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We shall study the class of real Banach spaces B with the following restricted Hahn-Banach extension property: For each Banach space C with a dense set of cardinality $\leqq$ some fixed cardinal $\mathfrak{N}$, and for each subspace A of C and bounded linear map $T_0\colon A \to B$, there exists an extension $T\colon C \to B$ such that $\|T\| = \|T_0\|$. Suprisingly, there exist Banach spaces in this class which are not isometrically isomorphic to $C(X)$ for a compact Hausdorff X!

The combined results of Goodner, Hasumi, Kelley and Nachbin show that those Banach spaces with the Hahn-Banach extension property, that is, those Banach spaces which are injective in the category $\mathscr{B}_1$ of Banach spaces and linear maps of norm $\leqq 1$, are precisely the Banach spaces of the form $C(X)$, where X is compact Hansdorff and extremally disconnected [5], [6], [7], [11]. In this paper, we wish to study those Banach spaces which enjoy a restricted Hahn-Banach extension property, where the existence of an extension is only required for spaces which are relatively small.

To be more precise, let $\mathfrak{N}$ be an infinite cardinal. We shall say that a Banach space C is $\mathfrak{N}$-separable if C has a dense subset of cardinality $\mathfrak{N}$. As usual, the word "separable" standing alone means $\mathfrak{N}_0$ separable. We shall call a Banach space B $\mathfrak{N}$-injective if B has the following restricted Hahn-Banach extension property: Let C be an $\mathfrak{N}$-separable Banach space, let A be a subspace of C, let $i\colon A \hookrightarrow C$ be the inclusion map, and let $T_0\colon A \to B$ be a bounded linear map. Then there exists a bounded linear map T with $\|T\| = \|T_0\|$, making the following diagram commute:

$$
\begin{tikzcd}
A \arrow[rr, hook, "i"] \arrow[dr, "T_0"'] & & C \arrow[dl, "T"] \\
& B &
\end{tikzcd}
\tag{1}
$$

We shall study the $\mathfrak{N}$-injective Banach spaces in this paper. We shall only consider real Banach spaces here. We shall characterize the Banach spaces of type $C(X)$ which are $\mathfrak{N}$-injective. We shall also show that there are a good many other $\mathfrak{N}$-injective Banach spaces! Finally, we shall show that if an $\mathfrak{N}$-injective Banach space also happens to be $\mathfrak{N}$-separable, then it is in fact injective in the full category $\mathscr{B}_1$. This contrasts rather sharply with the situation in the category $\mathscr{B}$ of Banach spaces and bounded linear maps. Sobczyk showed that c_0 satisfies diagram 1, with $c_0 = B$ and $\mathfrak{N} = \mathfrak{N}_0$, except that the extension T may have a larger norm than T_0. On the other hand, Phillips showed that there is no continuous linear projection of m onto c_0, so c_0 is not injective in the full category $\mathscr{B}$ (cf. [3, p. 25], [13]).

$\mathfrak{N}$-Injectives of Type $C(X)$.

First, let us prove a theorem which will enable us to characterize the $\mathfrak{N}$-injective spaces of type $C(X)$. To motivate this theorem, the reader should recall the Stone-Nakano theorem, which says, among other things, that a compact Hausdorff space X is extremally disconnected if and only if $C(X)$ is a boundedly complete vector lattice under the usual ordering [12]. Thus, the Goodner-Hasumi-Kelley-Nachbin theorem may be rephrased to assert that the injectives in $\mathscr{B}_1$ are exactly the $C(X)$'s which are also boundedly complete vector lattices. It is thus not surprising that a property similar to lattice completeness would play a role in the study of $\mathfrak{N}$-injectives. We shall say that an ordered normed linear space B satisfies condition $a_{\mathfrak{N}}$ if the following is true:

($a_{\mathfrak{N}}$) For each $\mathfrak{N}$-separable subspace V of B, the following is true: Given a subset $\mathscr{F}$ of V which is bounded above in norm by m and of cardinality $\leqq \mathfrak{N}$, there exists at least one $b \in B$ such that $\|b\| \leqq m$, $f \leqq b$ for all $f \in \mathscr{F}$, and if $v \in V$ and $f \leqq v$ for all $f \in \mathscr{F}$, then $b \leqq v$.

THEOREM 1. *Let X be a compact Hausdorff space and let B be a closed subspace of $C(X)$. Then B is $\mathfrak{N}$-injective if B satisfies property $a_{\mathfrak{N}}$.*

Conversely, suppose B is $\mathfrak{N}$-injective, and suppose B contains a subset Q with the following properties. Q consists of nonnegative functions none of which are identically 0, Q contains a dense set of carinality $\leqq \mathfrak{N}$, and the set of points at which the function in Q attain their suprema is dense in X. Then B satisfies condition $a_{\mathfrak{N}}$.

Proof. First suppose that B satisfies condition $a_{\mathfrak{N}}$. The proof that B is $\mathfrak{N}$-injective follows Goodner's idea of replacing real valued sublinear functionals with C(X) valued sublinear functionals in Banach's original proof of the Hahn-Banach theorem [3, pp. 135–137], [5]. Let A be a subspace of C, let C be $\mathfrak{N}$-separable, and let $T_0: A \rightarrow B$ be a bounded linear map. Let $p: C \rightarrow B$ be defined as follows: Let $\mathscr{K}$ be a dense subset of cardinality $\leqq \mathfrak{N}$ of the unit ball of A. We know that the set $T_0(\mathscr{K})$ has cardinality $\leqq \mathfrak{N}$ and is bounded above in norm by $\|T_0\|$. Clearly $T_0(A)$ is $\mathfrak{N}$-separable. By condition $a_{\mathfrak{N}}$, there exists $u \in B$ which bounds $T_0(\mathscr{K}) \cup \{0\}$ from above, and which satisfies $\|u\| \leqq \|T_0\|$. Because $\mathscr{K}$ is dense in the unit ball of A, we have $T_0(c) \leqq \|c\| u$ for all $c \in C$. Let $p(c) = \|c\| u$. Then p is a sublinear map, and T_0 is dominated by p. Furthermore, if S is any linear map from a subspace of C to B which is dominated by p, then S is continuous. In fact, $S(c) \leqq \|c\| u$ and $-S(c) = S(-c) \leqq \|c\| u$, so $\|S\| \leqq \|T_0\|$.

Now suppose A' is a proper subspace of C containing A, and suppose T' is an extension of T_0 to A' which is dominated by p. Let $\mathscr{K}$ be a dense subset of the unit ball of A' of cardinality $\leqq \mathfrak{N}$. Let $z \in C \sim A'$. As in Banach's proof of the Hahn-Banach theorem, we obtain for each $(x, y) \in A' \times A'$, $-p(-y - z) - T'(y) \leqq p(x + z) - T'(x)$. Let $V =$ the linear hull of $\{-p(-y - z) - T'(y): y \in A'\} \cup \{p(x + z) - T'(x): x \in A'\}$. The continuity of p and T' together with the $\mathfrak{N}$-separability of A' implies that V is also $\mathfrak{N}$-separable. We would like to apply condition $a_{\mathfrak{N}}$ to a set $\mathscr{F} = \{-p(-y - z) - T'(y): y \in$ some dense set in $A'\}$ to obtain the existence of $c \in B$, such that $-p(-y - z) - T'(y) \leqq c \leqq p(x + z) - T'(x)$ for all $(x, y) \in A' \times A'$. But such a set $\mathscr{F}$ would not be bounded in norm, so we shall consider a sequence of sets $\mathscr{F}_n = \{-p(-y - z) - T'(y): y \in n\mathscr{K}\}$, $n = 1, 2, \cdots$. Since $\mathscr{K}$ is a subset of the unit ball of A', each set $\mathscr{F}_n$ is bounded in norm. By condition $a_{\mathfrak{N}}$ applied to $\mathscr{F}_n$ and V, there exists $c_n \in B$ such that $w \leqq c_n \leqq p(x + z) - T'(x)$ for each $w \in \mathscr{F}_n$ and $x \in A'$. Since $\mathscr{K}$ is dense in U, the unit ball of A', we have $-p(-y - z) - T'(y) \leqq c_n \leqq p(x + z) - T'(x)$ for each $x \in A'$ and $y \in nU$. Let $W =$ the linear hull of V and $\{c_n: n = 1, 2, \cdots\}$. Pick $x_0 \in A'$ and $y_0 \in U$. Since $U \subseteqq nU$, $-p(-y_0 - z) - T'(y_0) \leqq c_n \leqq p(x_0 + z) - T'(x_0)$ for $n = 1, 2, \cdots$, so the set $\mathscr{C} = \{c_n: n = 1, 2, \cdots\}$ is bounded in norm. Clearly $\mathscr{C}$ is $\mathfrak{N}$-separable, so by condition $a_{\mathfrak{N}}$ applied to $\mathscr{C}$ and W, there exists $c \in B$ such that $c_n \leqq c \leqq p(x + z) - T'(x)$ for each n and each $x \in A'$. Hence $-p(-y-z)-T'(y) \leqq c \leqq p(x+z)-T'(x)$ for all $(x, y) \in A' \times A'$. The rest of the proof now follows from Zorn's lemma or transfinite induction exactly as in Banach's original proof (cf. [3, p. 10]).

Conversely, suppose that B is $\mathfrak{N}$-injective, and that $Q \subseteqq B$ contains a dense subset of cardinality $\leqq \mathfrak{N}$ and consists of nonnegative elements. Let $Y \subseteqq X$ be the set of points at which the elements of Q attain their suprema, and suppose Y is dense in X. We wish to show that B satisfies condition $a_{\mathfrak{N}}$. Let V be an $\mathfrak{N}$-separable subspace of B, and let $\mathscr{F}$ be a subset of V of cardinality $\leqq \mathfrak{N}$ which is bounded above by m in norm. Let $f \in l^\infty(X)$ be the pointwise supremum of the set $\mathscr{F}$. Then $\|f\|_\infty \leqq m$. Let A be the closed linear hull of $V \cup Q$, let C be the linear hull of A and $\{f\}$ in $l^\infty(X)$, and consider the commutative diagram

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "I"'] & & C \arrow[dl, "P"] \\ & B & \end{tikzcd}$$

Here, I is the inclusion map and P is the map guaranteed by $\mathfrak{N}$-injectivity. We assert that $b = P(f)$ is the element whose existence is required by condition $a_{\mathfrak{N}}$.

Clearly it suffices to show that the map P is positive. For then, if $v \in \mathscr{F}$, we have $v \leqq f$, which implies that $v = P(v) \leqq P(f)$. Also, if $v \in V$ and v is an upper bound for $\mathscr{F}$, then $f \leqq v$, which implies $P(f) \leqq P(v) = v$. Finally $\|P(f)\| \leqq \|f\|_\infty \leqq m$. But it is easy to show that P is positive. Let $c \in C$ be $\geqq 0$. Let $y \in Y$. There exists at least one $q \in Q$ such that $q \geqq 0$, and $q(y) = \|q\| > 0$. Let $\lambda > 0$ be such that $\|\lambda q\| = \|c\|$. Then $\|c\| \geqq \|\lambda q - c\| \geqq \|P(\lambda q - c)\| = \|\lambda q - P(c)\| \geqq \lambda q(y) - P(c)(y) = \|c\| - P(c)(y)$. Thus $P(c) \geqq 0$ on Y. Since $P(c) \in C(X)$ and Y is dense in X, we have $P(c) \geqq 0$ on X. This concludes the proof of Theorem 1.

If we pick $B = C(X)$ and $Q = \{1\}$ in Theorem 1, we immediately have

COROLLARY 1. *$C(X)$ is $\mathfrak{N}$-injective if and only if $C(X)$ satisfies condition $a_{\mathfrak{N}}$.*

Clearly, $C(X)$ satisfies condition $a_{\mathfrak{N}}$ for every cardinal $\mathfrak{N}$ if and only if $C(X)$ is a boundedly complete vector lattice. Thus we have another proof of the following result of Goodner and Nachbin [5], [11]:

COROLLARY 2. *$C(X)$ is injective in the category $\mathscr{B}_1$ if and only if $C(X)$ is a boundedly complete vector lattice. Equivalently, using the Stone-Nakano theorem, $C(X)$ is injective in the category $\mathscr{B}_1$ if and only if the compact Hausdorff space X is extremally disconnected.*

Of course, if $C(X)$ is a boundedly N-complete vector lattice, that is, if every set in $C(X)$ of cardinality $\leqq \mathfrak{N}$ which is bounded above has a least upper bound, then $C(X)$ clearly satisfies condition $a_{\mathfrak{N}}$. From the Stone-Nakano theorem again, we know that $C(X)$ is boundedly $\mathfrak{N}$-complete if and only if X is both totally disconnected and

$\mathfrak{N}$-disconnected. Here, $\mathfrak{N}$-disconnected means the closure of a union of at most $\mathfrak{N}$ clopen sets is again clopen, and the if part is true because we assume X is compact Hausdorff [11], [12]. Thus we have

COROLLARY 3. *$C(X)$ is $\mathfrak{N}$-injective if it is a boundedly $\mathfrak{N}$-complete vector lattice. Equivalently, $C(X)$ is $\mathfrak{N}$-injective if the compact Hausdorff space X is both totally disconnected and $\mathfrak{N}$-disconnected.*

We do not know whether there exist spaces $C(X)$ which satisfy condition $a_{\mathfrak{N}}$, but which are not boundedly $\mathfrak{N}$-complete vector lattices. We conjecture that there are such spaces.

Other examples of $\mathfrak{N}$-injectives. Kelley showed that every injective in the category $\mathscr{B}_1$ is of type $C(X)$ [7]. This is not true for $\mathfrak{N}$-injectives! We with to give a general example of a proper subspace of $C(X)$ which is N-injective, but which is not of type $C(Z)$ for any compact Hausdorff Z. To do this, we shall need an example of a compact Hausdorff space X with special properties. We shall call a point p of a topological space Y an $\mathfrak{N} - P$ point if the intersection of $\mathfrak{N}$ neighborhoods of p is again a neighborhood of p. The standard name for an $\mathfrak{N}_0 - P$ point is just P-point (cf. [4, Chapter 4]).

First, let us manufacture an example of a compact Hausdorff space X which contains a nonisolated $\mathfrak{N} - P$ point x_0, and which is totally disconnected and $\mathfrak{N}$-disconnected, but not extremally disconnected. For $\mathfrak{N} = \mathfrak{N}_0$, such examples are fairly ubiquitous in point set topology, and may be manufactured from the Stone space of suitable Booolean algebras, or by other means (cf. [2] for a recent and interesting example). The example we shall give is taken from Gillman and Jerison [4, Exercises 4N, 6M], who unfortunately only consider the case $\mathfrak{N} = \mathfrak{N}_0$. We shall consider the case of general $\mathfrak{N}$, which introduces slight additional difficulties. We want to thank A. Hager for suggesting this and other examples of nonisolated $\mathfrak{N} - P$ points.

Let Y be a set of cardinality $> \mathfrak{N}$. Let $x_0 \in Y$. Topologize Y as follows: A set $U \subseteqq Y$ is open if $x_0 \notin U$, or if $x_0 \in U$ and $Y \sim U$ has cardinality $\leqq \mathfrak{N}$. This amounts to giving $Y \sim \{x_0\}$ the discrete topology, and letting neighborhoods of x_0 be complements of sets of cardinality $\leqq \mathfrak{N}$. Clearly a set F is closed in Y if $x_0 \in F$, or if $x_0 \notin F$ and the cardinality of F is $\leqq \mathfrak{N}$. From this, we see immediately that Y is normal. Clearly x_0 is a nonisolated $\mathfrak{N} - P$ point of Y. To see that Y is not extremally disconnected, let U be a subset of Y such that $x_0 \notin U$ and both U and $Y \sim U$ have cardinality $> \mathfrak{N}$. Then U is open, and the smallest closed set containing U is $U \cup \{x_0\}$. But the closure of U, $U \cup \{x_0\}$, is not open, so Y is not extremally disconnected. By the Stone-Nakano theorem, $C_b(Y)$, the bounded continuous functions on Y, do not form a boundedly complete vector lattice.

On the other hand, it is easy to show directly that $C_b(Y)$ is a boundedly $\mathfrak{N}$-complete vector lattice. Clearly, $f\colon Y \to \boldsymbol{R}$ is continuous if and only if, given $\varepsilon > 0$, there exists a set F_ε of cardinality $\leqq \mathfrak{N}$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \notin F_\varepsilon$. Suppose $\mathscr{F} \subseteqq C_b(Y)$ has cardinality $\leqq \mathfrak{N}$. Let f be the pointwise supremum of $\mathscr{F}$. Let $\varepsilon > 0$. For each $h \in \mathscr{F}$, let $F_{h,\varepsilon}$ be a set of cardinality $\leqq \mathfrak{N}$ such that $|h(x) - h(x_0)| < \varepsilon$ for $x \notin F_{h,\varepsilon}$. Let $F_\varepsilon = \bigcup \{F_{h,\varepsilon}\colon h \in \mathscr{F}\}$. The F_ε has cardinality $\leqq \mathfrak{N}$, and if $x \notin F_\varepsilon$, then $|h(x) - h(x_0)| \leqq \varepsilon$ for all $h \in \mathscr{F}$, so $|f(x) - f(x_0)| \leqq \varepsilon$ as well. Thus f is continuous, and so $C_b(Y)$ is boundedly $\mathfrak{N}$-complete.

Let X be the Stone-Čech compactification of Y. Then $C(X)$ is isometrically isomorphic, and isomorphic as an ordered Banach space, to $C_b(Y)$. Thus $C(X)$ is not boundedly complete, so X is not extremally disconnected. On the other hand, $C(X)$ is boundedly $\mathfrak{N}$-complete, so X is totally disconnected and $\mathfrak{N}$-disconnected. Finally Y is dense in X, so x_0, being an $\mathfrak{N} - P$ point of Y, is also an $\mathfrak{N} - P$ point of X. Clearly, x_0 is not an isolated point of X.

By taking finite disjoint unions of copies of X, we may construct compact Hausdorff spaces with at least n nonisolated $\mathfrak{N} - P$ points, which are totally disconnected and $\mathfrak{N}$-disconnected, but not extremally disconnected. Incidentally, $C(X)$ is a good example of a Banach space which is $\mathfrak{N}$-injective, but not iujective in $\mathscr{B}_1$. So is $C_0(X) = \{f \in C(X)\colon f(x_0) = 0\}$! Because x_0 is an $\mathfrak{N} - P$ point, $C_0(X)$ is a boundedly $\mathfrak{N}$-complete vector sublattice of $C(X)$, and so satisfies condition $a_{\mathfrak{N}}$. Hence, by Theorem 1, it is $\mathfrak{N}$-injective. However, because x_0 is not isolated, $C_0(X)$ is not isometrically isomorphic to any $C(Z)$, and thus by the Goodner-Hasumi-Kelley-Nachbin theorem is not injetive in $\mathscr{B}_1$. We shall not go into greater detail, because this example will be subsumed under the promised general example, which we shall now give in the form of a theorem:

THEOREM 2. *Let X be a compact Hausdorff space which is totally disconnected and $\mathfrak{N}$-disconnected, and which contains n nonisolated $\mathfrak{N} - P$ points, $x_1, \cdots, x_n$. Let $x_0 \in X$, let $c_1, \cdots, c_n \in [-1, 1)$, and let $B = \{f \in C(X)\colon f(x_i) = c_i f(x_0), i = 1, \cdots, n\}$. Assume $n \geqq 1$ and x_0, x_1, x_n are all distinct. Then B is $\mathfrak{N}$-injective. However if at least one $c_i \neq -1$, then B is not isometrically isomorphic to $C(Z)$ for any compact Hausdorff space Z.*

Proof. The key to the proof is the fact that for any $f \in C(X)$ and $c \in \boldsymbol{R}$, $\{x\colon f(x) = c\}$ is a G_δ. Thus f is constant in a neighborhood of any P point of X. Let C be $\mathfrak{N}$-separable and let A be a subspace of C. Let $T_0: A \to B$ be a bounded linear map. Let V be the closure of $T_0(A)$, and let $\mathscr{K}$ be a dense subset of V of cardinality $\leqq \mathfrak{N}$. For $k \in \mathscr{K}$, let $G_{i,k}$ be a neighborhood of the $\mathfrak{N} - P$ point x_i in which k is constant. Since there are at most $\mathfrak{N}$ $G_{i,k}$'s, the set $\bigcap \{G_{i,k}: k \in \mathscr{K}\}$ is also a neighborhood of x_i. For $i = 1, 2, \cdots, n$, let G_i be a clopen neighborhood of x_i contained in $\bigcap \{G_{i,k}: k \in \mathscr{K}\}$, and assume $x_0 \notin G_i$ and $G_i \cap G_j = \emptyset$ for $i \neq j$. Since $\mathscr{K}$ is dense in V, not only is each $f \in \mathscr{K}$ constant on G_i, but so is each $f \in V$.

Let $Y = X \sim \bigcup_{i=1}^n G_i$ and let $r: C(X) \to C(Y)$ be the restriction map. Clearly Y is not only compact, but also open in X. Thus Y is both totally disconnected and $\mathfrak{N}$-disconnected, and hence $C(Y)$ is boundedly $\mathfrak{N}$-complete and hence is $\mathfrak{N}$-injective. Consequently, $rT_0: A \to C(Y)$ has an extension $S: C \to C(Y)$ with the same norm. Define $T: C \to B$ as follows: $T(c)(y) = S(c)(y)$ on Y, and $T(c)(x) = c_i S(c)(x_0)$ on G_i. Since each G_i is clopen, $T(c) \in C(X)$. Clearly $T(C) \subseteqq B$, T is linear, and $\|T(c)\| = \|S(c)\|$, so $\|T\| = \|T_0\|$. Finally, if $a \in A$, then $T(a) = T_0(a)$, since $T_0(a)$ is constant on each G_i. Thus B is $\mathfrak{N}$-injective.

We still must show that B is not isomorphic to any $C(Z)$ if some $c_i \neq -1$. Let $\mathscr{E}$ be the set of extreme points of the unit ball of B^*, the dual of B, and endow $\mathscr{E}$ with the weak $*$ topology. Be renumbering the x_i if necessary, we may assume $\{x_i: c_i = -1\} =$ either $\emptyset$ or $\{x_{p+1}, \cdots, x_n\}$. In the former case, set $p = n$. Then $\mathscr{E}$ is homeomorphic to the union of two disjoint copies of $X \sim \{x_1, \cdots, x_p\}$ (corresponding to $\pm$ point evaluations) with x_0 in one copy identified with $\{x_{p+1}, \cdots, x_n\}$ in the other copy and vice-versa, if $p < n$. Since $x_1, \cdots, x_p$ are not isolated points of X, $\mathscr{E}$ is not compact. Therefore, B cannot be isometrically isomorphic to a $C(Z)$ [3, p. 113].

The reader should note that if some of the c_i's are < 0, then B is not even a sublattice of $C(X)$. Actually, we can say even more. If Z is a compact Hausdorff space and $\sigma: Z \to Z$ is a homeomorphism such that σ^2 is the identity map, then $C_\sigma(Z) = \{f \in C(Z): f \circ \sigma = -f\}$. If for some c_{i_0}, $c_{i_0} \neq 0$ and $-1 < c_i < 1$, then B is not even isometrically isomorphic to any $C_\sigma(Z)$! For the set S of extreme points of the dual unit ball of B which are in minimal facets of the dual unit ball is clearly all of $\mathscr{E}$, and point evaluation at x_{i_0} clearly lies in the weak $*$ closure of $\mathscr{E}$. But for all $b \in B$, we have $b(x_i) < \|b\|$. Thus B cannot be isomorphic to any $C_\sigma(Z)$ by a theorem of Jerison's [3, p. 121].

Each $\mathfrak{N}$-Injective is Almost of Type $C(X)$.

Despite the example we have just given, an $\mathfrak{N}$-injective Banach space is not too far removed from a space of type $C(X)$. First, the spaces of type $C(X)$ share with the space of the example we have just given the property that their duals are isometrically isomorphic to a space of type $L^1(\mu)$. (A long list of spaces which are preduals of spaces of type $L^1(\mu)$ is given in [8, pp. 180–181]. (I would like to thank Y. Benjamini and the referee for bringing the class of preduals of spaces of type $L^1(\mu)$ to my attention.) A well known result of Lindenstrauss's states that each Banach space enjoying a finite dimensional extension property (which is much weaker than the extension property of $\mathfrak{N}$-injectivity) is the predual of an $L^1(\mu)$ space [9, Theorem 6.1]. Hence the $\mathfrak{N}$-injective Banach space and the spaces of type $C(X)$ all belong to the rather large family of preduals of spaces of type $L^1(\mu)$. But an $\mathfrak{N}$-injective space is more closely related to the spaces of type $C(X)$ than this. In fact, it turns out that if B is $\mathfrak{N}$-injective, then B is the direct limit of its $\mathfrak{N}$-separable subspaces of type $C(X)$.

We may prove this, and more, essentially by means of a slight modification of Kelley's original proof that a Banach space which is injective in the category $\mathscr{B}_1$ is of type $C(X)$ [7]. In what follows, if E is a Banach space, then E^* shall denote its dual, and U_E shall denote the closed unit ball of E. If K is a convex subset of E, then ext K shall denote the set of extreme points of K. If $Y \subseteqq E$, then Cl Y shall denote the closure of Y. The topology with respect to which the closure is taken will be specified whenever it is not clear from context. Finally, if Y is a compact Hausdorff space and $y \in Y$, then $e_y \in C(Y)^*$ shall denote evaluation at the point y.

LEMMA 1. *Let M and N be Banach spaces, and let $S: M \to N$ be a linear map of norm $\leqq 1$. Let p be an extreme point of U_N, and let $L = S^{-1}(p) \cap U_M$. Then either $L = \varnothing$ or L is a support of U_M.*

Lemma 1 is a standard fact (cf. [7]).

THEOREM 3. *Let B be an $\mathfrak{N}$-injective Banach space. Let A be an $\mathfrak{N}$-separable subspace of B, and let $i: A \hookrightarrow B$ be the inclusion map. Let W be a weak $*$ relatively open subset of* $\mathrm{Cl}\,\mathrm{ext}\,U_{A^*}$, *such that $W \cap (-W) = \varnothing$ and* $\mathrm{Cl}\,(W \cup (-W)) = \mathrm{Cl}\,\mathrm{ext}\,U_{A^*}$. *Let* $Y = \mathrm{Cl}\,W$. *Here, the closures are taken with respect to the weak $*$ topology. Endow Y with the weak $*$ topology, and let $j: A \to C(Y)$ be the natural isometric injection. Then there exists an isometric injection $p: C(Y) \to B$ such that $p \circ j = i$.*

Before proving Theorem 3, three comments are in order. First, as Kelley observed, it is easy to produce such sets W: Simply apply Zorn's lemma to produce a set W which is maximal with respect to the two properties $W \cap (-W) = \emptyset$, and W is weak $*$ open in $\mathrm{Cl}\,\mathrm{ext}\, U_{A^*}$ [7]. Second, from the Krein-Milman theorem, we know that if $a \in A$, then $\sup\{y(a): y \in U_{A^*}\}$ is actually attained at some $y \in \mathrm{Cl}\,\mathrm{ext}\, U_{A^*}$. Since $W \cap (-W)$ is dense in $\mathrm{Cl}\,\mathrm{ext}\, U_{A^*}$, it follows that j is an isometry. Finally, Y is clearly compact by the Alaoglu theorem.

Proof. First, we must show that $C(Y)$ is $\mathfrak{R}$-separable. Observe that $j(A)$ is $\mathfrak{R}$-separable, so the subalgebra in $\mathscr{A}$ in $C(Y)$ genarated by $j(A)$ and the function $\equiv 1$ is also $\mathfrak{R}$-separable. But $\mathscr{A}$ separates points of Y because A does. By the Stone-Weierstrass theorem, $\mathscr{A}$ is dense in $C(Y)$, so $C(Y)$ is also $\mathfrak{R}$-separable. From the $\mathfrak{R}$-injectivity of B, we conclude that there exists a linear map $p: C(Y) \to B$ of norm 1 such that $p \circ j = i$.

We will show that p is $1-1$ by showing that its adjoint $p^*: B^* \to C(Y)^*$ is onto. We assert that it suffices to show that $\{e_y: y \in W \cap \mathrm{ext}\, U_{A^*}\} \subseteqq p^*(U_{B^*})$. To see why, suppose this inclusion holds. We know p^* is weak $*$ continuous and U_{B^*} is weak $*$ compact, so $p^*(U_{B^*})$ is weak $*$ compact and hence weak $*$ closed in $C(Y)^*$. Thus $\mathrm{Cl}\,\{e_y: y \in W \cap \mathrm{ext}\, U_{A^*}\} \subseteqq p^*(U_{B^*})$. But the map $y \to e_y$, $y \in Y$, is a homeomorphism from Y onto the set $\{e_y: y \in Y\}$ endowed with the weak $*$ topology. Furthermore, because W is an open subset of $\mathrm{Cl}\,\mathrm{ext}\, U_{A^*}$, we know that $\mathrm{Cl}\,(W \cap \mathrm{ext}\, U_{A^*}) = \mathrm{Cl}\, W = Y$. Thus $\{e_y: y \in Y\} \subseteqq p^*(U_{B^*})$. From this, we conclude that $\mathrm{ext}\, U_{C(Y)^*} = \{\pm e_y: y \in Y\}$, as well as the closed convex hull of $\mathrm{ext}\, U_{C(Y)^*}$, are contained in $p^*(U_{B^*})$. By the Krein-Milman theorem, $U_{C(Y)^*} \subseteqq p^*(U_{B^*})$. Thus p^* is onto.

In fact, from this last inclusion, we may conclude that p is not only $1-1$, but is actually an isometry. Suppose p were not an isometry. We know that $\|p\| = 1$, so there exists $f \in C(Y)$ such that $\|f\| = 1$ and $\|p(f)\| < 1$. Let $\mu \in C(Y)^*$ be a linear functional of norm 1 such that $\mu(f) = 1$. If $\lambda \in U_{B^*}$, then $|p^*(\lambda)(f)| = |\lambda(p(f))| \leqq \|p(f)\| < 1$, so $\mu \notin p^*(U_{B^*})$. This is a contradiction.

We thus need only show that $\{e_y: y \in W \cap \mathrm{ext}\, U_{A^*}\} \subseteqq p^*(U_{B^*})$ in order to complete the proof of Lemma 2. We will do this by chasing the following commutative diagram of adjoint maps:

$$
\begin{tikzcd}
A^* & & B^* \arrow[ll, "i^*"'] \arrow[dl, "p^*"] \\
& C(Y)^* \arrow[ul, "j^*"] &
\end{tikzcd}
\tag{2}
$$

Note that all of the maps involved have norm $\leqq 1$. Let $y \in Y$. Clearly $j^*(e_y) = y$ for each $y \in Y$. We assert that if $y \in W \cap \mathrm{ext}\, U_{A^*}$, then $j^{*-1}(y) \cap U_{C(Y)^*} = \{e_y\}$. Let $L = j^{*-1}(y) \cap U_{C(Y)^*}$. By Lemma 1, L is a support of $U_{C(Y)^*}$. L is closed in $U_{C(Y)^*}$ and hence is compact. By the Krein-Milman theorem, L has extreme points. Because L is a support of $U_{C(Y)^*}$, each extreme point w of L is also an extreme point of $U_{C(Y)^*}$, and hence is of the form $\pm e_z$ for some $z \in Y$. If $w = -e_z$, then for each $a \in A$ we have $y(a) = j^*(-e_z)(a) = -e_z(j(a)) = -z(a)$, so $y = -z$. Hence $y \in W \cap (-\mathrm{Cl}\, W)$. But $W \cap (-\mathrm{Cl}\, W) = \varnothing$, since $W \cap (-W) = \varnothing$ and W is open. Thus w is not $= -e_z$ for any $z \in Y$. If $w = e_z$ for some $z \in Y$, it is immediate that $z = y$. Hence the only extreme point of L is e_y. By the Krein-Milman theorem, $L = \{e_y\}$.

Now let $y \in W \cap \operatorname{ext} U_{A^*}$, and observe that $i^{*-1}(y) \cap U_{B^*} \neq \varnothing$ by the Hahn-Banach theorem. Let $z \in i^{*-1}(y) \cap U_{B^*}$. By the commutativity of diagram 2, $j^*p^*(z) = i^*(z) = y$, so $p^*(z) \in j^{*-1}(y) \cap U_{B^*}$, and hence $p^*(z) = e_y$. Consequently, $\{e_y : y \in W \cap \operatorname{ext} U_{A^*}\} \subseteq p^*(U_{B^*})$, which completes the proof of Theorem 3.

COROLLARY 1. *Suppose B is not only $\mathfrak{N}$-injective, but also $\mathfrak{N}$-separable. Then B is isometrically isomorphic to $C(X)$, for some extremally disconnected compact Hausdorff space X. Hence B is injective in the category $\mathscr{B}_1$.*

Proof. In the statement of Theorem 3, choose $A = B$ and $X = Y$. Then B is isometrically isomorphic to $C(X)$. By Theorem 1, Corollary 1, $C(X)$ satisfies condition $a_{\mathfrak{N}}$. But $C(X)$ is $\mathfrak{N}$-separable, so $C(X)$ satisfies condition $a_{\mathfrak{N}'}$ for every infinite cardinal $\mathfrak{N}'$, and thus $C(X)$ is a boundedly complete vector lattice. The remainder of the corollary now follows from Theorem 1, Corollary 2, and the Stone-Nakano theorem.

Corollary 1 may be rephrased to assert that if B is injective in the full subcategory of $\mathscr{B}_1$ whose objects are all $\mathfrak{N}$-separable spaces, then B is actually injective in $\mathscr{B}_1$. As we mentioned in the introduction, Corollary 1 is interesting because the situation is dramatically different in the category $\mathscr{B}$. It would be interesting to know if there are any nontrivial full subcategories of $\mathscr{B}_1$ in which new injectives can arise. As we shall see in Corollary 5, the full subcategory $\mathscr{C}_1$ of $\mathscr{B}_1$, whose objects are all weakly compactly generated spaces, is not such a subcategory.

COROLLARY 2. *B is injective in the category $\mathscr{B}_1$ if and only if B is isometrically isomorphic to $C(X)$, for some extremally disconnected compact Hausdorff space X.*

Proof. One way is Theorem 1, Corollary 2. Conversely, suppose B is injective in $\mathscr{B}_1$. B is $\mathfrak{N}$-separable for some $\mathfrak{N}$, and also $\mathfrak{N}$-injective. Apply Corollary 1.

Corollary 2 is the full Goodner-Kelley-Nachbin characterization of injectives in $\mathscr{B}_1$ [5], [7], [11]. Of course, Corollary 2 is a bit of a cheat, because the proof of Theorem 3 is essentially Kelley's proof (slightly modified) that each injective in $\mathscr{B}_1$ is of type $C(X)$. However, the following corollary is somewhat more interesting because it may provide the first step toward a complete characterization of $\mathfrak{N}$-injectives.

COROLLARY 3. *Let B be an $\mathfrak{N}$-injective Banach space. Then B is the direct limit of its $\mathfrak{N}$-separable subspaces of type $C(Y)$.*

Proof. It suffices to prove that B is the union of such subspaces, and that if A and C are two such $\mathfrak{N}$-separable subspaces, then there is an $\mathfrak{N}$-separable subspace D of type $C(Y)$ such that $A \cup C \subseteqq D$. Both assertion follow immediately from Theorem 3.

I am indebted to Y. Benjamini for pointing out that Corollary 1 is actually strong enough to imply the following important result:

COROLLARY 4. *There are no $\mathfrak{N}$-injective, infinite dimensional, weakly compactly generated spaces.*

Proof. Suppose B is an $\mathfrak{N}$-injective space which is also infinite dimensional and weakly compactly generated. Let E be an infinite dimensional separable subspace of B. By a fundamental result of Lindenstrauss's, there exists a separable subspace $D \supseteqq E$ and a projection $P: B \rightarrow D$ of norm 1 [10, pp. 170–171]. The existence of the projection P guarantees that D is $\mathfrak{N}$-injective. By Corollary 1, D is isometrically isomorphic to $C(X)$, where X is extremally disconnected and compact Hausdorff. Since $C(X)$ is separable, X is metrizable. Thus X has finite cardinality, so $D = C(X)$ is finite dimensional, which is impossible.

COROLLARY 5. *The only injectives in $\mathscr{C}_1$ are already injective in the larger category $\mathscr{B}_1$, and are in fact finite dimensional.*

Proof. Let B be an injective in $\mathscr{C}_1$. Since $\mathscr{C}_1$ contains every separable space, B is $\mathfrak{N}_0$-injective. Apply Corollary 4.

In conclusion, we would like to raise a question whose resolution probably awaits a complete characterization of N-injectives. Cohen showed that every Banach space has an injective envelope in the catogory $\mathscr{B}_1$ [1]. It would be interesting to know whether or not every Banach space also has an $\mathfrak{N}$-injective envelope.

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