Pacific Journal of Mathematics

ARCHIMEDEAN AND BASIC ELEMENTS IN COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

Robert Horace Redfield

Vol. 63, No. 1 March 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 1, 1976

ARCHIMEDEAN AND BASIC ELEMENTS IN COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

R. H. REDFIELD

It is known that the *bi*-prime group $B(G)$ of an l-group G contains the basic elements of G. We show that every l-group G possesses a unique, maximal, archimedean, convex l-subgroup $A(G)$, and that if G is completely distributive and if $A(G)^{\perp}$ is representable, then $B(G)$ has a basis.

1. Introduction. An element s of a lattice-ordered group (l-group) G is *basic* (see [4]) if $s > 0$ and the closed interval $[0, s]$ is totally ordered. An l-group G *has a basis* if every $g > 0$ exceeds some basic element (any maximal disjoint set of basic elements is then a *basis*). An l-group G is *completely distributive* (see [3], [4], [9], [10]) if the relation

$$\bigwedge \{\vee \{g_{ij} | j \in J\} | i \in I\} = \bigvee \{\wedge \{g_{i(if)} | i \in I\} | f \in J^I\}$$

holds whenever $\{g_{ij} | i \in I, j \in J\} \subseteq G$ is such that all the indicated joins and meets exist. By [5], p. 5.18, Theorem 5.8, every l-group which has a basis is completely distributive. For *archimedean* l-groups, i.e. those in which $a \geqq nb \geqq 0$ for all natural numbers n implies $b = 0$, more can be said: viz., an archimedean l-group has a basis if and only if it is completely distributive ([5], p, 5.21, Theorem 5.10). In [8], we constructed, via minimal prime subgroups, the *bi*-prime group $B(G)$ of an l-group G (see §3 below) which contains all the basic elements and which, if G is completely distributive and representable, has a basis. In this note, we introduce "archimedean elements" (see §2 below) in order to investigate possible connections among the above results. Thus, in §2, we show that every l-group G posseses a unique, maximal, archimedean, convex l-subgroup $A(G)$. (Kenny [7] independently proved this result for representable l-groups.) It follows that if $A(G)^{\perp} = \{0\}$, then G is completely distributive if and only if G has a basis. In §3, proving somewhat more general results, we show that $A(B(G)) = B(A(G))$ and hence that if G is completely distributive and if $A(G)^{\perp}$ is representable, then $B(G)$ has a basis. In §4, we construct two examples, one of which is of completely distributive, nonrepresentable l-group which has a basis and for which $A(G)^{\perp}$ is representable.

NOTATION AND TERMINOLOGY. We use $\square$ for the empty set and write functions on the right. We use N, Z, and R for the natural numbers, the integers and the real numbers, respectively. The cartesian product of the sets $\{S_\alpha | \alpha \in A\}$ is denoted by $\prod \{S_\alpha | \alpha \in A\}$. If $\{G_\alpha | \alpha \in A\}$ is a set of l-groups, then $|\prod| \{G_\alpha | \alpha \in A\} (|\sum| \{G_\alpha | \alpha \in A\})$ denotes their cardinal product (sum); if $A = \{1, 2\}$, we use $G_1 | \times | G_2$ for the cardinal product.

Let G be an l-group. A subgroup H of G is *prime* if and only if it is a convex l-subgroup of G such that for all $a, b \in G^+ \backslash H$, $a \wedge b \in G^+ \backslash H$(see [5], pp. 1.13–1.16). If $g \in G \supseteqq A, B$, then $\langle A \rangle$ denotes the convex l-subgroup generated by A; $\langle A, B \rangle \equiv \langle A \cup B \rangle$; $G(g) \equiv \langle \{g\} \rangle$. For any $S \subseteqq G$, the *polar* of S, defined

$$S^\perp = \{g \in G \mid |g| \wedge |s| = 0 \quad \text{for all } s \in S\} ,$$

is a convex l-subgroup of G (see [8]). The following result will prove useful.

LEMMA 1.1. *Let H be a convex l-subgroup of an l-group G. If $\{h_\alpha\} \subseteqq H$ is such that $\bigvee_H h_\alpha$ exists in H, then $\bigvee_G h_\alpha$ exists in G and $\bigvee_G h_\alpha = \bigvee_H h_\alpha$. The dual statement also holds.*

Proof. Let $\{h_\alpha\} \subseteqq H$ be such that $\bigvee_H h_\alpha \in H$. Suppose that the join of $\{h_\alpha\}$ does not exist in G. Then, since $\bigvee_H h_\alpha$ is an upper bound of $\{h_\alpha\}$ in G, there exists $b \in G$ such that $h_\beta \leqq b < \bigvee_H h_\alpha$ for all β. Since H is convex, $b \in H$. This contradicts the minimality of $\bigvee_H h_\alpha$ among upper bounds of $\{h_\alpha\}$ in H and hence $\bigvee_G h_\alpha \in G$. Since $\bigvee_H h_\alpha \in G$ is an upper bound of $\{h_\alpha\}$, $h_\beta \leqq \bigvee_G h_\alpha \leqq \bigvee_H h_\alpha$ for all β, and hence $\bigvee_G h_\alpha \in H$. Therefore, $\bigvee_G h_\alpha = \bigvee_H h_\alpha$. The dual property follows from the above because G is an l-group.

For terminology left undefined, see Birkhoff [1], Fuchs [6], or Conrad [5].

2. Archimedean elements. Let G be an l-group. An element $a \in G$ is *archimedean* if $a \geqq 0$ and if for all $0 < g \leqq a$, there exists $n \in N$ such that $ng \nleqq a$. Clearly, G is archimedean if and only if every element of G^+ is archimedean. Let $P(G)$ be the set of all archimedean elements of G; let $A(G)$ be the l-subgroup of G generated by $P(G)$.

THEOREM 2.1. $A(G)^+ = P(G)$.

Proof. Clearly, $0 \in P(G)$ and $P(G)$ is convex. By [5], p. 1.5, Theorem 1.3, it therefore suffices to show that $P(G)$ is a subsemigroup of G^+.

The proof that $P(G)$ is a subsemigroup is by contradiction.

Suppose there exist $a, b \in P(G)$ such that $a + b \notin P(G)$. Then there exists $0 < t \leqq a + b$ such that $nt \leqq a + b$ for all $n \in N$. Since a is archimedean, there exists $m > 0$ such that $mt \nleqq a$. Then

$$s = (-a + mt) \vee 0 > 0 .$$

Since $nt \leqq a + b$ for all $n > 0$, $-a + nt \leqq b$ for all $n > 0$. Thus

$$(1) \qquad (-a + nt) \vee 0 \leqq b \quad \text{for all } n \in N ,$$

and in particular $0 < s \leqq b$. We will show by induction that

$$(2) \qquad ks \leqq (-a + kmt) \vee 0 \quad \text{for all} \quad k \in N .$$

Obviously,

$$s = (-a + mt) \vee 0 \leqq (-a + kmt) \vee 0$$

for all $k \in N$. Suppose $ks \leqq (-a+kmt) \vee 0$. Then

$$\begin{aligned}(k + 1)s &= (k + 1)[(-a + mt) \vee 0] \\ &= k[(-a + mt) \vee 0] + [(-a + mt) \vee 0] \\ &\leqq [(-a + kmt) \vee 0] + [(-a + mt) \vee 0] \\ &= (-a + kmt - a + mt) \vee (-a + kmt) \\ &\quad \vee (-a + mt) \vee 0 \\ &\leqq (-a + kmt + mt) \vee (-a + kmt) \vee 0 \\ &= (-a + (k + 1)mt) \vee (-a + kmt) \vee 0 \\ &= (-a + (k + 1)mt) \vee 0 .\end{aligned}$$

Then for all $k \in N$,

$$\begin{aligned}0 < ks &\leqq (-a + kmt) \vee 0 && \text{by (2)} \\ &\leqq b && \text{by (1)} .\end{aligned}$$

Therefore, $b \notin P(G)$, which contradicts our choice of b. Theorem 12. follows.

COROLLARY 2.2. *$A(G)$ is the unique, maximal, archimedean, convex l-subgroup of G.*

Proof. Since $A(G)^+ = P(G)$, $A(G)$ is archimedean. By definition of $P(G)$ any larger l-subgroup cannot be archimedean. That $A(G)$ is convex and unique is clear.

COROLLARY 2.3. *Let $g \in G^+$. Then g is archimedean if and only if $G(g)$ is archimedean.*

Proof. The proof of Theorem 2.1 shows that if g is archimedean, then ng is archimedean for all $n \in N$. Thus, $G(g)$ is archimedean. The converse is clear.

COROLLARY 2.4. $A(G) = \{g \in G \mid G(|g|) \text{ is archimedean}\}$.

Proof. If $g \in A(G)$, then $|g|$ is archimedean by Theorem 2.1, and thus $G(|g|)$ is archimedean by Corollary 2.3. Conversely, if $G(|g|)$ is archimedean, Corollary 2.3 implies that $|g|$ is archimedean. Hence by Theorem 2.1, $|g| \in A(G)^+$. Since $-|g| \leqq g \leqq |g|$ and $A(G)$ is convex, $g \in A(G)$.

Kenny [7] proved independently that for every representable l-group G, $\{g \in G \mid G(|g|) \text{ is archimedean}\}$ is the unique, maximal, archimedean, convex l-subgroup of G; this follows immediately from Corollaries 2.2 and 2.4 above.

PROPOSITION 2.5. *Let G be an l-group in which every strictly positive element exceeds a nonzero archimedean element. Then G is completely distributive if and only if G has a basis.*

Proof. By Lemma 1.1 if G is completely distributive, $A(G)$ is completely distributive. Since $A(G)$ is archimedean, this implies that $A(G)$ has a basis, and then G must have a basis because of the initial hypothesis. The converse follows from [5], p. 5.18, Theorem 5.8 (see §1).

3. The bi-prime group and $A(G)$. In [8], we defined the bi-prime group of an l-group G as follows: Let $\{P_\phi \mid \phi \in \Phi(G)\}$ be the set of minimal prime subgroups of G. The *bi-prime group* of G is the convex l-subgroup

$$B(G) = \bigcap \{\langle P_\phi, P_\omega\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} .$$

By [8], Theorem 3.1, $B(G)$ has a basis whenever G is both completely distributive and representable.

The following result is an easy consequence of [2], Theorem 3.5.

LEMMA 3.1. *Let $\{0\} \neq S$ be a convex l-subgroup of an l-group G. If Q is a minimal prime subgroup of S, then there exists a minimal prime subgroup P of G such that $Q = P \cap S$. If P is a minimal prime subgroup of G which does not contain S, then $P \cap S$ is a minimal prime subgroup of S.*

PROPOSITION 3.2. *Let G be an l-group and let H be a convex l-subgroup of G. Then $B(H) = B(G) \cap H$.*

Proof. By [5], p. 1.6, Theorem 1.4, the set of convex l-subgroups of an l-group, ordered by inclusion, is a (complete) distributive lattice. Combining this with Lemma 3.1, we have the following:

$$\begin{aligned} B(H) &= \bigcap \{\langle Q_\phi, Q_\omega\rangle \mid \phi, \omega \in \Phi(H), \phi \neq \omega\} \\ &= \bigcap \{\langle P_\phi \cap H, P_\omega \cap H\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega, P_\phi \ngeqq H \nleqq P_\omega\} \\ &= \bigcap \{\langle P_\phi \cap H, P_\omega \cap H\rangle \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} \\ &= \bigcap \{\langle P_\phi, P_\omega\rangle \cap H \mid \phi, \omega \in \Phi(G), \phi \neq \omega\} \\ &= B(G) \cap H. \end{aligned}$$

COROLLARY 3.3. *For any l-group G, $B(A(G)) = A(B(G))$.*

Proof. By definition of $P(B(G))$(cf. §2), $P(B(G)) = P(G) \cap B(G)$. Thus,

$$\begin{aligned} A(B(G)) &= \langle P(B(G))\rangle = \langle P(G) \cap B(G)\rangle \\ &= \langle P(G)\rangle \cap B(G) = A(G) \cap B(G) . \end{aligned}$$

By Proposition 3.2,

$$A(B(G)) = A(G) \cap B(G) = B(A(G)) .$$

PROPOSITION 3.4 *Let G be a completely distributive l-group. If G has a representable convex l-subgroup H such that $H^\perp = \{0\}$, then $B(G)$ has a basis.*

Proof. Since G is completely distributive, H is completely distributive by Lemma 1.1. Thus, since H is representable, $B(H)$ has a basis by [8], Theorem 3.1. By Proposition 3.2 above, $B(H) = H \cap B(G)$. If $g \in B(G)^+\backslash\{0\}$, then since $H^\perp = \{0\}$, there exists $h \in H$ such that $g \geqq h > 0$. But since $B(G)$ is convex, $h \in B(G)$ also, and thus $h \in B(H)$. Since $B(H)$ has a basis, h exceeds a basic element, and hence g exceeds a basic element. Therefore, $B(G)$ has a basis.

COROLLARY 3.5. *Let G be a completely distributive l-group. If $A(G)^\perp$ is representable, then $B(G)$ has a basis.*

Proof. Since $A(G)$ is archimedean, it is abelian and hence representable. Therefore, since $A(G)^\perp$ is representable, $H = \langle A(G), A(G)^\perp\rangle$ is representable (clearly H is l-isomorphic to $A(G) \mid \times \mid A(G)^\perp$). Clearly, $H^\perp = \{0\}$, and hence by Proposition 3.4, $B(G)$ has a basis.

COROLLARY 3.6. *Let G be a completely distributive l-group such that $A(G)^\perp$ is representable. Then G has a basis if and only if $B(G)^\perp = \{0\}$.*

4. Examples.

EXAMPLE 4.1. We construct an abelian, completely distributive l-group H such that $A(H) \subseteqq B(H)$ but $A(H) \neq B(H)$.

Let $V = \prod \{R \mid n \in N\}$, and $f, g \in V$; let $S(f, g) \equiv \{n \in N \mid (n)f \neq (n)g\}$. Then V becomes an o-group under (pointwise addition and) the relation: $f \leqq g$ if and only if $f = g$ or $f \neq g$ and $(\wedge S(f, g))f \leqq (\wedge S(f, g))g$. Clearly V, is completely distributive and abelian. Furthermore, if $f \in V^+ \backslash \{0\}$ and $h \in G$ is defined by

$$(n)h = \begin{cases} 0 & \text{if } n \leqq \wedge S(f, 0) \\ 1 & \text{otherwise}, \end{cases}$$

then for all $k \in N$,

$$\begin{aligned}(\wedge S(f, kh))(kh) = (\wedge S(f, 0))(kh) = k(\wedge S(f, 0))(h) = 0 \\ < (\wedge S(f, 0))f = (\wedge S(f, kh))f,\end{aligned}$$

and hence f is not archimedean. Thus, $A(V) = \{0\}$. Let $G = |\sum| \{R|n \in N\}$. Then clearly, G is completely distributive and abelian, and $A(G) = G$. It is also easy to show that any minimal prime subgroup of G has the form $\{f \mid nf = 0\}$ for some $n \in N$, and thus $B(G) = G$.

Let $H = V| \times |G$. Since V is an o-group, $V \subseteqq B(H)$; by Proposition 3.2, $G \subseteqq B(H)$. Thus, $B(H) = H$. Since $A(V) = \{0\}$ and $A(G) = G$, $A(H) = \{0\} \times G$. Thus $A(H)$ is properly contained in $B(H)$. Clearly, H is completely distributive and abelian.

REMARK 4.2. If $B(G)$ is strictly contained in G for some completely distributive, archimedean l-group G, then $H = V| \times |G$ (cf. Example 4.1) is an an abelian, completely distributive l-group for which $A(H)$ and $B(H)$ are incomparable. On the other hand, if $B(G) = G$ for all completely distributive, archimedean l-groups G, then Proposition 3.2 could be used to show that $A(G) \subseteqq B(G)$ for every completely distributive l-group G. Thus, it would be useful to have an answer to the following question: Does there exist a completely distributive, archimedean l-group G with distinct (minimal) prime subgroups P_1 and P_2 such that $G \neq \langle P_1, P_2 \rangle$?

EXAMPLE 4.3. We construct a non-representable l-group G which is completely distributive and has a basis and for which $A(G)^{\perp}$ is representable.

Let $G = Z\, Wr\, Z$ be the wreath product of Z by itself. Thus,

$G = Z \times (\prod_{i \in Z} Z_i)$, where each $Z_i = Z$, and group operation on G is defined as follows:

$$(i; \cdots, \alpha_j, \cdots) \oplus (k; \cdots, \beta_j, \cdots) = (i + k; \cdots, \gamma_j, \cdots),$$

where $\gamma_j = \alpha_{j-k} + \beta_j$. An element $(i; \cdots, \cdots, \alpha_j, \cdots)$ is positive in G if $i > 0$ or if $i = 0$ and $\alpha_j \geqq 0$ for all j. Clearly $A(G) = \{0\} \times (\prod_{i \in Z} Z_i) \cong |\prod|_{i \in Z} Z_i$. Thus, $A(G)^{\perp} = \{0\}$; hence $A(G)^{\perp}$ is representable and G satisfies the hypothesis of Proposition 2.5. Clearly, $A(G)$ has a basis so that G has a basis, and thus, by Proposition 2.5, G is completely distributive. It remains to show that G is not representable. By [5], p. 1.20, Theorem 1.8, for this it suffices to produce $a, x \in G^+ \backslash \{0\}$ such that $a \wedge (-x \oplus a \oplus x) = 0$. For $i \in Z$, let

$$\alpha_i = \begin{cases} 1 & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0, \end{cases} \qquad \gamma_i = \begin{cases} 1 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1, \end{cases} \qquad \delta_i = \begin{cases} -1 & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0. \end{cases}$$

Let $a = (0; \cdots, \alpha_i, \cdots)$ and $x = (1; \cdots, \gamma_i, \cdots)$. Then $-x = (-1; \cdots, \delta_i, \cdots)$, and hence $-x \oplus a \oplus x = (0; \cdots, \gamma_i, \cdots)$. Clearly $a \wedge (-x \oplus a \oplus x) = 0$ and $a > 0 < x$, and therefore, G is not representable.

Otis Kenny has found an example which supplies an affirmative answer to the question posed at the end of Remark 4.2.

References

1. Garrett Birkhoff, *Lattice Theory* (American Mathematical Society Colloquium Publications **25**, Providence, 1967 (third edition)).
2. Richard D. Byrd, *M-Polars in lattice-ordered groups*, Czech. Math. J., **18** (1968), 230-239.
3. Richard D. Byrd and Justin T. Lloyd, *Closed subgroups and complete distributivity in lattice-ordered groups*, Math. Zeit., **101** (1967), 123-130.
4. Paul Conrad, *Some structure theorems for lattice-ordered groups*, Trans. Amer. Math. Soc., **99** (1966), 212-240.
5. Paul Conrad, *Lattice-ordered Groups* (Tulane University, New Orleans, 1970).
6. László Fuchs, *Partially Ordered Algebraic Systems* (Pergamon Press (Addison-Wesley Publ. Co., Inc.), New York, 1963).
7. G. Otis Kenny, *Archimedean kernel of a representable l-group*, Notices Amer. Math. Soc., **21** (1974), #7, A-590.
8. R. H. Redfield, *Bases in completely distributive lattice-ordered groups*, Mich. Math. J., (to appear).
9. Elliot Carl Weinberg, *Higher degrees of distributivity in lattices of continuous functions*, Trans. Amer. Math. Soc., **104** (1962), 334-346.
10. Elliot Carl Weinberg, *Completely distributive lattice-ordered groups*, Pacific J. Math., **12** (1962), 1131-1137.

Received July 1, 1975.

SIMON FRASER UNIVERSITY
AND
MONASH UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 63, No. 1 March, 1976