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This paper represents a continuing effort to develop elements of a general theory of packing and covering by translates of a fixed subset of a group. For P a subset of a group X a subset L is a left P-packing if for any distinct elements x_1 and x_2 in L, $x_1P \cap x_2P$ is empty. A subset M of X is a left P-covering if $MP=X$. The Chabauty topology on the set of discrete subgroups of a locally compact group has been used only to a rather limited extent in the Geometry of Numbers but by its very definition is a natural one to work with in studying packing and covering problems. The main results of this paper are that the Chabauty topology extends to the family of all closed discrete subsets containing the identity and that if X is σ-compact then $S(X, P)$ the space of all left P-packings for a fixed neighborhood P of the identity is locally compact.

The Chabauty topology [1] extends in the following way. Let X be a Hausdorff locally compact group with identity e and $\mathscr{N}$ the family of open neighborhoods of e. We introduce the following notation: For any three subsets A, B and C of X we define $L(A, B, C)$ to be the set of subsets, A', of X satisfying

$$A' \cap B \subset AC \text{ and } A \cap B \subset A'C$$

and $R(A, B, C)$ as the set of subsets A' of X satisfying

$$A' \cap B \subset CA \text{ and } A \cap B \subset CA' .$$

DEFINITION 1. We denote by $\mathscr{S}(X)$ the space of closed discrete subsets of X containing e with the topology generated by $\{L(H, K, U)$: $H \in \mathscr{S}(X)$; K a compact subset of X; $U \in \mathscr{N}\}$.

The following two assertions are immediate consequences of the definition of L.

PROPOSITION 1. *If $U_1 \supset U_2$ then $L(H, K, U_1) \supset L(H, K, U_2)$.*

PROPOSITION 2. *If $K_1 \supset K_2$ then $L(H, K_1 U) \subset (H, K_2, U)$.*

As a further consequence we have

PROPOSITION 3. *$\{L(H, K, U)$: K compact; $U \in \mathcal{N}\}$ is a local basis.*

Proof. $L(H, K_1 \cup K_2, U_1 \cap U_2) \subset L(H, K_1, U_1) \cap L(H, K_2, U_2)$.

We shall require

LEMMA. *Given $H \in \mathcal{S}(X)$ and K compact there exists $V \in \mathcal{N}$ such that $HV \cap K = (H \cap K)V$.*

Proof. Choose U a compact neighborhood of e and consider the finite set $S = H \cap KU$. For $h \in S \backslash K$ there exists a symmetric $V_h \in \mathcal{N}$ with $V_h \subset U$ such that $hV_h \cap K = \varnothing$. Let $V = \bigcap V_h$, $h \in S \backslash K$ then V has the asserted property. For $SV \cap K = \varnothing$ and if $h \in H \backslash S$ then $h \notin KU$ so $h \notin KV$ hence $hV \cap K = \varnothing$ since V is symmetric. Thus for $h \in H$ we have that $hV \cap K \neq \varnothing$ if and only if $h \in K$ and the lemma follows.

Suppose now that $H_1 \in L(H, K, W)$ then there exists $V \in \mathcal{N}$; V relatively compact, $V \subset W$ for which $H_1 \in L(H, K, V)$. Namely, if $h \in H \cap K$ then $h \in h_1 W$ for some $h_1 \in H$ so there exists $V_h \in \mathcal{N}$; V_h relatively compact and $V_h \subset W$ such that $h \in h_1 V_h$. Let $V' = \bigcup V_h$, $h \in H \cap K$ then $H \cap K \subset H_1 V'$. Similarly there exists $V'' \in \mathcal{N}$; V'' relatively compact and $V'' \subset W$ such that $H_1 \cap K \subset HV''$. With $V = V' \cup V''$ the assertion follows.

If $k \in h_1 V$ then $h_1 \in kV^{-1}$ so that $H_1 V \cap K \subset (H_1 \cap KV^{-1})V$ and therefore $H \cap K \subset (H_1 \cap KV^{-1})V$. Similarly $H_1 \cap K \subset (H \cap KV^{-1})V$. Let $h_1 \in H_1 \cap K$ then $h_1 \in h'V$ for some $h' \in H \cap KV^{-1}$ so that $h' \in h_1 V^{-1}$. There exists $V_{h_1} \in \mathcal{N}$ such that $h'V_{h_1} \subset h_1 V^{-1}$ and so for any $h'' \in h'V_{h_1}$ we have that $h_1 \in h''V_{h_1}$. Let $V_1 = \bigcap V_{h_1}$, $h_1 \in H_1 \cap K$.

The lemma provides that there exists $V_2 \in \mathcal{N}$ such that $H_1 V_2 \cap K = (H_1 \cap K)V_2$.

Since $H_1 \cap K \subset HV$ there exists $V_3 \in \mathcal{N}$ such that $(H_1 \cap K)V_3 \subset HV$.

Let $U' = V_1 \cap V_2 \cap V_3$ and $U = U' \cap U'^{-1}$ and consider $H_2 \in L(H_1, K\overline{V^{-1}}, U)$. Since $H_1 \cap K\overline{V^{-1}} \subset H_2 U$ there is for each $h' \in H_1 \cap K\overline{V^{-1}}$ an element $h_2 \in H_2$ such that $h' \in h_2 U$, so $h_2 \in h'U$, hence $h_2 \in h'V_1$. It follows that $H \cap K \subset H_2 V$ by the argument motivating the choice of V_1.

On the other hand $H_2 \cap K \subset H_2 \cap K\overline{V^{-1}} \subset H_1 U$. Therefore $H_2 \cap K \subset H_1 U \cap K \subset (H_1 \cap K)U$ by choice of V_2. Our choice of V_3 now ensures that $(H_1 \cap K)U \subset HV$. Hence $H_2 \cap K \subset HV$. Thus $L(H_1, K\overline{V^{-1}}, U) \subset L(H, K, V)$. Recalling Proposition 3, we have proved

THEOREM 1. *The family* $\{L(H, K, V)$: *K compact* $V \subset \mathscr{N}\}$ *is a basis for the topology on* $\mathscr{S}(X)$.

As an immediate consequence in view of Proposition 1 and the lemma we have

COROLLARY. *The family* $\{L(H, K, V)$: *K compact, V relatively compact, symmetric and satisfying* $HV \cap K = (H \cap K)V\}$ *is a basis for the topology on* $\mathscr{S}(X)$.

That $\mathscr{S}(X)$ is Hausdorff we see as follows.

Let H and H' be distinct elements of $\mathscr{S}(X)$ then there exists $a \in H \Delta H'$ say $a \in H \backslash H'$. There exists a neighborhood $V \in \mathscr{N}$ which is symmetric and relatively compact for which $aV \cap H' = \varnothing$. Choose a symmetric $V_1 \in \mathscr{N}$ such that $V_1^2 \subset V$ and consider the neighborhoods $L(H', a\bar{V}_1, V_1)$ and $L(H, \{a\}, V_1)$. Let $H_1 \in L(H', a\bar{V}_1, V_1)$. Were $H_1 \in L(H, \{a\}, V_1)$ then $H \cap \{a\} \subset H_1V_1$ i.e. $a \in H_1V_1$ say $a \in h_1V_1$ for some $h_1 \in H_1$ implying that $h_1 \in aV_1$. Now $H \cap a\bar{V}_1 \subset H'V_1$ hence $h_1 \in H'V_1$. Thus $aV_1 \cap H'V_1 \neq \varnothing$ and therefore $H' \cap aV_1^2 \neq \varnothing$. But this implies that $aV \cap H' \neq \varnothing$ which is a contradiction. It follows that $L(H', a\bar{V}_1, V_1) \cap L(H, \{a\}, V_1) = \varnothing$.

THEOREM 2. *The topology on* $\mathscr{S}(X)$ *is equivalent to that generated by* $\{R(H, K, V)$: $H \in \mathscr{S}(X)$, *K compact* $V \in \mathscr{N}\}$.

Proof. Let $L(H, K, V)$ be such that V is symmetric and $HV \cap K = (H \cap K)V$.

By an analogous proof of the above lemma, there exists a symmetric $U \in \mathscr{N}$ satisfying $UH \cap K = U(H \cap K)$. Moreover $H \cap K$ being finite we can choose such a U for which $Uh \subset hV$ for all $h \in H \cap K$.

Let $H_1 \in R(H, K, U)$. Then $H_1 \cap K \subset UH \cap K \subset U(H \cap K) \subset (H \cap K)V \subset HV$. Also $H \cap K \subset UH_1 \cap K$ so that if $h \in H \cap K$ then $h \in Uh_1$ for some $h_1 \in H_1$; hence $h_1 \in Uh$ and so $h_1 \in hV$. Hence $h \in h_1V$. It follows that $H \cap K \subset H_1V$. Thus $R(H, K, U) \subset L(H, K, V)$. By a similar argument for any $R(H, K, U)$ there exists $W \in \mathscr{N}$ such that $L(H, K, W) \subset R(H, K, U)$.

We turn now to the concept of packing.

DEFINITION 2. For $V \in \mathscr{N}$, say that a set H is a (left) V packing if, for each pair of distinct elements h_1 and h_2 in H, $h_1V \cap h_2V = \varnothing$.

DEFINITION 3. Say that a subset H of X is uniformly discrete if there exists $U \in \mathscr{N}$ such that $hU \cup H = \{h\}$ for each $h \in H$. Denote the class of those uniformly discrete subsets of X which contain the identity by $\mathrm{S}(X)$.

The class of uniformly discrete subsets of X clearly coincides with the class of left V-packing for all $V \in \mathscr{N}$. For if $h_1 U \cap h_2 = \varnothing$ for any $h_1 \neq h_2$, h_1, $h_2 \in H$ then for any $V \in \mathscr{N}$ such that $VV^{-1} \subset U$ we have $h_1 V \cap h_2 V = \varnothing$. Conversely $h_1 V \cap h_2 V = \varnothing$ implies that $h_1 VV^{-1} \cap h_2 = \varnothing$.

We now restrict ourselves to $S(X)$ with the relative topology induced by $\mathscr{S}(X)$. We remark, however, that in the sequel some of our results concerning $S(X)$ will be true also of $\mathscr{S}(X)$ in view of

PROPOSITION 4. *$S(X)$ is everywhere dense in $\mathscr{S}(X)$.*

Proof. Suffice to observe that $H \cap K \in L(H, K, V)$.

Denoting by $p(K, V)$ the subset of $S(x) \times S(x)$ consisting of elements (H_1, H_2) which satisfy $H_1 \cap K \subset H_2 V$ and $H_2 \cap K \subset H_1 V$ we see that the family $U = \{p(K, V)$: K compact, $V \in \mathscr{N}\}$ is the basis of a uniform structure. For we have that $p(K_1 \cup K_2, V_1 \cap V_2) \subset p(K_1, V_1) \cap p(K_2, V_2)$, that each member of U contains the diagonal and that $p(K, V) \in U$ implies $p(K, V)^{-1} \in U$. Further, if $W^2 \subset V$, W symmetric and $\bar{W}$ compact then $(H_1, H_2) \in p(K\bar{W}, W)$ and $(H_2, H_3) \in p(K\bar{W}, W)$ implies that $H_1 \cap K \subset H_2 W \cap K$, so $H_1 \cap K \subset (H_2 \cap KW)W \subset (H_2 \cap K\bar{W})W \subset H_3 W$ and similarly $H_3 \cap K \subset H_1 W$. Thus $p(K\bar{W}, W) \circ p(K\bar{W}, W) \subset p(K, V)$. Clearly $S(X)$ is the uniform topology induced by U and therefore we have

PROPOSITION 5. *$S(X)$ is completely regular.*

Suppose now that K is compact, $V \in \mathscr{N}$ and $\bar{V}$ is compact. We shall show that $L(H, KV, \bar{V})$ is closed.

The complement $\mathscr{C}L(H, KV, \bar{V})$ consists of elements H_1 for which either of the following holds:

(i) $H_1 \cap KV \not\subset H\bar{V}$

(ii) $H \cap KV \not\subset H_1\bar{V}$.

In the case (i) there exists $h_1 \in H_1 \cap KV$ such that $h_1 \notin H\bar{V}$ so $h_1 \notin H\bar{V} \cap K\bar{V}$ hence $h_1 \notin (H \cap K\bar{V}\bar{V}^{-1})\bar{V}$. Since $H \cap K\bar{V}\bar{V}^{-1}$ is finite, $(H \cap K\bar{V}\bar{V}^{-1})\bar{V}$ is compact and there exists a symmetric neighborhood $U \in \mathscr{N}$ such that $h_1 U \cap (H \cap K\bar{V}\bar{V}^{-1})\bar{V} = \varnothing$. Moreover we can choose U so that $h_1 U \subset KV$. If $H_2 \in L(H_1, K\bar{V}, U)$ then $H_1 \cap K\bar{V} \subset H_2 U$ and there exists $h_2 \in H_2$ for which $h_1 \in h_2 U$ hence

$h_2 \in h_1 U$ and therefore $h_2 \in KV$. On the other hand, were $H_2 \in L(H, KV, \bar{V})$ we would find $h_2 \in H\bar{V} \cap K\bar{V}$ hence $h_2 \in (H \cap K\bar{V}\bar{V}^{-1})\bar{V}$ contradicting the fact that $h_1 U \cap (H \cap K\bar{V}\bar{V}^{-1})\bar{V} = \emptyset$. It follows that $L(H_1, K\bar{V}, U) \subset \mathscr{C}L(H, KV, \bar{V})$.

In the case (ii) there exists $h \in H \cap KV$ such that $h \notin H_1\bar{V} \cap K\bar{V}$ so $h \notin (H_1 \cap K\bar{V}\bar{V}^{-1})\bar{V}$. The latter set being compact, there exists a symmetric $U \in \mathscr{N}$ such that $Uh \cap (H_1 \cap K\bar{V}\bar{V}^{-1})\bar{V} = \emptyset$. Furthermore we can choose U small enough that $UH_1 \cap K\bar{V}\bar{V}^{-1} = U(H_1 \cap K\bar{V}\bar{V}^{-1})$. Let $H_2 \in R(H_1, K\bar{V}\bar{V}^{-1}, U)$ then

$$H_2 \cap K\bar{V}\bar{V}^{-1} \subset UH_1 \cap K\bar{V}\bar{V}^{-1}\,.$$

Were $H_2 \in L(H, KV, \bar{V})$ then $H \cap KV \subset H_2\bar{V}$, indeed $H \cap KV \subset H_2\bar{V} \cap K\bar{V}$ so that

$$h \in H_2\bar{V} \cap K\bar{V} = (H_2 \cap K\bar{V}\bar{V}^{-1})\bar{V}$$

hence

$$h \in (UH_1 \cap K\bar{V}\bar{V}^{-1})\bar{V} = U(H_1 \cap K\bar{V}\bar{V}^{-1})\bar{V}\,.$$

But this contradicts $Uh \cap (H_1 \cap K\bar{V}\bar{V}^{-1})\bar{V} = \emptyset$, hence $R(H_1, K\bar{V}\bar{V}^{-1}, U) \subset \mathscr{C}L(H, KV, \bar{V})$. Thus we have shown that $L(H, KV, \bar{V})$ is closed.

We next demonstrate that, under the further hypothesis that V is symmetric, $(H, KV, \bar{V})$ is uniformly bounded, namely for K' compact and $U \in \mathscr{N}$ there exists a finite subset $\{H_i\colon i = 1, \cdots, N\}$ of $L(H, KV, \bar{V})$ such that $\bigcup_{i=1}^{N} L(H_i, K', U) \supset L(H, KV, \bar{V})$.

It clearly suffices to assume that $K_1 \supset KV\bar{V}^{-1}$ and U is symmetric. Let $H \cap KV\bar{V}^{-1} = \{h_1, \cdots, h_m\}$. Each of the sets $h_i\bar{V}\,(i = 1, \cdots, m)$ admits a finite covering of the form $\bigcup_{s=1}^{t_i} h_{i,s} U$ with $h_{i,s} \in h_i\bar{V}$ $(s = 1, \cdots, t_i;\ i = 1, \cdots, m)$ and $K' \backslash KV\bar{V}^{-1}$ admits a finite covering $\bigcup_{j=1}^{r} k_j U$ with $k_j \in K' \backslash KV\bar{V}^{-1}$ $(j = 1, \cdots, r)$. We choose a finite set of elements H' as follows: H' is the union of non-empty, not necessarily proper, subsets of each of the sets $\{h_{i,s}\colon s = 1, \cdots, t_i\}$ $(i = 1, \cdots, m)$ together with a prossibly empty, not necessarily proper, subset of $\{k_i\colon i = 1, \cdots, r\}$. We claim that these $\prod_{i=1}^{m}(2^{t_i} - 1)\cdot 2^r$ sets satisfy the required condition. For the symmetry of V ensures that each such set is in $L(H, KV, \bar{V})$; moreover if $H_1 \in L(H, KV, \bar{V})$ then certainly there exists H' in our set for which $H_1 \cap K' \subset H'U$. If we choose H' to be minimal in the sense that no proper subset of H' satisfies this condition we will ensure that $H' \cap K' \subset H_1 U$ and hence $H_1 \in L(H', K', U)$.

DEFINITION 4. An equi-discrete subspace of $S(X)$ is one each member of which is a left P-packing for a fixed $P \in \mathscr{N}$. We denote this subspace by $S(X, P)$.

PROPOSITION 6. *For each* $P \in \mathscr{N}$, $S(X, P)$ *is a closed subspace of* $S(X)$.

Proof. Let $H \in \mathscr{C}S(X, P)$. Then there exists $x, y \in H$, $x \neq y$ such that $xP \cap yP \neq \emptyset$.

Choose $V_1 = V_1^{-1} \in \mathscr{N}$ for which $xV_1 \cap yV_1 = \emptyset$ then $H' \in L(H, \{x, y\}, V_1)$ implies that $\{x, y\} \subset H'V_1$. Hence, if $x \in x'V_1$ and $y \in y'V_1$ for $x', y' \in H'$ then $x' \neq y'$.

Suppose that $z \in xP \cap yP$ then there exists a symmetric $W \in \mathscr{N}$ such that $Wz \subset xP \cap yP$. Let $V_2 = V_2^{-1} \in \mathscr{N}$ be such that $xV_2 \subset Wx$ and $V_3 = V_3^{-1} \in \mathscr{N}$ satisfy $yV_3 \subset Wy$. With $V = V_1 \cap V_2 \cap V_3$ we have for $H' \in L(H, \{x, y\}, V)$ that there exist distinct x' and y' in H' such that $x' \in xV$ and $y' \in yV$. Moreover since $x' \in xV_2$ it follows that $x' = w_1 x$ for some $w_1 \in W$. Then $xx'^{-1}z = w_1^{-1}z \in xP$ so $z \in x'P$. Similarly $y' \in yV_3$ implies that $y' \in Wy$ and we find that $z \in y'P$. Thus $z \in x'P \cap y'P$ and we have shown that $L(H, \{x, y\}, V)$ is contained in $\mathscr{C}S(P, X)$ from which the proposition follows.

We conclude this paper with a proof of

THEOREM 3. *If* X *is* σ*-compact and satisfies the first axiom of countability then* $S(P, X)$ *is locally compact.*

Proof. For any $H \in S(P, X)$ we set $L(P; H, K, V) = L(H, K, V) \cap S(P, X)$. We have that $L(P; H, KV, \bar{V})$ is closed and for symmetric V that $L(P; H, KV, \bar{V})$ is uniformly bounded in $S(P, X)$ in that $L(H, KV, \bar{V})$ is uniformly bounded in $S(X)$. The theorem will follow upon showing that $S(P, X)$ is complete.

Let (K_i) be a sequence of compact sets containing e such that $K_{i+1} \supset K_i$ $(i = 1, 2, \cdots)$ and $X = \bigcup_{i=1}^{\infty} K_i$. Let (V_j) be a decreasing sequence of bounded, symmetric elements in $\mathscr{N}$ such that $\bigcap_{j=1}^{\infty} = \{e\}$. If a sequence (H_n) of elements in $S(P, X)$ is Cauchy then for each (i, j) there exists $n(i, j)$ such that, for all $n \geqq m \geqq n(i, j)$, $(H_m, H_n) \in p(K_i, V_j)$. It clearly suffices that this condition be met for all (i, j) will $i = j$ and we shall write $n(i)$ for $n(i, i)$.

If $(H_n) = \{e\}$ for all $n > n_0$ then H_n is certainly Cauchy and converges to $\{e\}$. Moreover if (H_n) is Cauchy and i is such that $H_{n(i)} \cap K_i V_i = \{e\}$ then $H_m \cap K_i = \{e\}$ for all $m \geqq n(i)$. If indeed i is such that, for all $j \geqq i$, $H_{n(j)} \cap K_j V_j = \{e\}$ then again (H_n) converges to $\{e\}$. It remains to consider (H_n) for which this is not the case. Choose i large enough that $V_i \subset P$. Then, for $n \geqq m \geqq n(i)$, $(H_m, H_n) \in p(K_i, V_i)$ and we may suppose i large enough that $H_m \cap K_i \neq \emptyset$. If $h_m \in H_m \cap K_i$ then for each $n > m$ there exists

$h_n \in H_n$ for which $h_m \in h_n V_i$. We claim that h_n is uniquely determined for each $n > m$. For h_n, $h'_n \in h_m V_i$ would imply that $h_m \in h_n V_i \cap h'_n V_i$ hence $h_n P \cap h'_n P \neq \emptyset$ and therefore $h_n = h'_n$. Thus for each $n > m$ there is a unique element in $H_n \cap K_i V_i$ which is contained in $h_m V_i$. We have, therefore, for each element h_m in $H_m \cap K_i$ a sequence $\{h_{m,n}: n = 1, 2, \cdots\}$ which is moreover Cauchy in that for each $j > i$ there exists $n(j)$ such that, for $k > l > n(j)$, $h_{m,k} h_{m,l}^{-1} \in V_j$. Since X is locally compact it is complete and therefore there exists a unique limit $\bar{h}_m$. Suppose that $\bar{h}_m$ and $\bar{h}'_m$ are limits of two such sequences. We claim that $\bar{h}_m P \cap \bar{h}'_m P = \emptyset$. Otherwise there would exist a symmetric neighborhood $V \in \mathscr{N}$ such that $V \bar{h}_m^{-1} \bar{h}'_m V \subset PP^{-1}$. If we take $s > i$ and large enough that $V_s \subset V$ then for $r > n(s)$ we would find $H_r \cap \bar{h}_m V_s = h_r$ and $H_r \cap \bar{h}'_m V_s = h'_r$ with $h_r \neq h'_r$. Moreover $h_r^{-1} h'_r \in PP^{-1}$ implying that $h_r P \cap h'_r P \neq \emptyset$ contradicting $H_r \in S(P, X)$. It follows that the limit set $\bar{H}$ whose existence we have shown is an element of $S(P, X)$ and the proof is complete.

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