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WEIGHTED SIDON SETS

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A weighted generalisation of Sidon sets, W -Sidon sets, is introduced and studied for compact abelian groups. Firstly W -Sidon sets are characterised analogously to Sidon sets and variations of these characterisations shown to lead back to Sidon sets. For the circle group W -Sidon sets are constructed which are not $A(1)$ and hence not Sidon. The algebra of all W 's making a set W -Sidon is investigated and Sidon and p -Sidon sets cast in terms of it. Finally analytic properties of W -Sidon sets are pursued and a necessary condition on the growth of W^2 obtained.

Throughout this paper G denotes a compact abelian Hausdorff topological group and X denotes its (discrete) dual group. Both are written multiplicatively with identities e and 1 respectively.

We write $(L^p(G), \|\cdot\|_p)$ for the Lebesgue space derived from the normalised Haar measure on G and $(C(G), \|\cdot\|_\infty)$ for the space of (complex-valued) functions continuous on G with the supremum norm. However for $\Delta \subseteq X$ and counting measure on Δ we denote the Lebesgue spaces $(l^p(\Delta), \|\cdot\|_p)$ and use $c_0(\Delta)$ for the subset of $l^\infty(\Delta)$ of functions tending to zero at infinity.

If A and B are sets we write B^A for the set of all functions from A to B ; if $f \in B^A$ and $C \subseteq A$ (\subset is reserved for strict inclusion) we write $f|_C$ for the restriction of f to C ; ξ_A is the characteristic function of A ; $\mathfrak{F}(A)$ denotes the set of all finite subsets of A ; $\mathfrak{P}(A)$ denotes the power set of A ; $\nu(A)$ is the cardinality of A ; and we write \square for the empty set.

The sets of complex numbers, real numbers, integers and natural numbers will be written \mathbb{C} , \mathbb{R} , \mathbb{Z} , and \mathbb{N} respectively and we write \mathfrak{T} for the topological group of unimodular complex numbers. If $c \in \mathbb{C}$, c denotes the constant function with value c , whose domain will be clear from the context.

For $\Delta \subseteq X$, $\phi \in \mathbb{C}^{\Delta}$ and $A \subseteq \mathbb{C}^{\Delta}$ we write ϕA for $\{\phi\psi: \psi \in A\}$.

We denote the Fourier transform of $f \in L^1(G)$ by \hat{f} . If E is a Banach space we write E' for its dual. Let $A(G) = \{f \in C(G): \hat{f} \in l^1(X)\}$ be normed by $\|f\|_A = \|\hat{f}\|_1$ and set the space of pseudomeasures on G , $(PM(G), \|\cdot\|_{PM})$, equal to $A(G)'$ so that it contains $(M(G), \|\cdot\|)$, the space of measures on G . For $\pi \in PM(G)$ we write $\hat{\pi}$ for its Fourier transform and $sp\pi$ for its spectrum, i.e. $\{\chi \in X: \hat{\pi}(\chi) \neq 0\}$. If $E \subseteq PM(G)$ and $\Delta \subseteq X$ we let $E_\Delta = \{\pi \in E: sp\pi \subseteq \Delta\}$ and call its members Δ -spectral pseudomeasures. We also write E^\wedge for $\{\hat{\pi}: \pi \in E\}$.

The set of trigonometric polynomials on G will be denoted $T(G)$.

A subset \mathcal{A} of X is called

(i) a Sidon set iff

$$\sup \{ \sum_{\chi \in \mathcal{A}} |\hat{t}(\chi)| : t \in T_{\mathcal{A}}(G) \text{ and } \|t\|_{\infty} \leq 1 \} < \infty, \text{ and}$$

(ii) a $\mathcal{A}(p)$ set, for $0 < p < \infty$ (written $\mathcal{A} \in \mathcal{A}(p)$) iff for some r with $0 < r < p$, $L^p_{\mathcal{A}}(G) = L^r_{\mathcal{A}}(G)$. The reader is referred to [2] for an exposition of Sidon and $\mathcal{A}(p)$ sets.

1. W -Sidon sets.

DEFINITIONS 1.0. If $\mathcal{A} \subseteq X$ and $W \in \mathbb{C}^{\mathcal{A}}$ we let

$$\|W\|_{\mathcal{A}} = \sup \{ \sum_{\chi \in \mathcal{A}} |W(\chi)\hat{t}(\chi)| : t \in T_{\mathcal{A}}(G) \text{ and } \|t\|_{\infty} \leq 1 \}$$

and say \mathcal{A} is W -Sidon iff this is finite. Set

$$\mathfrak{B}(\mathcal{A}) = \{ W \in \mathbb{C}^{\mathcal{A}} : \|W\|_{\mathcal{A}} < \infty \}.$$

Evidently $\|W\|_{\mathcal{A}}$ equals the least constant for which, whenever $t \in T_{\mathcal{A}}(G)$, $\sum_{\chi \in \mathcal{A}} |W(\chi)\hat{t}(\chi)| \leq k\|t\|_{\infty}$.

The letter W is used to suggest a weight function and W -Sidon sets should not be confused with p -Sidon sets ([4]) or V -Sidon sets ([13]).

1.1. Taking $\chi \in \mathcal{A}$ as t above we see $\|W\|_{\infty} \leq \|W\|_{\mathcal{A}}$. So \mathcal{A} is Sidon iff $\mathfrak{B}(\mathcal{A}) = l^{\infty}(\mathcal{A})$ and the Sidon constant of \mathcal{A} equals $\|1\|_{\mathcal{A}}$.

1.2. For any $\mathcal{A} \subseteq X$, $l^2(\mathcal{A}) \subseteq \mathfrak{B}(\mathcal{A})$.

For if $t \in T(G)$ the Cauchy-Schwarz inequality followed by Parseval's identity shows

$$\sum_{\chi \in \mathcal{A}} |W(\chi)\hat{t}(\chi)| \leq \|W\|_2 \|\hat{t}\|_2 = \|W\|_2 \|t\|_2 \leq \|W\|_2 \|t\|_{\infty}.$$

Thus $\|W\|_{\mathcal{A}} \leq \|W\|_2$.

In the W -Sidon theory to follow, sets \mathcal{A} for which $W \in l^2(\mathcal{A})$ behave very like finite sets in the Sidon theory. We refer to them as trivial W -Sidon sets.

Examples of \mathcal{A} and W for which $W \notin l^2(\mathcal{A})$ yet \mathcal{A} is W -Sidon and not Sidon are given in 2.3, and some infinite \mathcal{A} 's which are W -Sidon only for $W \in l^2(\mathcal{A})$ in 3.4.

1.3. In 1.0 we have not referred directly to the group X . The following result excuses this. Let X_1 and X_2 be discrete abelian groups with $\mathcal{A} \subseteq X_1$ and X_1 a subgroup of X_2 .

THEOREM. For $W \in \mathbb{C}^{\mathcal{A}}$, \mathcal{A} is W -Sidon as a subset of X_1 iff it

is W -Sidon as a subset of X_2 .

Proof. Suppose that G_i is the dual of X_i for $i \in \{1, 2\}$ and define an equivalence relation α on G_1 by $(x, y) \in \alpha$ iff $\chi(x) = \chi(y)$ for all $\chi \in X_1$. Writing A for $\{x \in G_1: \chi(x) = 1 \text{ for all } \chi \in X_1\}$, the kernel of α , A is a closed subgroup of G_1 and G_1/A is isomorphic to G_2 by [10], 2.1.

For $t \in T_d(G_2)$ define $t^* \in T_d(G_1/A)$ by

$$t^*(\alpha(x)) = \sum_{\chi \in \mathcal{A}} \hat{t}(\chi)\chi(x).$$

By definition of α , the map $\beta: T_d(G_2) \rightarrow T_d(G_1/A)$ given by $\beta(t) = t^*$ is well defined. It is easily seen to be a vector space isomorphism, $\|\cdot\|_\infty$ -isometric and to satisfy

$$(\beta(t))^\wedge(\chi) = \hat{t}(\chi) \text{ for all } t \in T_d(G_2) \text{ and } \chi \in \mathcal{A}.$$

Consequently

$$\begin{aligned} & \sup \left\{ \sum_{\chi \in \mathcal{A}} |W(\chi)\hat{t}(\chi)| : t \in T_d(G_2) \text{ with } \|t\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \sum_{\chi \in \mathcal{A}} |W(\chi)\hat{u}(\chi)| : u \in T_d(G_1/A) \text{ with } \|u\|_\infty \leq 1 \right\} \end{aligned}$$

and the conclusion follows.

1.4. To see how W -Sidon sets are affected by group operations on X we extend 1.3 as follows. If ϕ is a function from one discrete abelian group X_1 to another, X_2 , (with duals G_i) it induces a map ϕ^* from $T(G_1)$ to $T(G_2)$ by

$$\sum_{\chi \in X_1} \hat{t}(\chi)\chi \longmapsto \sum_{\chi \in X_1} \hat{t}(\chi)\phi(\chi).$$

When ϕ^* is $\|\cdot\|_\infty$ -isometric, ϕ is injective so given $\mathcal{A} \subseteq X$ and $W \in \mathbb{C}^{\mathcal{A}}$ there is a map $W_\phi \in \mathbb{C}^\phi$ defined by

$$W_\phi(\phi(\chi)) = W(\chi) \text{ for all } \chi \in \mathcal{A}.$$

THEOREM. *If ϕ^* is $\|\cdot\|_\infty$ -isometric, \mathcal{A} is W -Sidon iff $\phi(\mathcal{A})$ is W_ϕ -Sidon.*

Proof. Now ϕ^* maps $T_d(G_1)$ onto $T_{\phi(\mathcal{A})}(G_2)$ and whenever $t \in T_d(G_1)$ and $\chi \in \mathcal{A}$,

$$W(\chi)\hat{t}(\chi) = W_\phi(\phi(\chi))(\phi^*t)^\wedge(\phi(\chi)).$$

Consequently, using 1.3 to move from the group $\phi(X_1)$ to X_2 ,

$$\begin{aligned} \|W\|_{\mathcal{A}} &= \sup \left\{ \sum_{\chi \in \mathcal{A}} |W(\chi)\hat{t}(\chi)| : t \in T_{\mathcal{A}}(G_1) \text{ and } \|t\|_{\infty} \leq 1 \right\} \\ &= \sup \left\{ \sum_{\xi \in \phi(\mathcal{A})} |W_{\phi}(\xi)\hat{u}(\xi)| : u \in T_{\phi(\mathcal{A})}(G_2) \text{ and } \|u\|_{\infty} \leq 1 \right\} \\ &= \|W_{\phi}\|_{\phi(\mathcal{A})}. \end{aligned}$$

1.5. (i) For example take as ϕ the map $\tau_{\chi_0}: X \rightarrow X$ (for $\chi_0 \in X$) given by $\tau_{\chi_0}(\chi) = \chi_0\chi$. If $t \in T(G)$,

$$\|\tau_{\chi_0}^*(t)\|_{\infty} = \left\| \sum_{\chi \in X} \hat{t}(\chi)\chi_0\chi \right\|_{\infty} = \left\| \sum_{\chi \in X} \hat{t}(\chi)\chi \right\|_{\infty}$$

whence $\tau_{\chi_0}^*$ is $\|\cdot\|_{\infty}$ -isometric. For any $\mathcal{A} \subseteq X$, $\chi_0 \in X$ and $W \in \mathbb{C}^{\mathcal{A}}$, provided we define $W_0 \in \mathbb{C}^{\chi_0\mathcal{A}}$ by $W_0(\chi_0\chi) = W(\chi)$ for all $\chi \in \mathcal{A}$, 1.4 guarantees

$$\mathfrak{B}(\chi_0\mathcal{A}) = \{W_0 : W \in \mathfrak{B}(\mathcal{A})\}.$$

(ii) Similarly if we define $\rho: X \rightarrow X$ by $\rho(\chi) = \chi^{-1}$ then provided we set $W_{\rho} \in \mathbb{C}^{\mathcal{A}^{-1}}$ to be $W_{\rho}(\chi^{-1}) = W(\chi)$, 1.4 shows

$$\mathfrak{B}(\mathcal{A}^{-1}) = \{W_{\rho} : W \in \mathfrak{B}(\mathcal{A})\}.$$

(iii) Note that for $W \in \mathbb{C}^{\mathcal{A} \cup \chi_0\mathcal{A}}$, 1.5(i) does not claim \mathcal{A} is W -Sidon iff $\chi_0\mathcal{A}$ is W -Sidon (and similarly for 1.5(ii)).

If \mathcal{A} is an infinite proper subgroup of X (it can be chosen for \mathfrak{B} say) and $\chi_0 \in X \setminus \mathcal{A}$ then clearly $\chi_0\mathcal{A} \cap \mathcal{A} = \emptyset$. So we may choose $W \in \mathbb{C}^{\mathcal{A} \cup \chi_0\mathcal{A}}$ such that $W|_{\mathcal{A}} \in l^2(\mathcal{A})$ yet $W|_{\chi_0\mathcal{A}} \in l^{\infty}(\chi_0\mathcal{A}) \setminus l^2(\chi_0\mathcal{A})$. A premature glance at 3.3 now shows, together with 1.5(i), that $\mathfrak{B}(\mathcal{A}) = l^2(\mathcal{A})$ and $\mathfrak{B}(\chi_0\mathcal{A}) = l^2(\chi_0\mathcal{A})$. Thus \mathcal{A} is W -Sidon yet $\chi_0\mathcal{A}$ is not W -Sidon (taking restrictions for granted).

1.6. Suppose E is a Banach space contained in $PM(G)$, with norm $\|\cdot\|_E$ stronger than $\|\cdot\|_{PM}$. For $\mathcal{A} \subseteq X$ define $\delta: E \rightarrow E^{\wedge} | \mathcal{A}$ by $\delta(\pi) = \hat{\pi} | \mathcal{A}$. Since δ is a vector space morphism, $\ker \delta$ is a subspace of E . This subspace is closed since if $\pi \in E$ and $\{\pi_n : n \in \mathfrak{N}\} \subseteq \ker \delta$ with $\|\pi - \pi_n\|_E \rightarrow 0$ then $\|\hat{\pi} - \hat{\pi}_n\|_{\infty} \rightarrow 0$ hence $\hat{\pi} | \mathcal{A} = 0$.

Thus $E/\ker \delta$ is a Banach space under the quotient norm. Equivalently, $E^{\wedge} | \mathcal{A}$ is a Banach space with norm

$$\|\phi\|_{\delta} = \inf \{ \|\pi\|_E : \pi \in E \text{ and } \hat{\pi} | \mathcal{A} = \phi \}.$$

Evidently for all $\pi \in E$,

$$\|\hat{\pi}\|_{\infty} \leq \|\hat{\pi} | \mathcal{A}\|_{\delta} \leq \|\pi\|_E.$$

(See also 3.7.)

If E is a Banach subalgebra of $PM(G)$ (not necessarily with identity) then so too is $E^{\wedge} | \mathcal{A}$.

When considering E' rather than E we write δ' in place of δ .

1.7. Our dependence on Δ -spectral functions makes the following result useful. Refer to [7], Chapter 1, (2.10) for the definition of a homogeneous Banach space on G , replacing \mathfrak{X} there by G .

Suppose E is a homogeneous Banach space on G and E' is the dual of E under a pairing $\langle f, \psi \rangle$ (for $f \in E$ and $\psi \in E'$). If $\psi \in E'$ and $\chi \in X \cap E$ then the Fourier coefficient is defined to be

$$\hat{\psi}(\chi) = \langle \chi, \overline{\psi} \rangle$$

and satisfies $|\hat{\psi}(\chi)| \leq \|\psi\|_{E'} \|\chi\|_E$.

THEOREM. *Let $\Delta \subseteq X$, let E be a homogeneous Banach space on G containing Δ and suppose that, restricted to Δ , $\|\cdot\|_E$ is weaker than $\|\cdot\|_\Delta$. Then there is a canonical isomorphism from $(E_\Delta)'$ to $(E')^\wedge | \Delta$ (the latter being normed by $\|\cdot\|_{\delta'}$) whose norm is less than or equal to one.*

Proof. Since

$$\|\hat{f}\|_\infty \leq \|f\|_1 \leq \|f\|_E, \text{ for all } f \in E,$$

E_Δ is a closed subspace of E . So the canonical map

$$J: (E_\Delta)' \longrightarrow E'/(E_\Delta)^0$$

is an isomorphism of norm less than or equal to 1, where $(E_\Delta)^0$, the annihilator of E_Δ , is $\{\psi \in E': \psi(f) = 0 \text{ for all } f \in E_\Delta\}$ (see [8], p. 93).

Now $|\hat{\psi}(\chi)| \leq \|\psi\|_{E'}$ whenever $\psi \in E'$ and $\chi \in \Delta$ thus by 1.6 it remains to show that $(E_\Delta)^0 = \ker \delta'$. If $\psi \in (E_\Delta)^0$ then $\psi(\chi) = 0$ for all $\chi \in \Delta$ hence $\langle \chi, \psi \rangle = 0$ so that $\hat{\psi} | \Delta = 0$ whence $\psi \in \ker \delta'$. Conversely if $\hat{\psi}(\chi) = 0$ for all $\chi \in \Delta$ then $\psi(t) = 0$ for all $t \in \text{span}(\Delta)$. But $\text{span}(\Delta)$ is dense in E_Δ (by the method of [7], Chapter 1, (2.12)) hence $\psi(f) = 0$ whenever $f \in E_\Delta$, whence $\psi \in (E_\Delta)^0$.

Consequently $(E_\Delta)'$ is isomorphic to $(E')^\wedge | \Delta$ under J followed by the Fourier transform lifted to $E'/\ker \delta'$.

COROLLARY 1.8. *Let $\Delta \subseteq X$. Then*

- (i) *if $1 \leq p < \infty$, there is a canonical isomorphism from $L^p_\Delta(G)'$ to $L^p(G)^\wedge | \Delta$ whose norm is dominated by 1,*
- (ii) *there is a canonical isomorphism from $C_\Delta(G)'$ to $M(G)^\wedge | \Delta$ whose norm is dominated by 1, and*
- (iii) *if $1 \leq p < \infty$, there is a canonical isomorphism from $(L^p(G)^\wedge | \Delta)'$ to $L^p_\Delta(G)$.*

Proof. (i) and (ii) follow immediately from 1.7.

If $1 < p < \infty$, $L^p_2(G)$, being a closed subspace of the reflexive space $L^p(G)$, is also reflexive. So by (i) the dual of $L^p(G)^\wedge | \Delta$ is canonically isomorphic to $L^p_2(G)''$, i.e. to $L^p_2(G)$.

For $p = 1$ we are forced to resort to the method of 1.7. Any $\psi \in (L^1(G)^\wedge | \Delta)'$ lifts to a continuous linear map $\Psi: L^1(G) \rightarrow \mathbb{C}$ which is constant on cosets of $\ker \delta'$ and which may be identified with an element of $L^\infty(G)$, giving $\|\Psi\|_\infty \leq \|\psi\|$. Consequently if $\chi \in X \setminus \Delta$,

$$\hat{\Psi}(\chi) = \int_G \psi \bar{\chi} = \int_G \psi \cdot \mathbf{0} = 0$$

so that $\Psi \in L^\infty_2(G)$. This yields a map from $(L^1(G)^\wedge | \Delta)'$ to $L^\infty_2(G)$ and the method of 1.7 completes the proof.

REMARKS 1.9. (i) Obviously $A_\Delta(G)'$ is isometrically isomorphic to $PM(G)^\wedge | \Delta$ as is $L^2_\Delta(G)'$ to $L^2(G)^\wedge | \Delta$.

(ii) In (i) and (iii) above it suffices to take $\Delta = X$ to see the falsity for $p = \infty$. However $L^1_\Delta(G)$ can still be embedded canonically in $(L^\infty(G)^\wedge | \Delta)'$, as can $C_\Delta(G)$ in $(M(G)^\wedge | \Delta)'$.

THEOREM 1.10. Let $\Delta \subseteq X$ and $W \in \mathbb{C}^\Delta$. With the understanding that the constants in (ii), (iii), (iv) and (v) are the least possible, the following are equivalent:

- (i) Δ is W -Sidon with $\kappa = \|W\|_\Delta$,
- (ii) $f \in L^\infty_2(G)$ implies $\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| \leq \kappa \|f\|_\infty$,
- (iii) $f \in C_\Delta(G)$ implies $\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| \leq \kappa \|f\|_\infty$,
- (iv) for all $\phi \in l^\infty(\Delta)$ there exists $\mu \in M(G)$ with $\hat{\mu}|_\Delta = W\phi$ and $\|u\| \leq \kappa \|\phi\|_\infty$,
- (v) for all $\phi \in c_0(\Delta)$ there exists $f \in L^1(G)$ with $\hat{f}|_\Delta = W\phi$ and $\|f\|_1 \leq \kappa \|\phi\|_\infty$,
- (vi) $WL^\infty_2(G)^\wedge | \Delta \subseteq l^1(\Delta)$ (see section 0 for product notation),
- (vii) $WC_\Delta(G)^\wedge | \Delta \subseteq l^1(\Delta)$,
- (viii) $Wl^\infty(\Delta) \subseteq M(G)^\wedge | \Delta$, and
- (ix) $Wc_0(\Delta) \subseteq L^1(G)^\wedge | \Delta$.

Proof. (i) \Rightarrow (ii) follows by a straightforward modification of (a) \Rightarrow (b) in [10], 5.7.4.

(ii) \Rightarrow (iii) is obvious because $C_\Delta(G) \subseteq L^\infty_2(G)$.

(iii) \Rightarrow (iv). By hypothesis the map $f \mapsto W\hat{f}|_\Delta$ from $C_\Delta(G)$ to $l^1(\Delta)$ is linear and bounded by κ . Let $K: l^\infty(\Delta) \rightarrow M(G)^\wedge | \Delta$ denote the canonical isomorphism of 1.8(ii) composed with the adjoint of this map-evidently $\|K\| \leq \kappa$. For $\chi \in \Delta$,

$$K\phi(\chi) = \sum_{\xi \in X} \phi(\xi)(W(\chi)\hat{\chi})(\xi) = W(\chi)\phi(\chi),$$

so given $\phi \in l^\infty(\Delta)$, there is $\mu \in M(G)$ -namely $\mu \in \delta^{-1}(K\phi)$ -with $\hat{\mu}|_\Delta = W\phi$

and $\|\mu\| \leq \kappa \|\phi\|_\infty$.

(iv) \Rightarrow (v) follows by an easy alteration of (d) \Rightarrow (e) in [2], 15.1.4.

(v) \Rightarrow (i). By hypothesis the map $\phi \mapsto W\phi$ from $c_0(\Delta)$ to $L^1(G)^\wedge | \Delta$ is linear and bounded by κ . Let $K: L^\infty_\Delta(G) \rightarrow l^1(\Delta)$ denote the composition of its adjoint with the canonical isomorphism of 1.8(iii). Then K is linear and bounded by κ . If $\chi \in \Delta$ and $f \in L^\infty_\Delta(G)$ then

$$(Kf)(\chi) = \int_G W(\chi)f\bar{\chi} = W(\chi)\hat{f}(\chi)$$

hence $Kf = W\hat{f} | \Delta$, so (i) holds.

(ii) \Rightarrow (vi), (iii) \Rightarrow (vii), (iv) \Rightarrow (viii) and (v) \Rightarrow (ix) are obvious. Since the converses fall into similar pairs we show only one of each.

(vii) \Rightarrow (iii). In the following lemma take A to be $l^1(\Delta)$ with α the canonical injection, B to be $C_\Delta(G)$ with $\beta f = W\hat{f} | \Delta$ and C to be \mathbb{C}^Δ with the product topology. Now (vii) ensures $\beta(B) \subseteq \alpha(A) \subseteq C$ so by 1.11 to follow, there is a constant κ such that for all $f \in C_\Delta(G)$, there is $\phi \in l^1(\Delta)$ with $W\hat{f} | \Delta = \phi$ and $\|\phi\|_1 \leq \kappa \|f\|_\infty$. That is, (iii) holds.

(ix) \Rightarrow (v). In the following lemma take A to be $L^1(G)$ with $\alpha(f) = \hat{f} | \Delta$, B to be $c_0(\Delta)$ with $\beta(\phi) = W\phi$ and C to be \mathbb{C}^Δ with the product topology. Now (ix) assures us that the hypotheses of 1.11 hold and hence (v) results.

1.11. I am indebted to Professor R. E. Edwards for the following statement:

LEMMA. *If A and B are Banach spaces, C a Hausdorff topological vector space, $\alpha: A \rightarrow C$ and $\beta: B \rightarrow C$ continuous linear maps and if $\beta(B) \subseteq \alpha(A)$ then there is a constant κ such that for all $b \in B$ there exists $a \in A$ with $\alpha(a) = \beta(b)$ and $\|a\|_A \leq \kappa \|b\|_B$.*

Proof. Let $\bar{A} = A/\ker \alpha$ and endow it with the quotient topology in which $\|\bar{a}\| = \inf \{\|c\|: c \in \bar{a}\}$ for each $\bar{a} \in \bar{A}$. Since C is Hausdorff, $\{0\}$ is closed in C and since α is continuous $\bar{0} = \alpha^{-1}(\{0\})$ is closed in A . Thus \bar{A} is again a Banach space and α induces a continuous injection $\bar{\alpha}: \bar{A} \rightarrow C$ defined by $\bar{\alpha}(\bar{a}) = \alpha(a)$, for $\bar{a} \in \bar{A}$.

Define $\gamma: B \rightarrow \bar{A}$ by $\gamma(b) = \bar{\alpha}^{-1} \circ \beta(b)$, for $b \in B$. By hypothesis γ is well defined-it clearly suffices to show it is bounded. Evidently γ is linear, so it remains to show it has a closed graph. If $b_n \rightarrow 0$ in B and $\gamma(b_n) \rightarrow \bar{a}$ in \bar{A} then $\beta(b_n) \rightarrow \beta(0) = 0$ in C . Thus, since $\bar{\alpha}$ is also continuous and linear,

$$\bar{\alpha}(\lim_n \gamma(b_n)) = \bar{\alpha}(\lim_n \bar{\alpha}^{-1} \circ \beta(b_n)) = \lim_n \beta(b_n) = \bar{\alpha}(\bar{a})$$

and so

$$0 = \lim_n \beta(b_n) = \bar{\alpha}(\bar{a}) .$$

Finally by injectivity of $\bar{\alpha}$, $\bar{a} = 0$.

1.12. We shall also use this lemma in another direction.

THEOREM. *Let A and B be Banach spaces, let Δ be a set and suppose \mathbb{C}^Δ has the product topology. Let $\alpha: A \rightarrow \mathbb{C}^\Delta$ and $\beta: B \rightarrow \mathbb{C}^\Delta$ be continuous and linear with*

(i) *there is $\lambda > 0$ such that for all $a \in A$ and all $\chi \in \Delta$,*

$$|\alpha(a)(\chi)| \leq \lambda \|a\|_A, \text{ and}$$

(ii) *there exist $\{b_\chi: \chi \in \Delta\} \subseteq B$ with*

$$\beta(b_\chi)(\xi) = \begin{cases} 1 & \text{if } \xi = \chi \\ 0 & \text{otherwise} \end{cases}, \text{ and } \sup \{\|b_\chi\|_B: \chi \in \Delta\} < \infty .$$

Suppose finally that $\psi \in \mathbb{C}^\Delta$ with $\psi\beta(B) \subseteq \alpha(A)$. Then $\psi \in l^\infty(\Delta)$.

Proof. Applying 1.11 there is a constant κ such that for all $b \in B$, there exists $a \in A$ with $\alpha(a) = \psi\beta(b)$ and $\|a\|_A \leq \kappa \|b\|_B$. If we write a_χ for an element of A corresponding to b_χ by this process we have

$$|\psi(\chi)| = |\psi(\chi)\beta(b_\chi)(\chi)| = |\alpha(a_\chi)(\chi)| \leq \lambda \|a_\chi\|_A \leq \kappa\lambda \|b_\chi\|_B .$$

Consequently $\|\psi\|_\infty < \infty$ as required.

1.13. The next result is helpful when showing a set is W -Sidon.

THEOREM. *If $\Delta \subseteq X$ and $W \in \mathbb{C}^\Delta$ the following are equivalent:*

- (i) Δ is W -Sidon,
- (ii) $f \in C_\Delta(G)$ with $\hat{f} \in \mathfrak{R}^X$ implies $\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| < \infty$, and
- (iii) whenever $\phi \in l^\infty(\Delta) \cap \mathfrak{R}^X$ there is $\mu \in M(G)$ with $\hat{\mu}|_\Delta = W\phi$.

Proof. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) follow from 1.10.

(iii) \Rightarrow (i). If $\phi \in l^\infty(\Delta)$ we may write $\phi = \phi_1 + i\phi_2$ where, by (iii), there is $\mu_j \in M(G)$ with $\hat{\mu}_j|_\Delta = W\phi_j$ for $j \in \{1, 2\}$. Thus taking $\mu = \mu_1 + i\mu_2$ gives $\mu \in M(G)$ and $\hat{\mu}|_\Delta = W\phi$, so (i) results by 1.10.

1.14. One important respect in which 1.10 differs from the analogous result for Sidon sets is that we only claim inclusions like 1.10(viii) rather than $Wl^\infty(\Delta) = M(G)^\wedge|_\Delta$. The reasons for this are embodied in:

THEOREM. *Suppose $\Delta \subseteq X$ and $W \in \mathbb{C}^\Delta$. Then Δ is Sidon when-*

ever one of the following holds:

- (i) $Wl^\infty(\mathcal{A}) = M(G)^\wedge | \mathcal{A}$,
- (ii) $Wc_0(\mathcal{A}) = L^1(G)^\wedge | \mathcal{A}$,
- (iii) $WC_{\mathcal{A}}(G)^\wedge | \mathcal{A} = l^1(\mathcal{A})$,
- (iv) $WL_{\mathcal{A}}^\infty(G)^\wedge | \mathcal{A} = l^1(\mathcal{A})$.

Proof. (i) Taking the Dirac measure at e we see $1 \in Wl^\infty(\mathcal{A})$. Thus $l^\infty(\mathcal{A}) \subseteq Wl^\infty(\mathcal{A}) \subseteq l^\infty(\mathcal{A})$ hence $l^\infty(\mathcal{A}) = Wl^\infty(\mathcal{A}) = M(G)^\wedge | \mathcal{A}$ so \mathcal{A} is Sidon.

(ii) By hypothesis we cannot have $W(\chi) = 0$ for any $\chi \in \mathcal{A}$, so $W^{-1}L^1(G)^\wedge | \mathcal{A} = c_0(\mathcal{A})$. Now in 1.12 we take $A \equiv c_0(\mathcal{A})$ with norm $\|\cdot\|_\infty$, α the canonical injection, $B \equiv L^1(G)$ with norm $\|\cdot\|_1$, $\beta(\hat{f}) = \hat{f} | \mathcal{A}$ and $\psi \equiv W^{-1}$. The hypotheses are readily verified so we conclude that $\|W^{-1}\|_\infty < \infty$. Applying 1.10, whenever $t \in T_{\mathcal{A}}(G)$,

$$\sum_{\chi \in \mathcal{A}} |\hat{t}(\chi)| \leq \|W^{-1}\|_\infty \kappa \|t\|_\infty.$$

So \mathcal{A} is Sidon.

(iii) Again, W is never zero so we may apply 1.12 taking $A \equiv C_{\mathcal{A}}(G)$, $B \equiv l^1(\mathcal{A})$, $\alpha(f) = \hat{f} | \mathcal{A}$, β the canonical injection and $\psi \equiv W^{-1}$. As in (ii) we deduce that \mathcal{A} is Sidon.

(iv) Apply the same method as (iii).

NOTE. The converse to each of these assertions is false. Even if \mathcal{A} is replaced by $\mathcal{A}_0 \equiv \{\chi \in \mathcal{A} : W(\chi) \neq 0\}$ and \mathcal{A}_0 is Sidon, these inclusions are strict if \mathcal{A}_0 is infinite and $W \in c_0(\mathcal{A})$.

THEOREM 1.15. *Let $\mathcal{A} \subseteq X$, $W \in \mathbb{C}^{\mathcal{A}}$ and \mathcal{A}_0 be as above. Assuming the constants in (ii), (iii) and (iv) to be the least possible, these are equivalent:*

- (i) \mathcal{A}_0 is Sidon with constant κ ,
- (ii) $f \in L_{\mathcal{A}_0}^\infty(G)$ implies $\sum_{\chi \in \mathcal{A}_0} W(\chi)\hat{f}(\chi)\chi \in A_{\mathcal{A}_0}(G)$ and $\|W\hat{f}\|_1 \leq \kappa \|\sum_{\chi \in \mathcal{A}_0} W(\chi)\hat{f}(\chi)\chi\|_\infty$,
- (iii) $t \in T_{\mathcal{A}_0}(G)$ implies $\|W\hat{t}\|_1 \leq \kappa \|\sum_{\chi \in \mathcal{A}_0} W(\chi)\hat{t}(\chi)\chi\|_\infty$, and
- (iv) for all $\phi \in l^\infty(\mathcal{A}_0)$ there is $\mu \in M(G)$ such that $\hat{\mu} | \mathcal{A}_0 = W\phi$ and $\|\mu\| \leq \kappa \|W\phi\|_\infty$.

Proof. (i) \Rightarrow (ii). If $f \in L_{\mathcal{A}_0}^\infty(G)$ then

$$\|W\hat{f}\|_1 = \sum_{\chi \in \mathcal{A}_0} |W(\chi)\hat{f}(\chi)| \leq \|W\|_\infty \|\hat{f}\|_1$$

so that if \mathcal{A}_0 is Sidon, (ii) follows.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i). If $t \in T_{\mathcal{A}_0}(G)$ define $u \in T_{\mathcal{A}_0}(G)$ by taking

$$\hat{u}(\chi) = W^{-1}(\chi)\hat{t}(\chi) \text{ for all } \chi \in \Delta_0.$$

They by (iii),

$$\sum_{\chi \in \Delta_0} |\hat{t}(\chi)| = \sum_{\chi \in \Delta_0} |W(\chi)\hat{u}(\chi)| \leq \kappa \left\| \sum_{\chi \in \Delta_0} W(\chi)\hat{u}(\chi) \right\|_{\infty} \leq \kappa \|t\|_{\infty}$$

so (i) follows.

(i) \Rightarrow (iv). If $\phi \in l^{\infty}(\Delta)$ and $W \in \mathfrak{B}(\Delta)$ then $\phi W \in l^{\infty}(\Delta)$ hence (iv) results from (i) and 1.11.

(iv) \Rightarrow (i). If $\psi \in l^{\infty}(\Delta_0)$ and $\Phi \in \mathfrak{F}(\Delta_0)$ let

$$\psi_{\Phi}(\chi) = \begin{cases} W^{-1}(\chi)\psi(\chi) & \text{if } \chi \in \Phi \\ 0 & \text{if } \chi \in \Delta_0 \setminus \Phi \end{cases}, \text{ so that } \psi_{\Phi} \in c_0(\Delta_0).$$

By hypothesis there is $\mu_{\Phi} \in M(G)$ with $\hat{\mu}_{\Phi}|_{\Delta_0} = W\psi_{\Phi}$ and

$$\|\mu_{\Phi}\| \leq \kappa \|W\psi_{\Phi}\|_{\infty} \leq \kappa \|\psi\|_{\infty}.$$

Thus $\{\mu_{\Phi}: \Phi \in \mathfrak{F}(\Delta)\}$ is bounded in $M(G)$ hence by Alaoglu's theorem it has a weakly convergent subnet. So there is $\mu \in M(G)$ with $\hat{\mu}|_{\Delta_0} = \psi$, and Δ_0 must be Sidon.

1.16. Many characterisations of Sidon sets have weighted analogues, like 1.10. More of these may be found in [11].

2. Thick W -Sidon sets.

2.0. To find W -Sidon sets which are not Sidon it suffices, by 1.2, to take $\Delta \subseteq X$ not Sidon and then choose $W \in l^2(\Delta)$ (such Δ exist since infinite subgroups are not Sidon). It is the purpose of this section to exhibit non-Sidon sets Δ which are W -Sidon for some $W \notin l^2(\Delta)$. These sets are in the dual of the circle group and are not even $\Lambda(1)$.

The proof relies on Riesz products and therefore requires a sort of independence condition on Δ . Recall $\Delta^2 = \{\chi\xi: \chi, \xi \in \Delta\}$ whenever $\Delta \subseteq X$.

THEOREM 2.1. *Suppose $\Delta = \bigcup \{\Delta_n: n \in \mathfrak{N}\}$ where $0 < \nu(\Delta_n) < \aleph_0$ and*

- (i) $1 \notin \Delta_0$,
- (ii) $\Delta_n^{-1} = \Delta_n$,
- (iii) $\Delta_{n+1} \subseteq X \setminus \bigcup \{\Delta_0^{\varepsilon_0} \Delta_1^{\varepsilon_1} \dots \Delta_n^{\varepsilon_n}: \varepsilon_i \in \{0, 1, 2\} \text{ for } 0 \leq i \leq n \text{ and at most one } \varepsilon_i \text{ equal to } 2\}$, and
- (iv) $\Delta_{n+1}^2 \subseteq X \setminus \bigcup \{\Delta_0^{\varepsilon_0} \Delta_1^{\varepsilon_1} \dots \Delta_n^{\varepsilon_n}: \varepsilon_i \in \{0, 1\} \text{ for } 0 \leq i \leq n \text{ and } \sum_{i=0}^n \varepsilon_i \geq 1\}$

Define $W: \mathcal{A} \rightarrow (0, 1]$ to equal $\nu(\mathcal{A}_n)^{-1}$ on \mathcal{A}_n . The conclusion is that \mathcal{A} is W -Sidon.

Proof. Suppose $\phi \in \mathfrak{R}^d$ with $\|\phi\|_\infty \leq 1$. For $n \in \mathfrak{N}$ define $t_n \in T(G)$ by

$$t_n = (2\nu(\mathcal{A}_n))^{-1} \left(\sum_{\substack{\chi \in \mathcal{A}_n \\ x^2 \neq 1}} \phi(\chi)(\chi + \bar{\chi}) + \sum_{\substack{\chi \in \mathcal{A}_n \\ x^2 = 1}} \phi(\chi)\chi \right).$$

It is easy to see that

(2.1.1) t_n is real-valued

(2.1.2) $\|t_n\|_\infty \leq 1$

(2.1.3) and, by (ii), $\hat{t}_n(\chi) = \begin{cases} (2\nu(\mathcal{A}_n))^{-1}\phi(\chi) & \text{if } \chi \in \mathcal{A}_n \\ 0 & \text{if } \chi \in X \setminus \mathcal{A}_n \end{cases}$.

Next for $N \in \mathfrak{N}$ set $P_N = \prod_{n=0}^N (1 + t_n)$ so that $P_N = 1 + \sum_{n=0}^N t_n + Q_N$ where

(2.1.4)
$$Q_N = \sum_{0 \leq n_1 < n_2 \leq N} t_{n_1} t_{n_2} + \sum_{0 \leq n_1 < n_2 \leq n_3 \leq N} t_{n_1} t_{n_2} t_{n_3} + \dots + t_0 t_1 \dots t_N.$$

(2.1.5) Now $\hat{P}_N | \mathcal{A}_n = \hat{t}_n | \mathcal{A}_n$ if $0 \leq n \leq N$

provided that whenever $0 \leq n \leq N$,

$$\mathcal{A}_n \subseteq X \setminus [sp(1) \cup \bigcup \{ \mathcal{A}_m : 0 \leq m \leq N \text{ and } m \neq n \} \cup sp(Q_N)].$$

Consequently the lemma to follow ensures this for each $N \in \mathfrak{N}$.

By (2.1.1), (2.1.2) and (2.1.3), for each N , if we have

$$1 \notin \bigcup \{ \mathcal{A}_n : 0 \leq n \leq N \} \cup sp(Q_N)$$

then

(2.1.6) $\|P_N\|_1 = \int_G P_N = 1 + \sum_{n=0}^N \int_G t_n + \int_G Q_N = 1.$

Again, the lemma assures us of this.

So by (2.1.6), $\{P_N : N \in \mathfrak{N}\}$ is bounded in $M(G)$ and thus has a weak cluster point $\tau \in M(G)$; let $\mu = 2\tau$. Then for each $n \in \mathfrak{N}$ and $\chi \in \mathcal{A}_n$,

$$\begin{aligned} \hat{\mu}(\chi) &= 2\hat{\tau}(\chi) = 2\hat{t}_n(\chi) \text{ by (2.1.5)} \\ &= \nu(\mathcal{A}_n)^{-1}\phi(\chi) \text{ by (2.1.3)} \\ &= W(\chi)\phi(\chi) \text{ by definition of } W. \end{aligned}$$

Thus $\hat{\mu} | \mathcal{A} = W\phi$ so by 1.13(iii), \mathcal{A} is W -Sidon.

LEMMA 2.2. Suppose $\{\Delta_n: n \in \mathfrak{N}\} \subseteq \mathfrak{B}(X)$ satisfies conditions (i) to (iv) of the previous theorem. Then with Q_N given by (2.1.4), for each $N \in \mathfrak{N}$,

(i) $0 \leq n \leq N$ implies

$$\Delta_n \subseteq X \setminus [\{1\} \cup \bigcup \{\Delta_m: 0 \leq m \leq N \text{ and } m \neq n\} \cup sp(Q_N)], \text{ and}$$

(ii) $1 \notin \bigcup \{\Delta_n: 0 \leq n \leq N\} \cup sp(Q_N)$.

Proof. By (2.1.4) and (2.1.3),

$$sp(Q_N) \subseteq \bigcup \left\{ \Delta_0^{\varepsilon_0} \Delta_1^{\varepsilon_1} \dots \Delta_N^{\varepsilon_N}: \varepsilon_i \in \{0, 1\} \text{ for } 0 \leq i \leq N \text{ and } \sum_{i=0}^N \varepsilon_i \geq 2 \right\}.$$

For brevity define

$$A(N, n) = \{1\} \cup \bigcup \{\Delta_m: 0 \leq m \leq N \text{ and } m \neq n\} \text{ for } 0 \leq n \leq N,$$

and

$$B(N, j) = \bigcup \{ \Delta_0^{\varepsilon_0} \Delta_1^{\varepsilon_1} \dots \Delta_N^{\varepsilon_N}: \varepsilon_i \in \{0, 1\} \text{ and } \sum_{i=0}^N \varepsilon_i \geq j \} \text{ for } j \in \{1, 2\}.$$

In these terms we have to prove, for each $N \in \mathfrak{N}$, $0 \leq n \leq N$ implies $\Delta_n \subseteq X \setminus [A(N, n) \cup B(N, 2)]$, and

$$1 \notin \bigcup \{\Delta_n: 0 \leq n \leq N\} \cup B(N, 2).$$

A straightforward induction, relying heavily on 2.1(ii), completes the argument.

THEOREM 2.3. There is a subset Δ of \mathfrak{B} which is W -Sidon for some $W \in l^\infty(\Delta) \setminus l^2(\Delta)$ yet which is not $\Delta(1)$.

Proof. Take $m_0 \neq 0$ and let $\Delta_0 = \{\pm m_0\}$. Supposing $\Delta_0, \dots, \Delta_n$ have been defined so as to satisfy the hypotheses of 2.1, let $m \in \mathfrak{N}$ be the supremum of the finite set

$$\bigcup \{ \varepsilon_0 \Delta_0 + \dots + \varepsilon_n \Delta_n: \varepsilon_i \in \{0, 1, 2\} \text{ with at most one } \varepsilon_i = 2 \}.$$

Now if $n = 0$ set $\Delta_1 = \{\pm(m + 1)\}$ and if $n \geq 1$ take

$$\Delta_{n+1} = \{ \pm j(m + 1): 1 \leq j \leq [(n + 1)/2] \}.$$

Since $\Delta_{n+1} + \Delta_{n+1}$ is also disjoint from the finite set above, it is disjoint from

$$\bigcup \left\{ \varepsilon_0 \Delta_0 + \dots + \varepsilon_n \Delta_n: \varepsilon_i \in \{0, 1\} \text{ with } \sum_{i=0}^n \varepsilon_i \geq 1 \right\}.$$

Consequently 2.1 shows $\Delta \equiv \bigcup \{\Delta_n: n \in \mathfrak{N}\}$ is W -Sidon where

$$\sum_{\chi \in \mathcal{A}} |W(\chi)|^2 \geq \sum_{n \in \mathfrak{N}} (1+n)^{-1} = \infty$$

so $W \notin l^2(\mathcal{A})$.

By construction \mathcal{A} contains arbitrarily long arithmetic progressions hence it is not $\mathcal{A}(1)$ by [9], (4.1).

2.4. Using multiplier notation from 4.2, by 3.3 to follow,

$$l^2(\mathcal{A}) = (C_{\mathcal{A}}(G), A_{\mathcal{A}}(G))$$

whenever \mathcal{A} is a subgroup of X . If $\mathcal{A} \subseteq X$, Parseval's identity shows

$$l^2(\mathcal{A}) \subseteq (C_{\mathcal{A}}(G), A_{\mathcal{A}}(G)) .$$

To find \mathcal{A} for which this inclusion is strict it suffices to take \mathcal{A} an infinite Sidon set so that $1 \in (C_{\mathcal{A}}(G), A_{\mathcal{A}}(G)) \setminus l^2(\mathcal{A})$. However 2.3 provides examples of non-Sidon sets \mathcal{A} in \mathfrak{B} for which the strict inclusion holds. It also indicates the impossibility of extending [1], Theorem 1 to arbitrary subsets of X .

3. The algebra of weight functions.

3.0. From 1.10 we may read off more expressions for $\|W\|_{\mathcal{A}}$:

$$\begin{aligned} \|W\|_{\mathcal{A}} &= \sup \{ \sum_{\chi \in \mathcal{A}} |W(\chi)\hat{f}(\chi)| : f \in C_{\mathcal{A}}(G) \text{ with } \|f\|_{\infty} \leq 1 \} \\ &= \sup \{ \inf \{ \|f\|_1 : f \in L^1(G) \text{ with } \hat{f}|_{\mathcal{A}} = W\phi : \phi \in c_0(\mathcal{A}) \text{ and } \|\phi\|_{\infty} \leq 1 \} \} \\ &= \sup \{ \inf \{ \|\mu\| : \mu \in M(G) \text{ with } \hat{\mu}|_{\mathcal{A}} = W\phi : \phi \in l^{\infty}(\mathcal{A}) \text{ and } \|\phi\|_{\infty} \leq 1 \} \} . \end{aligned}$$

THEOREM 3.1. $\mathfrak{B}(\mathcal{A})$ is a commutative Banach algebra under $\|\cdot\|_{\mathcal{A}}$ and pointwise operations. It has an identity iff \mathcal{A} is Sidon.

Proof. The following straightforward formulae establish that $\|\cdot\|_{\mathcal{A}}$ makes $\mathfrak{B}(\mathcal{A})$ into a commutative normed algebra under pointwise operations.

Suppose $W_1, W_2 \in \mathfrak{B}(\mathcal{A})$, $\alpha \in \mathbb{C}$ and $t \in T_{\mathcal{A}}(G)$ with $\|t\|_{\infty} \leq 1$. Then

$$\begin{aligned} \sum_{\chi \in \mathcal{A}} |(W_1(\chi) + W_2(\chi))\hat{t}(\chi)| &\leq \sum_{\chi \in \mathcal{A}} |W_1(\chi)\hat{t}(\chi)| + \sum_{\chi \in \mathcal{A}} |W_2(\chi)\hat{t}(\chi)| \\ &\leq \|W_1\|_{\mathcal{A}} + \|W_2\|_{\mathcal{A}} ; \\ \sum_{\chi \in \mathcal{A}} |\alpha W_1(\chi)\hat{t}(\chi)| &= |\alpha| \sum_{\chi \in \mathcal{A}} |W_1(\chi)\hat{t}(\chi)| \leq |\alpha| \|W_1\|_{\mathcal{A}} ; \\ \sum_{\chi \in \mathcal{A}} |W_1(\chi)W_2(\chi)\hat{t}(\chi)| &\leq \|W_1\|_{\infty} \sum_{\chi \in \mathcal{A}} |W_2(\chi)\hat{t}(\chi)| \leq \|W_1\|_{\mathcal{A}} \|W_2\|_{\mathcal{A}} \text{ by 1.1 ;} \end{aligned}$$

and if $\|W\|_{\mathcal{A}} = 0$ then $\|W\|_{\infty} = 0$ hence $W = 0$.

Suppose $\{W_n : n \in \mathfrak{N}\} \subseteq \mathfrak{B}(\mathcal{A})$ is a Cauchy sequence. Then by 1.1 again, $\|W_n - W_m\|_{\infty} \rightarrow 0$ hence there is $W \in l^{\infty}(\mathcal{A})$ for which $\|W - W_n\|_{\infty} \rightarrow 0$.

If $\varepsilon > 0$, there is $N \in \mathfrak{N}$ such that $n \geq N$ implies, for all $t \in T_{\Delta}(G)$ with $\|t\|_{\infty} \leq 1$,

$$\sum_{\chi \in \Delta} |(W_n(\chi) - W_m(\chi))\hat{t}(\chi)| < \varepsilon.$$

Letting $m \rightarrow \infty$, the same inequality holds with W replacing W_m . So $n \geq N$ implies $\|W_n - W\|_{\Delta} < \varepsilon$. Furthermore

$$\|W\|_{\Delta} - \|W_N\|_{\Delta} \leq \|W - W_N\|_{\Delta} < \varepsilon$$

hence $\|W\|_{\Delta} < \varepsilon + \|W_N\|_{\Delta} < \infty$. Thus $W_n \rightarrow W$ in $\mathfrak{B}(\Delta)$.

Finally $\mathfrak{B}(\Delta)$ has an identity iff $1 \in \mathfrak{B}(\Delta)$ iff Δ is Sidon.

3.2. From 1.1 we have: Δ is Sidon iff $\mathfrak{B}(\Delta) = l^{\infty}(\Delta)$. Our next few results consider $\mathfrak{B}(\Delta)$ contained in $c_0(\Delta)$.

THEOREM. *If $L^1(G) \wedge \Delta \subseteq \mathfrak{B}(\Delta)$ (in particular, if $\mathfrak{B}(\Delta) = c_0(\Delta)$) then Δ is Sidon.*

Proof. Suppose $f \in C_{\Delta}(G)$ —we show $\|\hat{f}\|_1 < \infty$ by using the boundedness principle 1.11. Take therein $A \equiv l^1(\Delta)$ with α the identity, $B \equiv L^1(G)$ with $\beta(g) = \hat{f}\hat{g} \upharpoonright \Delta$ and $C \equiv \mathfrak{C}^{\Delta}$ with the product topology. Then for some constant κ , for all $g \in L^1(G)$, there is $\phi \in l^1(\Delta)$ such that $\phi = \hat{f}\hat{g} \upharpoonright \Delta$ and $\sum_{\chi \in \Delta} |\phi(\chi)| \leq \kappa \|g\|_1$. In other words, $\sum_{\chi \in \Delta} |\hat{f}(\chi)\hat{g}(\chi)| \leq \kappa \|g\|_1$.

Allowing g to vary over an approximate identity,

$$\sum_{\chi \in \Delta} |\hat{f}(\chi)| < \infty$$

as required.

3.3. At the other end of the spectrum we can have equality in 1.2.

THEOREM. *If Δ is a subgroup of X then $\mathfrak{B}(\Delta) = l^2(\Delta)$.*

Proof. Obviously $l^2(\Delta) \subseteq \mathfrak{B}(\Delta)$ by 1.2.

If $W \in \mathfrak{B}(\Delta)$ then by 1.3 we may suppose $\Delta = X$. Now by 1.10(iii) and [1], 2.1(a), it follows that $W \in l^2(\Delta)$. This completes the proof.

REMARKS 3.4. From 3.3 it follows that if Δ is cofinite in some subgroup of X then $\mathfrak{B}(\Delta) = l^2(\Delta)$.

Similarly by [10], 8.7.8, if Δ is cofinite in the positive cone of the ordered dual of a compact connected abelian group then $\mathfrak{B}(\Delta) = l^2(\Delta)$.

THEOREM 3.5. *For $\Delta \subseteq X$, $\mathfrak{B}(\Delta)$ is an ideal in $M(G)^\wedge | \Delta$ which is improper iff Δ is Sidon. For each $W \in \mathfrak{B}(\Delta)$, $\|W\|_s \leq \|W\|_\Delta$ (see 1.6 for notation).*

Proof. If $W \in \mathfrak{B}(\Delta)$ by applying 1.10(iv) to $\phi = 1$, there is $\nu \in M(G)$ with $\hat{\nu} | \Delta = W$ and $\|\nu\| \leq \|W\|_\Delta$. So $\mathfrak{B}(\Delta) \subseteq M(G)^\wedge | \Delta$ and for all $W \in \mathfrak{B}(\Delta)$, $\|W\|_s \leq \|W\|_\Delta$.

Obviously the algebraic operations on these spaces coincide and if $\mu \in M(G)$, for all $t \in T_\Delta(G)$ with $\|t\|_\infty \leq 1$,

$$\sum_{\chi \in \Delta} |W(\chi)\hat{\mu}(\chi)\hat{t}(\chi)| \leq \|\hat{\mu}\|_\infty \|W\|_\Delta .$$

Thus $W\hat{\mu} | \Delta \in \mathfrak{B}(\Delta)$ which, by 3.1, is consequently an ideal in $M(G)^\wedge | \Delta$ which is improper iff Δ is Sidon.

NOTE. By 3.3, $\mathfrak{B}(\Delta)$ need not be closed in $M(G)^\wedge | \Delta$.

3.6. As algebras, for $\Delta \subseteq X$,

$$l^2(\Delta) \subseteq \mathfrak{B}(\Delta) \subseteq M(G)^\wedge | \Delta \subseteq l^\infty(\Delta) .$$

Each is endowed with a norm—they are $\|\cdot\|_2$, $\|\cdot\|_\Delta$, $\|\cdot\|_s$ and $\|\cdot\|_\infty$ respectively. When Δ is a subgroup of X , $\|\cdot\|_2$ and $\|\cdot\|_\Delta$ are actually equivalent (by 3.3 and the open mapping theorem or [1], (2.1)(b)) on $\mathfrak{B}(\Delta)$.

A different proof of the inequality $\|\cdot\|_s \leq \|\cdot\|_\Delta$ (established above) follows by the method in [10], 1.9.1 which yields the characterisation: for $W \in \mathfrak{B}(\Delta)$,

$$\|W\|_s = \sup \{ \left| \sum_{\chi \in \Delta} W(\chi)\hat{t}(\chi) \right| : t \in T_\Delta(G) \text{ and } \|t\|_\infty \leq 1 \} .$$

This shows why, in 1.0, we kept the modulus signs inside the sum.

We now consider when pairs of these norms are equivalent.

THEOREM 3.7. *For $\Delta \subseteq X$ these are equivalent:*

- (i) Δ is Sidon,
- (ii) $\|\cdot\|_\infty$ and $\|\cdot\|_\Delta$ are equivalent on $\mathfrak{B}(\Delta)$,
- (iii) $\|\cdot\|_s$ and $\|\cdot\|_\Delta$ are equivalent on $M(G)^\wedge | \Delta$,
- (vi) $\|\cdot\|_s$ and $\|\cdot\|_\infty$ are equivalent on $M(G)^\wedge | \Delta$.

Proof. (a) If Δ is Sidon and $W \in \mathfrak{B}(\Delta)$ and $t \in T_\Delta(G)$ with $\|t\|_\infty \leq 1$ then

$$\sum_{\chi \in \Delta} |W(\chi)\hat{t}(\chi)| \leq \|W\|_\infty \sum_{\chi \in \Delta} |\hat{t}(\chi)| \leq \|W\|_\infty \|1\|_\Delta .$$

Thus whenever $W \in \mathfrak{B}(\Delta) = M(G)^\wedge | \Delta$,

$$\|W\|_\infty \leq \|W\|_s \leq \|W\|_\Delta \leq \|1\|_\Delta \|W\|_\infty \leq \|1\|_\Delta \|W\|_s,$$

so the norms are pairwise equivalent.

(b) If Δ is not Sidon then by 3.2, $l^2(\Delta) \subseteq \mathfrak{B}(\Delta) \subset c_0(\Delta)$. Since $l^2(\Delta)$ contains all finite linear combinations of characteristic functions of singleton subsets of Δ and these are dense in $c_0(\Delta)$, $\mathfrak{B}(\Delta)$ cannot be closed in $c_0(\Delta)$. Thus $\mathfrak{B}(\Delta)$ cannot be complete under the restriction of $\|\cdot\|_\infty$. So by 3.1, $\|\cdot\|_\infty$ and $\|\cdot\|_\Delta$ cannot be equivalent on $\mathfrak{B}(\Delta)$.

(c) If $\|\cdot\|_s$ and $\|\cdot\|_\Delta$ are equivalent on $M(G)^\wedge | \Delta$ then $\mathfrak{B}(\Delta) = M(G)^\wedge | \Delta$ hence by 3.5, Δ is Sidon.

(d) If $\|\cdot\|_s$ and $\|\cdot\|_\infty$ are equivalent on $M(G)^\wedge | \Delta$ then it is complete under $\|\cdot\|_\infty$ and hence $c_0(\Delta) \subseteq M(G)^\wedge | \Delta$. So by 1.9(ii), $C_-(G)^\wedge | \Delta \subseteq l^2(\Delta)$ and so Δ is Sidon.

REMARKS 3.8. (i) As a Banach algebra, $\mathfrak{B}(\Delta)$ is neither separable nor a B^* -algebra in general. The former follows by 1.1 and the latter by 3.3.

(ii) Considering $C_\Delta(G)^\wedge | \Delta$ as a sequence space, $\mathfrak{B}(\Delta)$ is its α -dual (see [8], § 30). However 3.3 shows that $C_\Delta(G)^\wedge | \Delta$ is not, in general, a perfect sequence space.

3.9. Refer to [4], 1.1 for the definition of a p -Sidon set.

THEOREM. *Let $\Delta \subseteq X$ and $1 \leq p < 2$. Then Δ is p -Sidon iff $l^{p'}(\Delta) \subseteq \mathfrak{B}(\Delta)$.*

Proof. For $p = 1$ this is just 1.1 (it is trivial when $p = 2$). If $1 < p < 2$ and Δ is p -Sidon then by [4], 1.2(ii), $f \in C_\Delta(G)$ implies $\hat{f} | \Delta \in l^p(\Delta)$. So if $W \in l^{p'}(\Delta)$, Hölder's inequality shows

$$\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| < \infty$$

hence by 1.10, $W \in \mathfrak{B}(\Delta)$.

Conversely if $l^{p'}(\Delta) \subseteq \mathfrak{B}(\Delta)$ then by 3.5, $l^{p'}(\Delta) \subseteq M(G)^\wedge | \Delta$. So by [4], 1.2(iv), Δ is p -Sidon.

From this follows, by the Hausdorff-Young theorem, a converse of 3.2 for $p > 1$.

COROLLARY. *If $1 < p < 2$ and Δ is p -Sidon then $L^p(G)^\wedge | \Delta \subseteq \mathfrak{B}(\Delta)$.*

4. Multipliers and W -Sidon sets.

4.0. When Δ is Sidon, spaces of Δ -spectral functions collapse. Not only is $L_1^\infty(G) = A_\Delta(G)$ but $M_\Delta(G) = \bigcap \{L_2^p(G) : 1 \leq p < \infty\}$. In this

section we investigate analogues for W -Sidon sets.

In this context it is natural to consider the trigonometric series $\sum_{\chi \in \Delta} W(\chi) \hat{\mu}(\chi) \chi$ for $\mu \in M_\Delta(G)$ (see for instance 1.15.) To ensure such objects make sense we define, for $\Delta \subseteq X$,

$$T: l^\infty(\Delta) \times PM_\Delta(G) \longrightarrow PM_\Delta(G)$$

by

$$T(\phi, \pi) = \sum_{\chi \in \Delta} \phi(\chi) \hat{\pi}(\chi) \chi .$$

When ϕ is fixed we shall use the single variable notation T_ϕ even for its restriction to some subset of $PM_\Delta(G)$.

If $\phi \in l^\infty(\Delta)$ let $\pi_\phi \in PM_\Delta(G)$ be given by

$$\hat{\pi}_\phi(\chi) = \begin{cases} \phi(\chi) & \text{if } \chi \in \Delta \\ 0 & \text{if } \chi \in X \setminus \Delta \end{cases} .$$

Then $T(\phi, \pi) = \pi_\phi * \pi$, for all $\pi \in PM_\Delta(G)$, so T is just convolution from $PM_\Delta(G) \times PM_\Delta(G)$ into $PM_\Delta(G)$. From this it is evident that T is bilinear, continuous and behaves nicely under translation and convolution.

THEOREM 4.1. *If Δ is W -Sidon and $t \in T_\Delta(G)$ then*

$$(4.1.1) \quad \|T_w t\|_p \leq 2 \|W\|_\Delta p^{1/2} \|t\|_2 \quad \text{if } 2 < p < \infty$$

and

$$(4.1.2) \quad \|T_w t\|_2 \leq 8 \|W\|_\Delta \|t\|_1 .$$

Proof. We modify Rudin's proof for Sidon sets. For an exposition of the Rademacher functions $\{r_n: n \in \mathfrak{N}\}$ refer to [2], Chapter 14. By redefining r_n on a set of measure zero so that it is right continuous at each dyadic rational and left continuous at 1, we ensure $r_n \in \{\pm 1\}^{[0,1]}$.

For $t \in T_\Delta(G)$ let $j \in X^{\mathfrak{N}}$ be an injection with $sp(t) \subseteq j(\mathfrak{N})$, and define $R: X \rightarrow \{\pm 1\}^{[0,1]}$ by

$$R_x = \begin{cases} r_j^{-1}(\chi) & \text{if } \chi \in j(\mathfrak{N}) \\ r_0 & \text{if } \chi \in X \setminus j(\mathfrak{N}) \end{cases} .$$

Now let $f: G \times [0, 1] \rightarrow \mathbb{C}$ be given by

$$f(x, \rho) = \sum_{\chi \in X} \hat{t}(\chi) R_x(\rho) \chi(x) .$$

Using single variable notation we have $f_\rho \in T_\Delta(G)$ for all $\rho \in [0, 1]$ and for all $x \in G$, $f_x = \sum_{n \in \mathfrak{N}} \hat{t}(j(n)) j(n)(x) r_n$ which is a Rademacher series.

Since f is a finite sum of functions which are measurable on $G \times [0, 1]$ each dominated by the constant $\|t\|_\infty$, f is integrable and we may use Fubini's theorem.

Suppose $\rho \in [0, 1]$. By 1.10(iv), there is $\mu_\rho \in M(G)$ such that $\hat{\mu}_\rho(\chi) = W(\chi)R_x(\rho)$, for all $\chi \in \mathcal{A}$ and $\|\mu_\rho\| \leq \|W\|_{\mathcal{A}} \|R.(\rho)\|_\infty = \|W\|_{\mathcal{A}}$. So for $\chi \in \mathcal{A}$,

$$\hat{\mu}_\rho(\chi)\hat{f}_\rho(\chi) = W(\chi)R_x(\rho)\hat{t}(\chi)R_x(\rho) = W(\chi)\hat{t}(\chi) = (T_w t)^\wedge(\chi) :$$

and if $\chi \in X \setminus \mathcal{A}$,

$$(T_w t)^\wedge(\chi) = 0 = \hat{f}_\rho(\chi) .$$

Thus $T_w t = \mu_\rho * f_\rho$ hence $\|T_w t\|_p \leq \|\mu_\rho\| \|f_\rho\|_p \leq \|W\|_{\mathcal{A}} \|f_\rho\|_p$.

So when $p = 2m$ (for some $m \in \mathfrak{N}$),

$$(4.1.3) \quad \int_G |T_w t|^{2m} \leq \|W\|_{\mathcal{A}}^{2m} \int_G |f_\rho|^{2m} .$$

But a property of Rademacher series ([2], 14.2.1) ensures that for all $x \in G$,

$$\int_0^1 |f_x|^{2m} \leq (4m)^m \left(\sum_{\chi \in X} |\hat{t}(\chi)\chi(x)|^2 \right)^m .$$

So using Fubini's theorem to integrate (4.1.3) along $[0, 1]$,

$$(4.1.4) \quad \int_G |T_w t|^{2m} \leq \|W\|_{\mathcal{A}}^{2m} (4m)^m \left(\sum_{\chi \in \mathcal{A}} |\hat{t}(\chi)|^2 \right)^m .$$

Now given any $p \in (2, \infty)$ choose $m \in \mathfrak{N}$ such that $2(m - 1) < p \leq 2m$ and $1 < m \leq p$. Then (4.1.4) guarantees

$$\|T_w t\|_p \leq \|T_w t\|_{2m} \leq 2 \|W\|_{\mathcal{A}} m^{1/2} \|t\|_2 \leq 2 \|W\|_{\mathcal{A}} p^{1/2} \|t\|_2$$

which yields (4.1.1).

To prove (4.1.2) we argue similarly, except that for $t \in T_{\mathcal{A}}(G)$ we redefine $f(x, \rho) = \sum_{\chi \in \mathcal{A}} W(\chi)\hat{t}(\chi)R_x(\rho)\chi(x)$.

NOTATION 4.2. When $E, F \subseteq PM(G)$ and $\mathcal{A} \subseteq X$ we shall write $(E_{\mathcal{A}}, F_{\mathcal{A}})$ for the set of all $\phi \in \mathfrak{C}^{\mathcal{A}}$ such that $\pi \in E_{\mathcal{A}}$ implies $\phi\hat{\pi} |_{\mathcal{A}} \in F_{\mathcal{A}} |_{\mathcal{A}}$. Writing (E, F) for (E_X, F_X) we return to the standard multiplier notation.

4.3. Exploiting the conclusions of 4.1 we have

THEOREM. *If $1 \leq p, q \leq \infty$ with $p \neq \infty$ and $q \neq 1$, these are equivalent:*

- (i) $\sup \{ \|T_w t\|_q : t \in T_{\mathcal{A}}(G) \text{ and } \|t\|_p \leq 1 \} < \infty,$

- (ii) $f \in L^p_2(G)$ implies $T_w f \in L^q_2(G)$,
- (iii) $W \in (L^p_2(G), L^q_2(G))$, and
- (iv) $WL^{q'}(G)^\wedge|_\Delta \subseteq L^{p'}(G)^\wedge|_\Delta$.

Proof. (i) \Rightarrow (ii). Let $\{t_\alpha\} \subseteq T(G)$ be an approximate identity (see [6], (28.53)). If $f \in L^p_2(G)$ then $t_\alpha * f \in T_\Delta(G)$ hence by (i), for some $\kappa > 0$

$$\|T_w(t_\alpha * f)\|_q \leq \kappa \|t_\alpha * f\|_p \leq \kappa \|f\|_p.$$

By the weak compactness of norm balls in $L^q(G)$ ($q \neq 1$) there exists $g \in L^q(G)$ with $\|g\|_q \leq \kappa \|f\|_p$ and $\hat{g} = W\hat{f}$. So by the uniqueness theorem, $T_w f = g \in L^q_2(G)$.

(ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (iv). By hypothesis and the boundedness result 1.11, $T_w: L^p_2(G) \rightarrow L^q_2(G)$ is bounded and linear. So by 1.8 and 1.9 there is a bounded linear map $K: L^{q'}(G)^\wedge|_\Delta \rightarrow L^{p'}(G)^\wedge|_\Delta$ for which, whenever $f \in L^{q'}(G)$ and $\chi \in \Delta$, $K(\hat{f}|_\Delta)(\chi) = W(\chi)\hat{f}(\chi)$.

(iv) \Rightarrow (i) follows similarly.

4.4. It is usually hard to identify (E_Δ, F_Δ) even when (E, F) is known (for $E, F \subseteq PM(G)$) so we pause to combine the approach of 3.1 with the result above.

COROLLARY. Let $1 \leq p, q \leq \infty$ with $p \neq \infty$ and $p \neq 1$. Then $W \in (L^p_2(G), L^q_2(G))$ iff $\sup\{\inf\{\|g\|_p: g \in L^{p'}(G) \text{ and } \hat{g}|_\Delta = W\hat{f}|_\Delta\}: f \in L^{q'}(G) \text{ and } \|f\|_{q'} \leq 1\} \equiv \sup\{\|T_w t\|_q: t \in T_\Delta(G) \text{ with } \|t\|_p \leq 1\} < \infty$. $(L^p_2(G), L^q_2(G))$ is a Banach space and when $p \leq q$ it is a commutative Banach algebra which has an identity iff $\Delta \in A(q)$.

REMARKS. (i). Although $(L^p_2(G), L^q_2(G))$ is unknown in general, special cases yield: $W \in (L^p_2(G), L^q_2(G))$ iff $W \in l^\infty(\Delta)$; and for $1 \leq p < \infty$, $W \in (L^p_2(G), L^q_2(G))$ iff $W \in L^{p'}(G)^\wedge|_\Delta$ by [2], 16.7.5.

(ii). Conditions sufficient to ensure membership to $(L^p(\mathfrak{X}), L^q(\mathfrak{X}))$ are known and yield:

$$\text{if } 1 < p \leq 2 < q < \infty \text{ and } W \in \mathbb{C}^\Delta \text{ with} \\ \sup\{|W(n)|(1 + |n|)^{1/p-1/q}: n \in \Delta\} < \infty$$

then $W \in (L^p_2(\mathfrak{X}), L^q_2(\mathfrak{X}))$ —see [2], 16.4.6(3). More involved conditions apply when $q = p$.

4.5. When $p = 1$, 4.3 can be extended ‘at each end’.

COROLLARY. For $1 < q < \infty$ these are equivalent:

- (i) $W \in (L^2_\Delta(G), L^2_\Delta(G))$,
- (ii) $WM_\Delta(G)^\wedge | \Delta \subseteq L^2_\Delta(G)^\wedge | \Delta$,
- (iii) $WL^{q'}(G)^\wedge | \Delta \subseteq L^\infty(G)^\wedge | \Delta$,
- (iv) $WL^{q'}(G)^\wedge | \Delta \subseteq C(G)^\wedge | \Delta$.

Proof. (i) \Rightarrow (ii) follows as in 4.3(i) \Rightarrow (ii).

(ii) \Rightarrow (iii). Since (ii) \Rightarrow (i), 4.3 implies this.

(iii) \Rightarrow (iv). If $f \in L^{q'}(G)$, by [6], (32.30), there exist $g \in L^1(G)$ and $f_0 \in L^{q'}(G)$ with $f = g * f_0$. By (iii) there is $h_0 \in L^\infty(G)$ with $W\hat{f}_0 | \Delta = \hat{h}_0 | \Delta$. Setting $h = g * h_0$ gives $h \in C(G)$ and

$$\hat{h} | \Delta = \hat{g}\hat{h}_0 | \Delta = \hat{g}W\hat{f}_0 | \Delta = W\hat{f} | \Delta$$

as required.

4.6. More can also be said when $p = 2$.

THEOREM. For $1 < q \leq \infty$, $W \in (L^2_\Delta(G), L^2_\Delta(G))$ iff for all $f \in L^{q'}(G)$,

$$(4.6.1) \quad \left(\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)|^2 \right)^{1/2} \leq \kappa \|f\|_{q'}$$

for some constant κ .

Proof. (\Rightarrow) uses the adjoint of T_w as in 4.3(iii) \Rightarrow (iv).

(\Leftarrow). Parseval's identity with the hypothesis shows $WL^{q'}(G)^\wedge | \Delta \subseteq L^2(G)^\wedge | \Delta$ hence by 4.3(iv), $W \in (L^2_\Delta(G), L^2_\Delta(G))$.

NOTE. By choosing an approximate identity the method above shows $W \in (L^2_\Delta(G), L^\infty_\Delta(G))$ iff $W \in l^2(\Delta)$, as noted in 4.4(i).

Since $(L^2_\Delta(G), L^\infty_\Delta(G)) \subseteq (L^2_\Delta(G), L^\infty_\Delta(G))$ we have thus dealt with the case $q = \infty$ of 4.5. Alternatively,

$$(L^2_\Delta(G), L^\infty_\Delta(G)) \subseteq l^2(\Delta) \text{ when } 1 \leq p \leq 2.$$

See also 4.8.

4.7. Summarising what we have gleaned about W -Sidon sets by virtue of 4.1:

COROLLARY. If Δ is W -Sidon then

- (i) for all $\mu \in M_\Delta(G)$, $T_w\mu \in L^2_\Delta(G)$ and $\|T_w\mu\|_2 \leq 8 \|W\|_\Delta \|\mu\|$,
- (ii) for all $f \in L^2_\Delta(G)$, $T_w f \in L^2_\Delta(G)$ whenever $2 < p < \infty$ and $\|T_w f\|_p \leq 2 \|W\|_\Delta p^{1/2} \|f\|_2$,
- (iii) for all $\mu \in M_\Delta(G)$, $T_{w^2}\mu \in L^2_\Delta(G)$ whenever $2 < p < \infty$ and $\|T_{w^2}\mu\|_p \leq 16 \|W\|_\Delta^2 p^{1/2} \|\mu\|$,

- (iv) for all $\phi \in l^2(\Delta)$, there is $f \in C(G)$ such that $\hat{f}|_{\Delta} = W\phi$ and $\|f\|_{\infty} \leq 8 \|W\|_{\Delta} \|\phi\|_2$, and
- (v) if $1 < p \leq 2$ and $f \in L^p(G)$ then

$$\left(\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)|^2 \right)^{1/2} \leq 2 \|W\|_{\Delta} p^{-1/2} \|f\|_p.$$

Proof. All are obvious except possibly (iii). If $\mu \in M_{\Delta}(G)$ and $2 < p < \infty$, by (i) and (ii),

$$\begin{aligned} \|T_{W^2}\mu\|_p &= \|T_w(T_w\mu)\|_p \leq 2 \|W\|_{\Delta} p^{1/2} \|T_w\mu\|_2 \\ &\leq 16 \|W\|_{\Delta}^2 p^{1/2} \|\mu\|. \end{aligned}$$

4.8. For which W can 1.10(vi) be tightened to

$$(4.8.1) \quad WL^p_2(G)^{\wedge} |_{\Delta} \subseteq l^p(\Delta)$$

for some $p \in [1, \infty)$? We show that when $1 \leq p \leq 2$, (4.8.1) holds iff Δ is a trivial W -Sidon set, and we give a partial answer when $2 < p < \infty$.

THEOREM. *If $\Delta \subseteq X$ then*

- (i) $1 \leq p < \infty$ implies $(L^p_2(G), A_{\Delta}(G)) \subseteq L^{p'}(G)^{\wedge} |_{\Delta}$,
- (ii) $1 \leq p \leq 2$ implies $l^p(\Delta) \subseteq (L^p_2(G), A_{\Delta}(G))$,
- (iii) $2 < p < \infty$ implies $l^2(\Delta) \subseteq (L^p_2(G), A_{\Delta}(G))$, and
- (iv) $2 < p < \infty$ implies $(L^p_2(G), A_{\Delta}(G)) \cap (L^2_2(G), L^{p'}(G)) \subseteq l^p(\Delta)$.

Proof. (i) This follows by 4.4(i) but may be proved quickly as follows. If $W \in (L^p_2(G), A_{\Delta}(G))$ then letting K denote the composition of the isomorphism of 1.8(i) with T_w^* , we have $K: l^{\infty}(\Delta) \rightarrow L^{p'}(G)^{\wedge} |_{\Delta}$ and whenever $\phi \in l^{\infty}(\Delta)$ and $\chi \in \Delta$, $(K\phi)(\chi) = W(\chi)\phi(\chi)$. Taking $\phi = \mathbf{1}$ this gives

$$(4.8.2) \quad f \in L^{p'}(G) \text{ with } \hat{f}|_{\Delta} = W$$

as required.

(ii) If $1 \leq p \leq 2$ and $f \in L^p_2(G)$ then by the Hausdorff-Young theorem and Hölder's inequality, whenever $W \in l^p(\Delta)$,

$$\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| \leq \|W\|_p \|f\|_p < \infty.$$

(iii) If $2 < p < \infty$ and $f \in L^p_2(G)$ then $\hat{f}|_{\Delta} \in l^2(\Delta)$ hence when $W \in l^2(\Delta)$,

$$\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| \leq \|W\|_2 \|\hat{f}\|_2 < \infty.$$

(iv) Continuing from (4.8.2), if $2 < p < \infty$, 4.6 shows

$$\left(\sum_{\chi \in \Delta} |W(\chi)|^4 \right)^{1/2} \leq 2 \|W\|_{\Delta} p^{1/2} \|f\|_{p'}$$

so $W \in l^4(\Delta)$.

REMARKS. (i) Taking W constant, (4.8.2) shows there can be no infinite Sidon sets Δ with $L^2_\Delta(G)^\wedge | \Delta \subseteq l^4(\Delta)$ when $1 \leq p < \infty$.

(ii) Results (i) and (ii) above combine to show that trivial W -Sidon sets are precisely the W -Sidon sets for which (4.8.1) holds when $p \in [1, 2]$.

Results (iii) and (iv) do not interlock in this way but show, thanks to 4.7(v), that when $p \in (2, \infty)$, (4.8.1) cannot hold when Δ is W -Sidon and $W \notin l^4(\Delta)$.

(iii) For comparison, $(L^2_\Delta(G), A_\Delta(G))$ is identified when Δ is a subgroup of X in [6], (36.20) via the method of 1.3.

4.9. When $W = 1$ the inclusions implied by 4.7 for Sidon sets are, by Parseval's identity, equalities. In fact these are the only W -Sidon sets with equality:

THEOREM. Δ is Sidon whenever it is W -Sidon and one of these holds.

- (i) $l^2(\Delta) \subseteq WM_\Delta(G)^\wedge | \Delta$,
- (ii) $L^\infty(G)^\wedge | \Delta \subseteq Wl^2(\Delta)$,
- (iii) $C(G)^\wedge | \Delta \subseteq Wl^2(\Delta)$,
- (iv) $L^2_\Delta(G)^\wedge | \Delta \subseteq Wl^2(\Delta)$, for some $p \in (2, \infty)$ and
- (v) $l^2(\Delta) \subseteq WL^p(G)^\wedge | \Delta$, for some $p \in (1, 2)$.

Proof. Theorem 1.12 as used in 1.14 makes short work of these.

4.10. So far we have discussed the behaviour of $T_w\pi$ when π is a Δ -spectral measure of L^p -function and Δ is W -Sidon. Immediately from 1.10(viii) we have: Δ is W -Sidon iff $WPM_\Delta(G)^\wedge | \Delta \subseteq M(G)^\wedge | \Delta$. From 1.14(i) this inclusion is proper whenever Δ is not Sidon.

Evidently $T_w(PM_\Delta(G)) \subseteq L^2_\Delta(G)$ iff Δ is a trivial W -Sidon set and if $T_w(PM_\Delta(G)) \subseteq M_\Delta(G)$ then $W \in l^4(\Delta)$.

4.11. We now deduce more about those W in $\mathfrak{B}(\Delta)$. Specialising to \mathfrak{X} (though (4.11.1) holds in general) we use:

THEOREM. Let $F \in \mathbb{C}^3$. If $\phi F \in \bigcap \{L^p(\mathfrak{X})^\wedge : 1 \leq p < \infty\}$ for all $\phi \in c_0(\mathfrak{Z})$ then for all $\alpha > 0$, $\sum_{n \neq 0} |n^{-\alpha} F(n)| < \infty$.

Proof. Successive applications of 1.11 and 1.8 show that if

$1 < p < \infty$, then $\phi F \in L^p(\mathfrak{Z})^\wedge$ for all $\phi \in c_0(\mathfrak{Z})$ implies $WL_2^p(G)^\wedge \triangleleft \subseteq l^1(\mathcal{A})$. So the hypothesis entails

$$(4.11.1) \quad \text{for all } p \in (1, \infty) \text{ and all } g \in L^p(\mathfrak{Z}), \sum_{n \in \mathfrak{Z}} |F(n)\hat{g}(n)| < \infty .$$

Now if $0 < \alpha < 1$ then by [2], Exercise 7.8, there exist $p \in (1, (1-\alpha)^{-1})$ and $g \in L^p(\mathfrak{Z})$ such that $\hat{g}(n) = n^{-\alpha}$ for $n \neq 0$. If $\alpha \geq 1$ then the map $n \mapsto n^{-\alpha}$ belongs to $l^2(\mathfrak{Z} \setminus \{0\})$ hence there is $g \in L^2(\mathfrak{Z})$ with $\hat{g}(n) = n^{-\alpha}$ whenever $n \neq 0$.

In either case, substitution into (4.11.1) yields

$$\sum_{n \neq 0} |F(n)n^{-\alpha}| < \infty$$

as required.

NOTES. (i) In [12] we show the converse of this theorem to be false.

(ii) The sum $\sum_{n \neq 0} |n^{-\alpha}F(n)|$ was first considered by Hardy and Littlewood in [5]. Their results imply that it is finite whenever $\alpha > 1/2$ and may be infinite otherwise, when $F \in \bigcap \{L^p(\mathfrak{Z})^\wedge : 1 \leq p < \infty\}$.

4.12. The information this gives about W is:

COROLLARY. *If $W \in \mathfrak{B}(\mathcal{A})$ then for all $\mu \in M_\mathcal{A}(\mathfrak{Z})$, if $\alpha > 0$ then*

$$\sum_{n \neq 0} |n^{-\alpha}\hat{\mu}(n)W^2(n)| < \infty .$$

Proof. In fact if $\phi \in l^\infty(\mathfrak{Z})$ (not merely $c_0(\mathfrak{Z})$) and \mathcal{A} is W -Sidon then evidently \mathcal{A} is $W\phi^{1/2}$ -Sidon. Hence by 4.7(ii), whenever $\mu \in M_\mathcal{A}(G)$,

$$\phi W^2\hat{\mu} \in \bigcap \{L_2^p(G)^\wedge : 1 \leq p < \infty\}$$

so the conclusion follows from 4.11.

4.13. Using $l^\infty(\mathfrak{Z})$ rather than $c_0(\mathfrak{Z})$ above seems to be stronger. However in this context they are equivalent.

THEOREM. *Let $F \in \mathfrak{C}^X$. Then ϕF belongs to $\bigcap \{L^p(G)^\wedge : 1 \leq p < \infty\}$ for all $\phi \in c_0(X)$ iff it does for all $\phi \in l^\infty(X)$.*

Proof. This follows readily upon taking the bidual of the map $K: c_0(X) \rightarrow L^p(G)$ given by $(K\phi)^\wedge = \phi F$.

4.14. It might be hoped that a tight necessary condition for W to belong to $\mathfrak{B}(\mathcal{A})$ follows from 4.12 by eliminating μ somehow to give a purely combinatorial property. However the \mathcal{A} -spectral

measures compensate for variations in the thickness of Δ , so we turn to other means for this.

Refer to [3], 3.1 for the definition of a test family of order m .

THEOREM. *If $W \in (L^p_\Delta(G), L^q_\Delta(G))$ where $1 \leq p \leq 2$ and $1 < q < \infty$, and \mathfrak{F} is a test family of order m then for each $\Phi \in \mathfrak{F}$,*

$$\sum_{\chi \in \Phi \cap \Delta} |W(\chi)|^2 \leq \kappa^2 m \nu(\Phi)^{2/q}$$

where κ is the unnamed constant in 4.4.

Proof. This is a routine modification of [3], 3.2 for which details appear in [11].

COROLLARY 4.15. *If Δ is W -Sidon and \mathfrak{F} is a test family of order m then for each $\Phi \in \mathfrak{F}$ with $\nu(\Phi) \geq 3$,*

$$\sum_{\chi \in \Phi \cap \Delta} |W(\chi)|^2 \leq 8e \|W\|_\Delta m \log \nu(\Phi).$$

Proof. By hypothesis and 4.7(ii), $W \in (L^2_\Delta(G), L^q_\Delta(G))$ whenever $q \in (2, \infty)$ and so by 4.14,

$$\sum_{\chi \in \Phi \cap \Delta} |W(\chi)|^2 \leq 4 \|W\|_\Delta^2 q m \nu(\Phi)^{2/q}.$$

Taking $q = 2 \log \nu(\Phi)$ so that $q > 2$ because $\nu(\Phi) \geq 3$, this entails the result.

NOTES. (i). This means that if $\varepsilon > 0$, the number of elements of Δ in Φ with $|W(\chi)| > \varepsilon$ remains small as Φ enlarges.

(ii). For $q = \infty$ the result above is overshadowed by the note to 4.6.

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