace $D \supseteq E$ and a projection $P: B \rightarrow D$ of norm 1 [10, pp. 170-171]. The existence of the projection $P$ guarantees that $D$ is $\mathcal{N}$-injective. By Corollary 1, $D$ is isometrically isomorphic to $C(X)$, where $X$ is extremally disconnected and compact Hausdorff. Since $C(X)$ is separable, $X$ is metrizable. Thus $X$ has finite cardinality, so $D = C(X)$ is finite dimensional, which is impossible.

**Corollary 5.** The only injectives in $\mathcal{B}_i$ are already injective in the larger category $\mathcal{B}_i$, and are in fact finite dimensional.

Proof. Let $B$ be an injective in $\mathcal{B}_i$. Since $\mathcal{B}_i$ contains every separable space, $B$ is $\mathcal{N}$-injective. Apply Corollary 4.

In conclusion, we would like to raise a question whose resolution probably awaits a complete characterization of $\mathcal{N}$-injectives. Cohen showed that every Banach space has an injective envelope in the
category $\mathcal{F}$ [1]. It would be interesting to know whether or not every Banach space also has an $\mathcal{F}$-injective envelope.

**REFERENCES**


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