ARCHIMEDEAN AND BASIC ELEMENTS IN COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

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It is known that the bi-prime group $B(G)$ of an $l$-group $G$ contains the basic elements of $G$. We show that every $l$-group $G$ possesses a unique, maximal, archimedean, convex $l$-subgroup $A(G)$, and that if $G$ is completely distributive and if $A(G)^\perp$ is representable, then $B(G)$ has a basis.

1. Introduction. An element $s$ of a lattice-ordered group ($l$-group) $G$ is basic (see [4]) if $s > 0$ and the closed interval $[0, s]$ is totally ordered. An $l$-group $G$ has a basis if every $g > 0$ exceeds some basic element (any maximal disjoint set of basic elements is then a basis). An $l$-group $G$ is completely distributive (see [3], [4], [9], [10]) if the relation

$$\bigwedge \{ \bigvee \{ g_{ij} \mid j \in J \} \mid i \in I \} = \bigvee \{ \bigwedge \{ g_{il} \mid i \in I \} \mid f \in J' \}$$

holds whenever $\{ g_{ij} \mid i \in I, j \in J \} \subseteq G$ is such that all the indicated joins and meets exist. By [5], p. 5.18, Theorem 5.8, every $l$-group which has a basis is completely distributive. For archimedean $l$-groups, i.e. those in which $a \geq nb \geq 0$ for all natural numbers $n$ implies $b = 0$, more can be said: viz., an archimedean $l$-group has a basis if and only if it is completely distributive ([5], p. 5.21, Theorem 5.10). In [8], we constructed, via minimal prime subgroups, the bi-prime group $B(G)$ of an $l$-group $G$ (see §3 below) which contains all the basic elements and which, if $G$ is completely distributive and representable, has a basis. In this note, we introduce “archimedean elements” (see §2 below) in order to investigate possible connections among the above results. Thus, in §2, we show that every $l$-group $G$ possesses a unique, maximal, archimedean, convex $l$-subgroup $A(G)$. (Kenny [7] independently proved this result for representable $l$-groups.) It follows that if $A(G) = \{ 0 \}$, then $G$ is completely distributive if and only if $G$ has a basis. In §3, proving somewhat more general results, we show that $A(B(G)) = B(A(G))$ and hence that if $G$ is completely distributive and if $A(G)^\perp$ is representable, then $B(G)$ has a basis. In §4, we construct two examples, one of which is of completely distributive, nonrepresentable $l$-group which has a basis and for which $A(G)^\perp$ is representable.

NOTATION AND TERMINOLOGY. We use $\square$ for the empty set and write functions on the right. We use $N$, $Z$, and $R$ for the natural
numbers, the integers and the real numbers, respectively. The cartesian product of the sets \( \{S_\alpha | \alpha \in A \} \) is denoted by \( \prod \{S_\alpha | \alpha \in A \} \).

If \( \{G_\alpha | \alpha \in A \} \) is a set of \( l \)-groups, then \( \| \{G_\alpha | \alpha \in A \} \| \sum \| \{G_\alpha | \alpha \in A \} \| \) denotes their cardinal product (sum); if \( A = \{1, 2\} \), we use \( G_1 \times G_2 \) for the cardinal product.

Let \( G \) be an \( l \)-group. A subgroup \( H \) of \( G \) is prime if and only if it is a convex \( l \)-subgroup of \( G \) such that for all \( a, b \in G^+ \setminus H \), \( a \wedge b \in G^+ \setminus H \) (see [5], pp. 1.13–1.16). If \( g \in G \supseteq A, B \), then \( \langle A \rangle \) denotes the convex \( l \)-subgroup generated by \( A \); \( \langle A, B \rangle \equiv \langle A \cup B \rangle \); \( G(g) \equiv \langle \{g\} \rangle \). For any \( S \subseteq G \), the polar of \( S \), defined
\[ S^\perp = \{ g \in G \mid g \wedge s = 0 \text{ for all } s \in S \}, \]
is a convex \( l \)-subgroup of \( G \) (see [8]). The following result will prove useful.

**Lemma 1.1.** Let \( H \) be a convex \( l \)-subgroup of an \( l \)-group \( G \). If \( \{h_\alpha \} \subseteq H \) is such that \( \vee_H h_\alpha \) exists in \( H \), then \( \vee_G h_\alpha \) exists in \( G \) and \( \vee_G h_\alpha = \vee_H h_\alpha \). The dual statement also holds.

**Proof.** Let \( \{h_\alpha \} \subseteq H \) be such that \( \vee_H h_\alpha \in H \). Suppose that the join of \( \{h_\alpha \} \) does not exist in \( G \). Then, since \( \vee_H h_\alpha \) is an upper bound of \( \{h_\alpha \} \) in \( G \), there exists \( b \in G \) such that \( h_\beta \leq b < \vee_H h_\alpha \) for all \( \beta \). Since \( H \) is convex, \( b \in H \). This contradicts the minimality of \( \vee_H h_\alpha \) among upper bounds of \( \{h_\alpha \} \) in \( H \) and hence \( \vee_G h_\alpha \in G \). Since \( \vee_H h_\alpha \in G \) is an upper bound of \( \{h_\alpha \} \), \( h_\beta \leq \vee_G h_\alpha \leq \vee_H h_\alpha \) for all \( \beta \), and hence \( \vee_G h_\alpha \in H \). Therefore, \( \vee_G h_\alpha = \vee_H h_\alpha \). The dual property follows from the above because \( G \) is an \( l \)-group.

For terminology left undefined, see Birkhoff [1], Fuchs [6], or Conrad [5].

2. Archimedean elements. Let \( G \) be an \( l \)-group. An element \( a \in G \) is archimedean if \( a \geq 0 \) and if for all \( 0 < g \leq a \), there exists \( n \in N \) such that \( ng \not\leq a \). Clearly, \( G \) is archimedean if and only if every element of \( G^+ \) is archimedean. Let \( P(G) \) be the set of all archimedean elements of \( G \); let \( A(G) \) be the \( l \)-subgroup of \( G \) generated by \( P(G) \).

**Theorem 2.1.** \( A(G)^+ = P(G) \).

**Proof.** Clearly, \( 0 \in P(G) \) and \( P(G) \) is convex. By [5], p. 1.5, Theorem 1.3, it therefore suffices to show that \( P(G) \) is a subsemigroup of \( G^+ \).

The proof that \( P(G) \) is a subsemigroup is by contradiction.
Suppose there exist \(a, b \in P(G)\) such that \(a + b \in P(G)\). Then there exists \(0 < t \leq a + b\) such that \(nt \leq a + b\) for all \(n \in N\). Since \(a\) is archimedean, there exists \(m > 0\) such that \(mt \not\leq a\). Then

\[
s = (-a + mt) \lor 0 > 0.
\]

Since \(nt \leq a + b\) for all \(n > 0\), \(-a + nt \leq b\) for all \(n > 0\). Thus

\[
(1) \quad (-a + nt) \lor 0 \leq b \quad \text{for all } n \in N,
\]

and in particular \(0 < s \leq b\). We will show by induction that

\[
(2) \quad ks \leq (-a + kmt) \lor 0 \quad \text{for all } k \in N.
\]

Obviously,

\[
s = (-a + mt) \lor 0 \leq (-a + kmt) \lor 0
\]

for all \(k \in N\). Suppose \(ks \leq (-a + kmt) \lor 0\). Then

\[
(k + 1)s = (k + 1)[(-a + mt) \lor 0]
\]

\[
= k[(-a + mt) \lor 0] + [(-a + mt) \lor 0]
\]

\[
\leq [(-a + kmt) \lor 0] + [(-a + mt) \lor 0]
\]

\[
= (-a + kmt - a + mt) \lor (-a + kmt)
\]

\[
\lor (-a + mt) \lor 0
\]

\[
\leq (-a + kmt + mt) \lor (-a + kmt) \lor 0
\]

\[
= (-a + (k + 1)mt) \lor (-a + kmt) \lor 0
\]

\[
= (-a + (k + 1)mt) \lor 0.
\]

Then for all \(k \in N\),

\[
0 < ks \leq (-a + kmt) \lor 0 \quad \text{by (2)}
\]

\[
\leq b \quad \text{by (1)}.
\]

Therefore, \(b \in P(G)\), which contradicts our choice of \(b\). Theorem 12. follows.

**Corollary 2.2.** \(A(G)\) is the unique, maximal, archimedean, convex \(l\)-subgroup of \(G\).

**Proof.** Since \(A(G)^+ = P(G)\), \(A(G)\) is archimedean. By definition of \(P(G)\) any larger \(l\)-subgroup cannot be archimedean. That \(A(G)\) is convex and unique is clear.

**Corollary 2.3.** Let \(g \in G^+\). Then \(g\) is archimedean if and only if \(G(g)\) is archimedean.

**Proof.** The proof of Theorem 2.1 shows that if \(g\) is archimedean,
then \( ng \) is archimedean for all \( n \in \mathbb{N} \). Thus, \( G(g) \) is archimedean. The converse is clear.

**Corollary 2.4.** \( A(G) = \{ g \in G \mid G(g) \) is archimedean}.

**Proof.** If \( g \in A(G) \), then \( |g| \) is archimedean by Theorem 2.1, and thus \( G(|g|) \) is archimedean by Corollary 2.3. Conversely, if \( G(|g|) \) is archimedean, Corollary 2.3 implies that \( |g| \) is archimedean. Hence by Theorem 2.1, \( |g| \in A(G)^+ \). Since \( -|g| \leq g \leq |g| \) and \( A(G) \) is convex, \( g \in A(G) \).

Kenny [7] proved independently that for every representable \( l \)-group \( G \), \( \{ g \in G \mid G(g) \) is archimedean} is the unique, maximal, archimedean, convex \( l \)-subgroup of \( G \); this follows immediately from Corollaries 2.2 and 2.4 above.

**Proposition 2.5.** Let \( G \) be an \( l \)-group in which every strictly positive element exceeds a nonzero archimedean element. Then \( G \) is completely distributive if and only if \( G \) has a basis.

**Proof.** By Lemma 1.1 if \( G \) is completely distributive, \( A(G) \) is completely distributive. Since \( A(G) \) is archimedean, this implies that \( A(G) \) has a basis, and then \( G \) must have a basis because of the initial hypothesis. The converse follows from [5], p. 5.18, Theorem 5.8 (see §1).

3. The bi-prime group and \( A(G) \). In [8], we defined the bi-prime group of an \( l \)-group \( G \) as follows: Let \( \{ P_\phi \mid \phi \in \Phi(G) \} \) be the set of minimal prime subgroups of \( G \). The bi-prime group of \( G \) is the convex \( l \)-subgroup

\[
B(G) = \bigcap \{ \langle P_\phi, P_\omega \rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega \}.
\]

By [8], Theorem 3.1, \( B(G) \) has a basis whenever \( G \) is both completely distributive and representable.

The following result is an easy consequence of [2], Theorem 3.5.

**Lemma 3.1.** Let \( \{0\} \neq S \) be a convex \( l \)-subgroup of an \( l \)-group \( G \). If \( Q \) is a minimal prime subgroup of \( S \), then there exists a minimal prime subgroup \( P \) of \( G \) such that \( Q = P \cap S \). If \( P \) is a minimal prime subgroup of \( G \) which does not contain \( S \), then \( P \cap S \) is a minimal prime subgroup of \( S \).

**Proposition 3.2.** Let \( G \) be an \( l \)-group and let \( H \) be a convex \( l \)-subgroup of \( G \). Then \( B(H) = B(G) \cap H \).
Proof. By [5], p. 1.6, Theorem 1.4, the set of convex $l$-subgroups of an $l$-group, ordered by inclusion, is a (complete) distributive lattice. Combining this with Lemma 3.1, we have the following:

$$B(H) = \bigcap \{ \langle Q_\phi, Q_\omega \rangle | \phi, \omega \in \Phi(H), \phi \neq \omega \}$$

$$= \bigcap \{ \langle P_\phi \cap H, P_\omega \cap H \rangle | \phi, \omega \in \Phi(G), \phi \neq \omega, P_\phi \not\subseteq H \not\subseteq P_\omega \}$$

$$= \bigcap \{ \langle P_\phi \cap H, P_\omega \cap H \rangle | \phi, \omega \in \Phi(G), \phi \neq \omega \}$$

$$= \bigcap \{ \langle P_\phi, P_\omega \cap H | \phi, \omega \in \Phi(G), \phi \neq \omega \}$$

$$= B(G) \cap H.$$

**Corollary 3.3.** For any $l$-group $G$, $B(A(G)) = A(B(G))$.

Proof. By definition of $P(B(G))$(cf. §2), $P(B(G)) = P(G) \cap B(G)$. Thus,

$$A(B(G)) = \langle P(B(G)) \rangle = \langle P(G) \cap B(G) \rangle$$

$$= \langle P(G) \rangle \cap B(G) = A(G) \cap B(G).$$

By Proposition 3.2,

$$A(B(G)) = A(G) \cap B(G) = B(A(G)).$$

**Proposition 3.4** Let $G$ be a completely distributive $l$-group. If $G$ has a representable convex $l$-subgroup $H$ such that $H^\perp = \{0\}$, then $B(G)$ has a basis.

Proof. Since $G$ is completely distributive, $H$ is completely distributive by Lemma 1.1. Thus, since $H$ is representable, $B(H)$ has a basis by [8], Theorem 3.1. By Proposition 3.2 above, $B(H) = H \cap B(G)$. If $g \in B(G) \setminus \{0\}$, then since $H^\perp = \{0\}$, there exists $h \in H$ such that $g \geq h > 0$. But since $B(G)$ is convex, $h \in B(G)$ also, and thus $h \in B(H)$. Since $B(H)$ has a basis, $h$ exceeds a basic element, and hence $g$ exceeds a basic element. Therefore, $B(G)$ has a basis.

**Corollary 3.5.** Let $G$ be a completely distributive $l$-group. If $A(G)^\perp$ is representable, then $B(G)$ has a basis.

Proof. Since $A(G)$ is archimedean, it is abelian and hence representable. Therefore, since $A(G)^\perp$ is representable, $H = \langle A(G), A(G)^\perp \rangle$ is representable (clearly $H$ is $l$-isomorphic to $A(G)| \times | A(G)^\perp$). Clearly, $H^\perp = \{0\}$, and hence by Proposition 3.4, $B(G)$ has a basis.

**Corollary 3.6.** Let $G$ be a completely distributive $l$-group such that $A(G)^\perp$ is representable. Then $G$ has a basis if and only if $B(G)^\perp = \{0\}$. 
4. Examples.

Example 4.1. We construct an abelian, completely distributive $l$-group $H$ such that $A(H) \subseteq B(H)$ but $A(H) \neq B(H)$.

Let $V = \prod \{ R \mid n \in N \}$, and $f, g \in V$; let $S(f, g) = \{ n \in N \mid (n) f \neq (n) g \}$. Then $V$ becomes an $o$-group under (pointwise addition and) the relation: $f \leq g$ if and only if $f = g$ or $f \neq g$ and $(\wedge S(f, g)) f \leq (\wedge S(f, g)) g$. Clearly $V$, is completely distributive and abelian. Furthermore, if $f \in V \setminus \{ 0 \}$ and $h \in G$ is defined by

$$(n) h = \begin{cases} 0 & \text{if } n \leq \wedge S(f, 0) \\ 1 & \text{otherwise} \end{cases},$$

then for all $k \in N$,

$$(\wedge S(f, kh))(kh) = (\wedge S(f, 0))(kh) = k(\wedge S(f, 0))(h) = 0$$

$$< (\wedge S(f, 0)) f = (\wedge S(f, kh)) f,$$

and hence $f$ is not archimedean. Thus, $A(V) = \{ 0 \}$. Let $G = \sum \{ R \mid n \in N \}$. Then clearly, $G$ is completely distributive and abelian, and $A(G) = G$. It is also easy to show that any minimal prime subgroup of $G$ has the form $\{ f \mid n f = 0 \}$ for some $n \in N$, and thus $B(G) = G$.

Let $H = V \times \{ G \}$. Since $V$ is an $o$-group, $V \subseteq B(H)$; by Proposition 3.2, $G \subseteq B(H)$. Thus, $B(H) = H$. Since $A(V) = \{ 0 \}$ and $A(G) = G$, $A(H) = \{ 0 \} \times G$. Thus $A(H)$ is properly contained in $B(H)$. Clearly, $H$ is completely distributive and abelian.

Remark 4.2. If $B(G)$ is strictly contained in $G$ for some completely distributive, archimedean $l$-group $G$, then $H = V \times \{ G \}$ (cf. Example 4.1) is an an abelian, completely distributive $l$-group for which $A(H)$ and $B(H)$ are incomparable. On the other hand, if $B(G) = G$ for all completely distributive, archimedean $l$-groups $G$, then Proposition 3.2 could be used to show that $A(G) \subseteq B(G)$ for every completely distributive $l$-group $G$. Thus, it would be useful to have an answer to the following question: Does there exist a completely distributive, archimedean $l$-group $G$ with distinct (minimal) prime subgroups $P_1$ and $P_2$ such that $A(G) \neq \langle P_1, P_2 \rangle$?

Example 4.3. We construct a non-representable $l$-group $G$ which is completely distributive and has a basis and for which $A(G)^\dagger$ is representable.

Let $G = Z \times Z$ be the wreath product of $Z$ by itself. Thus,
$G = Z \times (\prod_{i \in Z} Z_i)$, where each $Z_i = Z$, and group operation on $G$ is defined as follows:

$$(i; \ldots, \alpha_j, \ldots) \oplus (k; \ldots, \beta_j, \ldots) = (i + k; \ldots, \gamma_j, \ldots),$$

where $\gamma_j = \alpha_j + \beta_j$. An element $(i; \ldots, \alpha_j, \ldots)$ is positive in $G$ if $i > 0$ or if $i = 0$ and $\alpha_j \geq 0$ for all $j$. Clearly $A(G) = \{0\} \times (\prod_{i \in Z} Z_i) \cong \prod_{i \in Z} Z_i$. Thus, $A(G)^\downarrow = \{0\}$; hence $A(G)^\downarrow$ is representable and $G$ satisfies the hypothesis of Proposition 2.5. Clearly, $A(G)\downarrow$ has a basis so that $G$ has a basis, and thus, by Proposition 2.5, $G$ is completely distributive. It remains to show that $G$ is not representable. By [5], p. 1.20, Theorem 1.8, for this it suffices to produce $a, x \in G^\downarrow\setminus\{0\}$ such that $a \wedge (-x \oplus a \oplus x) = 0$. For $i \in Z$, let

$$\alpha_i = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}, \quad \gamma_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}, \quad \delta_i = \begin{cases} -1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}.$$ 

Let $a = (0; \ldots, \alpha_i, \ldots)$ and $x = (1; \ldots, \gamma_i, \ldots)$. Then $-x = (-1; \ldots, \delta_i, \ldots)$, and hence $-x \oplus a \oplus x = (0; \ldots, \gamma_i, \ldots)$. Clearly $a \wedge (-x \oplus a \oplus x) = 0$ and $a > 0 < x$, and therefore, $G$ is not representable.

Otis Kenny has found an example which supplies an affirmative answer to the question posed at the end of Remark 4.2.

References


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