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It is known that the *bi*-prime group $B(G)$ of an l-group G contains the basic elements of G. We show that every l-group G possesses a unique, maximal, archimedean, convex l-subgroup $A(G)$, and that if G is completely distributive and if $A(G)^{\perp}$ is representable, then $B(G)$ has a basis.

1. Introduction. An element s of a lattice-ordered group (l-group) G is *basic* (see [4]) if $s > 0$ and the closed interval $[0, s]$ is totally ordered. An l-group G *has a basis* if every $g > 0$ exceeds some basic element (any maximal disjoint set of basic elements is then a *basis*). An l-group G is *completely distributive* (see [3], [4], [9], [10]) if the relation

$$\bigwedge \{\vee \{g_{ij} \mid j \in J\} \mid i \in I\} = \bigvee \{\wedge \{g_{i(if)} \mid i \in I\} \mid f \in J^I\}$$

holds whenever $\{g_{ij} \mid i \in I, j \in J\} \subseteq G$ is such that all the indicated joins and meets exist. By [5], p. 5.18, Theorem 5.8, every l-group which has a basis is completely distributive. For *archimedean* l-groups, i.e. those in which $a \geqq nb \geqq 0$ for all natural numbers n implies $b = 0$, more can be said: viz., an archimedean l-group has a basis if and only if it is completely distributive ([5], p, 5.21, Theorem 5.10). In [8], we constructed, via minimal prime subgroups, the *bi*-prime group $B(G)$ of an l-group G (see §3 below) which contains all the basic elements and which, if G is completely distributive and representable, has a basis. In this note, we introduce "archimedean elements" (see §2 below) in order to investigate possible connections among the above results. Thus, in §2, we show that every l-group G posseses a unique, maximal, archimedean, convex l-subgroup $A(G)$. (Kenny [7] independently proved this result for representable l-groups.) It follows that if $A(G)^{\perp} = \{0\}$, then G is completely distributive if and only if G has a basis. In §3, proving somewhat more general results, we show that $A(B(G)) = B(A(G))$ and hence that if G is completely distributive and if $A(G)^{\perp}$ is representable, then $B(G)$ has a basis. In §4, we construct two examples, one of which is of completely distributive, nonrepresentable l-group which has a basis and for which $A(G)^{\perp}$ is representable.

NOTATION AND TERMINOLOGY. We use $\square$ for the empty set and write functions on the right. We use N, Z, and R for the natural numbers, the integers and the real numbers, respectively. The cartesian product of the sets $\{S_\alpha \mid \alpha \in A\}$ is denoted by $\prod \{S_\alpha \mid \alpha \in A\}$. If $\{G_\alpha \mid \alpha \in A\}$ is a set of l-groups, then $|\prod|\{G_\alpha \mid \alpha \in A\}(|\sum|\{G_\alpha \mid \alpha \in A\})$ denotes their cardinal product (sum); if $A = \{1, 2\}$, we use $G_1| \times |G_2$ for the cardinal product.

Let G be an l-group. A subgroup H of G is *prime* if and only if it is a convex l-subgroup of G such that for all $a, b \in G^+ \backslash H$, $a \wedge b \in G^+\backslash H$(see [5], pp. 1.13–1.16). If $g \in G \supseteqq A, B$, then $\langle A \rangle$ denotes the convex l-subgroup generated by A; $\langle A, B\rangle \equiv \langle A \cup B\rangle$; $G(g) \equiv \langle \{g\} \rangle$. For any $S \subseteqq G$, the *polar* of S, defined

$$S^\perp = \{g \in G \mid |g| \wedge |s| = 0 \quad \text{for all } s \in S\} ,$$

is a convex l-subgroup of G (see [8]). The following result will prove useful.

LEMMA 1.1. *Let H be a convex l-subgroup of an l-group G. If $\{h_\alpha\} \subseteqq H$ is such that $\bigvee_H h_\alpha$ exists in H, then $\bigvee_G h_\alpha$ exists in G and $\bigvee_G h_\alpha = \bigvee_H h_\alpha$. The dual statement also holds.*

Proof. Let $\{h_\alpha\} \subseteqq H$ be such that $\bigvee_H h_\alpha \in H$. Suppose that the join of $\{h_\alpha\}$ does not exist in G. Then, since $\bigvee_H h_\alpha$ is an upper bound of $\{h_\alpha\}$ in G, there exists $b \in G$ such that $h_\beta \leqq b < \bigvee_H h_\alpha$ for all β. Since H is convex, $b \in H$. This contradicts the minimality of $\bigvee_H h_\alpha$ among upper bounds of $\{h_\alpha\}$ in H and hence $\bigvee_G h_\alpha \in G$. Since $\bigvee_H h_\alpha \in G$ is an upper bound of $\{h_\alpha\}$, $h_\beta \leqq \bigvee_G h_\alpha \leqq \bigvee_H h_\alpha$ for all β, and hence $\bigvee_G h_\alpha \in H$. Therefore, $\bigvee_G h_\alpha = \bigvee_H h_\alpha$. The dual property follows from the above because G is an l-group.

For terminology left undefined, see Birkhoff [1], Fuchs [6], or Conrad [5].

2. Archimedean elements.

Let G be an l-group. An element $a \in G$ is *archimedean* if $a \geqq 0$ and if for all $0 < g \leqq a$, there exists $n \in N$ such that $ng \nleqq a$. Clearly, G is archimedean if and only if every element of G^+ is archimedean. Let $P(G)$ be the set of all archimedean elements of G; let $A(G)$ be the l-subgroup of G generated by $P(G)$.

THEOREM 2.1. $A(G)^+ = P(G)$.

Proof. Clearly, $0 \in P(G)$ and $P(G)$ is convex. By [5], p. 1.5, Theorem 1.3, it therefore suffices to show that $P(G)$ is a subsemigroup of G^+.

The proof that $P(G)$ is a subsemigroup is by contradiction.

Suppose there exist $a, b \in P(G)$ such that $a + b \notin P(G)$. Then there exists $0 < t \leqq a + b$ such that $nt \leqq a + b$ for all $n \in N$. Since a is archimedean, there exists $m > 0$ such that $mt \nleqq a$. Then

$$s = (-a + mt) \vee 0 > 0 .$$

Since $nt \leqq a + b$ for all $n > 0$, $-a + nt \leqq b$ for all $n > 0$. Thus

$$(1) \qquad (-a + nt) \vee 0 \leqq b \quad \text{for all } n \in N ,$$

and in particular $0 < s \leqq b$. We will show by induction that

$$(2) \qquad ks \leqq (-a + kmt) \vee 0 \quad \text{for all} \quad k \in N .$$

Obviously,

$$s = (-a + mt) \vee 0 \leqq (-a + kmt) \vee 0$$

for all $k \in N$. Suppose $ks \leqq (-a+kmt) \vee 0$. Then

$$\begin{aligned}(k + 1)s &= (k + 1)[(-a + mt) \vee 0] \\ &= k[(-a + mt) \vee 0] + [(-a + mt) \vee 0] \\ &\leqq [(-a + kmt) \vee 0] + [(-a + mt) \vee 0] \\ &= (-a + kmt - a + mt) \vee (-a + kmt) \\ &\quad \vee (-a + mt) \vee 0 \\ &\leqq (-a + kmt + mt) \vee (-a + kmt) \vee 0 \\ &= (-a + (k + 1)mt) \vee (-a + kmt) \vee 0 \\ &= (-a + (k + 1)mt) \vee 0 .\end{aligned}$$

Then for all $k \in N$,

$$\begin{aligned}0 < ks &\leqq (-a + kmt) \vee 0 && \text{by (2)} \\ &\leqq b && \text{by (1)} .\end{aligned}$$

Therefore, $b \notin P(G)$, which contradicts our choice of b. Theorem 12. follows.

Corollary 2.2. *$A(G)$ is the unique, maximal, archimedean, convex l-subgroup of G.*

Proof. Since $A(G)^+ = P(G)$, $A(G)$ is archimedean. By definition of $P(G)$ any larger l-subgroup cannot be archimedean. That $A(G)$ is convex and unique is clear.

Corollary 2.3. *Let $g \in G^+$. Then g is archimedean if and only if $G(g)$ is archimedean.*

Proof. The proof of Theorem 2.1 shows that if g is archimedean, then ng is archimedean for all $n \in N$. Thus, $G(g)$ is archimedean. The converse is clear.

COROLLARY 2.4. $A(G) = \{g \in G \mid G(|g|) \text{ is archimedean}\}$.

Proof. If $g \in A(G)$, then $|g|$ is archimedean by Theorem 2.1, and thus $G(|g|)$ is archimedean by Corollary 2.3. Conversely, if $G(|g|)$ is archimedean, Corollary 2.3 implies that $|g|$ is archimedean. Hence by Theorem 2.1, $|g| \in A(G)^+$. Since $-|g| \leqq g \leqq |g|$ and $A(G)$ is convex, $g \in A(G)$.

Kenny [7] proved independently that for every representable l-group G, $\{g \in G \mid G(|g|) \text{ is archimedean}\}$ is the unique, maximal, archimedean, convex l-subgroup of G; this follows immediately from Corollaries 2.2 and 2.4 above.

PROPOSITION 2.5. *Let G be an l-group in which every strictly positive element exceeds a nonzero archimedean element. Then G is completely distributive if and only if G has a basis.*

Proof. By Lemma 1.1 if G is completely distributive, $A(G)$ is completely distributive. Since $A(G)$ is archimedean, this implies that $A(G)$ has a basis, and then G must have a basis because of the initial hypothesis. The converse follows from [5], p. 5.18, Theorem 5.8 (see §1).

3. The bi-prime group and $A(G)$. In [8], we defined the bi-prime group of an l-group G as follows: Let $\{P_\phi \mid \phi \in \Phi(G)\}$ be the set of minimal prime subgroups of G. The *bi-prime group* of G is the convex l-subgroup

$$B(G) = \bigcap \{\langle P_\phi, P_\omega \rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} .$$

By [8], Theorem 3.1, $B(G)$ has a basis whenever G is both completely distributive and representable.

The following result is an easy consequence of [2], Theorem 3.5.

LEMMA 3.1. *Let $\{0\} \neq S$ be a convex l-subgroup of an l-group G. If Q is a minimal prime subgroup of S, then there exists a minimal prime subgroup P of G such that $Q = P \cap S$. If P is a minimal prime subgroup of G which does not contain S, then $P \cap S$ is a minimal prime subgroup of S.*

PROPOSITION 3.2. *Let G be an l-group and let H be a convex l-subgroup of G. Then $B(H) = B(G) \cap H$.*

Proof. By [5], p. 1.6, Theorem 1.4, the set of convex l-subgroups of an l-group, ordered by inclusion, is a (complete) distributive lattice. Combining this with Lemma 3.1, we have the following:

$$\begin{aligned} B(H) &= \bigcap \{\langle Q_\phi, Q_\omega\rangle \mid \phi, \omega \in \Phi(H), \phi \neq \omega\} \\ &= \bigcap \{\langle P_\phi \cap H, P_\omega \cap H\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega, P_\phi \ngeqq H \nsubseteq P_\omega\} \\ &= \bigcap \{\langle P_\phi \cap H, P_\omega \cap H\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} \\ &= \bigcap \{\langle P_\phi, P_\omega\rangle \cap H \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} \\ &= B(G) \cap H. \end{aligned}$$

COROLLARY 3.3. *For any l-group G, $B(A(G)) = A(B(G))$.*

Proof. By definition of $P(B(G))$(cf. §2), $P(B(G)) = P(G) \cap B(G)$. Thus,

$$\begin{aligned} A(B(G)) &= \langle P(B(G))\rangle = \langle P(G) \cap B(G)\rangle \\ &= \langle P(G)\rangle \cap B(G) = A(G) \cap B(G)\ . \end{aligned}$$

By Proposition 3.2,

$$A(B(G)) = A(G) \cap B(G) = B(A(G))\ .$$

PROPOSITION 3.4 *Let G be a completely distributive l-group. If G has a representable convex l-subgroup H such that $H^\perp = \{0\}$, then $B(G)$ has a basis.*

Proof. Since G is completely distributive, H is completely distributive by Lemma 1.1. Thus, since H is representable, $B(H)$ has a basis by [8], Theorem 3.1. By Proposition 3.2 above, $B(H) = H \cap B(G)$. If $g \in B(G)^+\backslash\{0\}$, then since $H^\perp = \{0\}$, there exists $h \in H$ such that $g \geqq h > 0$. But since $B(G)$ is convex, $h \in B(G)$ also, and thus $h \in B(H)$. Since $B(H)$ has a basis, h exceeds a basic element, and hence g exceeds a basic element. Therefore, $B(G)$ has a basis.

COROLLARY 3.5. *Let G be a completely distributive l-group. If $A(G)^\perp$ is representable, then $B(G)$ has a basis.*

Proof. Since $A(G)$ is archimedean, it is abelian and hence representable. Therefore, since $A(G)^\perp$ is representable, $H = \langle A(G), A(G)^\perp\rangle$ is representable (clearly H is l-isomorphic to $A(G) \mid \times \mid A(G)^\perp$). Clearly, $H^\perp = \{0\}$, and hence by Proposition 3.4, $B(G)$ has a basis.

COROLLARY 3.6. *Let G be a completely distributive l-group such that $A(G)^\perp$ is representable. Then G has a basis if and only if $B(G)^\perp = \{0\}$.*

4. Examples.

EXAMPLE 4.1. We construct an abelian, completely distributive l-group H such that $A(H) \subseteqq B(H)$ but $A(H) \neq B(H)$.

Let $V = \prod \{R \mid n \in N\}$, and $f, g \in V$; let $S(f, g) \equiv \{n \in N \mid (n)f \neq (n)g\}$. Then V becomes an o-group under (pointwise addition and) the relation: $f \leqq g$ if and only if $f = g$ or $f \neq g$ and $(\wedge S(f, g))f \leqq (\wedge S(f, g))g$. Clearly V, is completely distributive and abelian. Furthermore, if $f \in V^+ \backslash \{0\}$ and $h \in G$ is defined by

$$(n)h = \begin{cases} 0 & \text{if } n \leqq \wedge S(f, 0) \\ 1 & \text{otherwise}, \end{cases}$$

then for all $k \in N$,

$$\begin{aligned}(\wedge S(f, kh))(kh) = (\wedge S(f, 0))(kh) = k(\wedge S(f, 0))(h) = 0 \\ < (\wedge S(f, 0))f = (\wedge S(f, kh))f,\end{aligned}$$

and hence f is not archimedean. Thus, $A(V) = \{0\}$. Let $G = |\sum| \{R \mid n \in N\}$. Then clearly, G is completely distributive and abelian, and $A(G) = G$. It is also easy to show that any minimal prime subgroup of G has the form $\{f \mid nf = 0\}$ for some $n \in N$, and thus $B(G) = G$.

Let $H = V| \times |G$. Since V is an o-group, $V \subseteqq B(H)$; by Proposition 3.2, $G \subseteqq B(H)$. Thus, $B(H) = H$. Since $A(V) = \{0\}$ and $A(G) = G$, $A(H) = \{0\} \times G$. Thus $A(H)$ is properly contained in $B(H)$. Clearly, H is completely distributive and abelian.

REMARK 4.2. If $B(G)$ is strictly contained in G for some completely distributive, archimedean l-group G, then $H = V| \times |G$ (cf. Example 4.1) is an an abelian, completely distributive l-group for which $A(H)$ and $B(H)$ are incomparable. On the other hand, if $B(G) = G$ for all completely distributive, archimedean l-groups G, then Proposition 3.2 could be used to show that $A(G) \subseteqq B(G)$ for every completely distributive l-group G. Thus, it would be useful to have an answer to the following question: Does there exist a completely distributive, archimedean l-group G with distinct (minimal) prime subgroups P_1 and P_2 such that $G \neq \langle P_1, P_2 \rangle$?

EXAMPLE 4.3. We construct a non-representable l-group G which is completely distributive and has a basis and for which $A(G)^\perp$ is representable.

Let $G = Z\,Wr\,Z$ be the wreath product of Z by itself. Thus,

$G = Z \times (\prod_{i \in Z} Z_i)$, where each $Z_i = Z$, and group operation on G is defined as follows:

$$(i; \cdots, \alpha_j, \cdots) \oplus (k; \cdots, \beta_j, \cdots) = (i + k; \cdots, \gamma_j, \cdots),$$

where $\gamma_j = \alpha_{j-k} + \beta_j$. An element $(i; \cdots, \cdots, \alpha_j, \cdots)$ is positive in G if $i > 0$ or if $i = 0$ and $\alpha_j \geqq 0$ for all j. Clearly $A(G) = \{0\} \times (\prod_{i \in Z} Z_i) \cong |\prod|_{i \in Z} Z_i$. Thus, $A(G)^{\perp} = \{0\}$; hence $A(G)^{\perp}$ is representable and G satisfies the hypothesis of Proposition 2.5. Clearly, $A(G)$ has a basis so that G has a basis, and thus, by Proposition 2.5, G is completely distributive. It remains to show that G is not representable. By [5], p. 1.20, Theorem 1.8, for this it suffices to produce $a, x \in G^+ \backslash \{0\}$ such that $a \wedge (-x \oplus a \oplus x) = 0$. For $i \in Z$, let

$$\alpha_i = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, \end{cases} \quad \gamma_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1, \end{cases} \quad \delta_i = \begin{cases} -1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Let $a = (0; \cdots, \alpha_i, \cdots)$ and $x = (1; \cdots, \gamma_i, \cdots)$. Then $-x = (-1; \cdots, \delta_i, \cdots)$, and hence $-x \oplus a \oplus x = (0; \cdots, \gamma_i, \cdots)$. Clearly $a \wedge (-x \oplus a \oplus x) = 0$ and $a > 0 < x$, and therefore, G is not representable.

Otis Kenny has found an example which supplies an affirmative answer to the question posed at the end of Remark 4.2.

References

1. Garrett Birkhoff, *Lattice Theory* (American Mathematical Society Colloquium Publications **25**, Providence, 1967 (third edition)).
2. Richard D. Byrd, *M-Polars in lattice-ordered groups*, Czech. Math. J., **18** (1968), 230-239.
3. Richard D. Byrd and Justin T. Lloyd, *Closed subgroups and complete distributivity in lattice-ordered groups*, Math. Zeit., **101** (1967), 123-130.
4. Paul Conrad, *Some structure theorems for lattice-ordered groups*, Trans. Amer. Math. Soc., **99** (1966), 212-240.
5. Paul Conrad, *Lattice-ordered Groups* (Tulane University, New Orleans, 1970).
6. László Fuchs, *Partially Ordered Algebraic Systems* (Pergamon Press (Addison-Wesley Publ. Co., Inc.), New York, 1963).
7. G. Otis Kenny, *Archimedean kernel of a representable l-group*, Notices Amer. Math. Soc., **21** (1974), #7, A-590.
8. R. H. Redfield, *Bases in completely distributive lattice-ordered groups*, Mich. Math. J., (to appear).
9. Elliot Carl Weinberg, *Higher degrees of distributivity in lattices of continuous functions*, Trans. Amer. Math. Soc., **104** (1962), 334-346.
10. Elliot Carl Weinberg, *Completely distributive lattice-ordered groups*, Pacific J. Math., **12** (1962), 1131-1137.

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