NORM ATTAINING OPERATORS ON $L^1[0, 1]$ AND THE RADON-NIKODÝM PROPERTY

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Let $Y$ be a strictly convex Banach space. Then norm 
attaining operators mapping $L^1[0, 1]$ to $Y$ are dense in the 
space of all linear operators from $L^1[0, 1]$ to $Y$ if and only 
if $Y$ has the Radon-Nikodým property.

Bishop and Phelps [1] have asked the general question—for 
which Banach spaces $X$ and $Y$ is the collection of norm 
attaining operators from $X$ to $Y$ dense in the space $B(X, Y)$ of all bounded 
(linear) operators from $X$ to $Y$. Lindenstrauss in [8] investigated 
this question and related this question to existence of extreme points 
and exposed points in the closed unit ball of $X$. In the course of 
his paper Lindenstrauss showed that for some space $Y$ the norm 
attaining operators in $B(L^1[0, 1], Y)$ are not dense in $B(L^1[0, 1], Y)$ 
due to the lack of extreme points in the closed unit ball of $L^1[0, 1]$. 
Left open is the following question: For which Banach spaces $Y$ 
are the norm attaining operators dense in $B(L^1[0, 1], Y)$? Based on 
Lindenstrauss’s work, one is led to believe that if the closed unit 
bond of $Y$ has a rich extreme point or exposed point structure, then 
the norm attaining operators may be dense in $B(L^1[0, 1], Y)$. On 
the other hand the Radon-Nikodým property is intimately connected 
with extreme point structure (Rieffel [12], Maynard [10], Huff [6], 
Davis and Phelps [2], Phelps [11], Huff and Morris [7]). So there 
is some prima facie evidence to support the belief that the norm 
attaining operators are dense in $B(L^1[0, 1], Y)$ if and only if $Y$ has 
the Radon-Nikodým property. The purpose of this paper is to verify 
this for strictly convex Banach spaces $Y$.

First a few well known results will be collected.

**Lemma A** [4, 5]. If $(Ω, Σ, μ)$ is a finite measure space and 
g: $Ω → Y$ is $μ$-essentially bounded Bochner integrable function, then

$$T(f) = \text{Bochner} - \int fg dμ$$

defines a member $T$ of $B(L^1(μ), Y)$ with $\| T \| = \text{ess sup} \| g \|_Y$.

**Lemma B** [3]. Any one of the following statements about $Y$ 
implies all the others.

(i) $Y$ has the Radon-Nikodým property.

(ii) If $(Ω, Σ, μ)$ is a finite measure space and $G: Σ → Y$ is a
μ-continuous countably additive measure of bounded variation, then there exists a μ-Bochner integrable

\( g: \Omega \rightarrow Y \) with \( G(E) = \int_E gd\mu \) for all \( E \in \Sigma \).

(iii) If \( \mu \) is Lebesgue measure on \([0, 1]\), then for each \( T \in B(L'[0, 1], Y) \) there is a μ-essentially bounded \( g: [0, 1] \rightarrow Y \) with

\[
T(f) = \int_{[0,1]} fg \, d\mu \quad \text{for all } f \in L'([0, 1], Y)
\]

Moreover, if \( Y \) has the Radon-Nikodym property statement (iii) is true for any finite measure space.

The first theorem is a straightforward observation that is based on the definition of a measurable function.

**Theorem 1.** If \( Y \) has the Radon-Nikodym property and if \((\Omega, \Sigma, \mu)\) is a finite measure space, then the norm attaining operators are dense in \( B(L'(\mu), Y) \).

**Proof.** Let \( T \in B(L'(\mu), Y) \) and \( \varepsilon > 0 \). Then there exists an essentially bounded Bochner integrable \( g: \Omega \rightarrow Y \) such that \( T(f) = \int g \, d\mu \) for all \( f \in L'(\mu) \) and there exists a countably valued function

\[
h: \Omega \rightarrow X, \quad h = \sum_{i=1}^{\infty} x_i \chi_{E_i}, \quad x_i \in X,
\]

\[
E_i \in \Sigma, \quad \mu(E_i) > 0, \quad E_i \cap E_j = \emptyset
\]

for \( i \neq j \), such that \( \text{ess sup } ||g - h|| < \varepsilon/2 \). Define \( T_1: L'(\mu) \rightarrow Y \) by \( T_1(f) = \int f h \, d\mu \), \( f \in L'(\mu) \). Then \( ||T - T_1|| < (\varepsilon/2) \).

Now \( T_1 \) will be approximated within \( \varepsilon/2 \) by an operator which attains its norm. If \( T_1 = 0 \), there is nothing to prove. Otherwise \( \beta = \sup ||y_i|| > 0 \). Choose \( i_0 \) such that \( \beta - ||y_{i_0}|| < \varepsilon/2 \) and \( \alpha > 1 \) such that \( \varepsilon/4 < (\alpha - 1) ||y_{i_0}|| < \varepsilon/2 \) and define

\[
T_2(f) = \int_{\bigcup_{E_i \cap E_j \neq \emptyset}} f h \, d\mu + \alpha y_{i_0} \int_{E_{i_0}} f \, d\mu.
\]

It is easy to verify that \( ||T_1 - T_2|| < \varepsilon/2 \) and that \( ||T_2|| = \alpha ||y_{i_0}|| = ||T_1(x_{i_0}||\mu(E_{i_0}))|| \). Hence \( T_2 \) attains its norm and \( ||T - T_2|| < \varepsilon \), as required.

The operator \( T_2 \) constructed in the proof of Theorem 1 has two important properties. First it attains its norm and second there
exists $E \in \Sigma$, $\mu(E) > 0$ and $y_0 \in Y$ with $\|y_0\| = \|T\|$ and $T(f\chi_E) = \int_E f d\mu y_0$ for all $f \in L'(\mu)$. If $Y$ is strictly convex and real, this property is shared by all norm attaining operators in $B(L'(\mu), Y)$.

**Lemma 2.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $Y$ be a strictly convex Banach space. If $T \in B(L'(\mu), Y)$ attains its norm then there exists a set $E_0 \in \Sigma$ with $\mu(E_0) > 0$, $g \in L^\infty(\mu)$ with $|g| = 1$ on $E_0$, and $y_0 \in Y$ with $\|y_0\| = \|T\|$ such that

$$T(f \chi_{E_0}) = \int_{E_0} f g d\mu y_0$$

for all $f \in L'(\mu)$.

If $Y$ is a real Banach space, $g$ may be taken as the constant function $1$.

**Proof.** If $\|T\| = 0$, there is nothing to prove.

Otherwise, choose $f_0 \in L'(\mu)$ with $\|T(f_0)\| = \|T\|$ and $\|f_0\| = 1$. With the help of the Hahn-Banach theorem, choose $y^* \in Y^*$ with $\|y^*\| = 1$ and $y^* T(f_0) = \|T(f_0)\| = \|T\|$.

Next choose $h \in L^\infty(\mu)$ with $\|h\|_\infty = \|T\|$ such that

$$y^* T(f) = \int_\Omega f h d\mu$$

for all $f \in L'(\mu)$. A simple computation reveals that $h = \text{sgn} f_0/\|T\|$ on the support of $f_0$. (Here $\text{sgn} f_0 = f_0/|f_0|$.) Let $E_0$ be the support of $f_0$. Thus if $f \in L'(\mu)$,

$$y^* T(f \chi_{E_0}) = \int_{E_0} f \text{sgn} f_0 \|T\| d\mu.$$

Next suppose $E \subset E_0$, $E \in \Sigma$ and $\mu(E), \mu(E_0 - E) > 0$. (The rest of the proof is trivial if $E_0$ is an atom of $\mu$.) Then

$$y^* T\left(\frac{\chi_E}{\mu(E)} \text{sgn} f_0\right) = \int_{E_0} \frac{\chi_E}{\mu(E)} \|T\| d\mu = \|T\|,$$

$$y^* T\left(\frac{\chi_{E_0 - E}}{\chi(E)} \text{sgn} f_0\right) = \int_{E_0} \frac{\chi_{E_0 - E}}{\mu(E_0 - E)} \|T\| d\mu = \|T\|,$$

and

$$y^* T\left(\frac{\chi_{E_0}}{\mu(E_0)} \text{sgn} f_0\right) = \int_{E_0} \frac{\chi_{E_0}}{\mu(E_0)} \|T\| d\mu = \|T\|.$$
From these equalities, one obtains
\[ \| T \| \mu(E_0) = \| T(\chi_{E_0} \text{sgn } f_0) \| = \| T(\chi_{E} \text{sgn } f_0) + T(\chi_{E_0-E} \text{sgn } f_0) \| \leq \| T(\chi_{E} \text{sgn } f_0) \| + \| T(\chi_{E_0-E} \text{sgn } f_0) \| = \| T \| \mu(E) + \| T \| \mu(E_0 - E) = \| T \| \mu(E_0). \]

This combined with the fact that \( Y \) is strictly convex shows that \( T(\chi_{E} \text{sgn } f_0) \) and \( T(\chi_{E_0-E} \text{sgn } f_0) \) are multiples of each other. Since \( T(\chi_{E_0} \text{sgn } f_0) = T(\chi_{E} \text{sgn } f_0) + T(\chi_{E_0-E} \text{sgn } f_0), \) \( T(\chi_{E} \text{sgn } f_0) \) is a scalar multiple of \( T(\chi_{E_0} \text{sgn } f_0); \) i.e., \( T(\chi_{E} \text{sgn } f_0) = \gamma T(\chi_{E_0} \text{sgn } f_0) \) for some scalar \( \gamma. \) On the other hand
\[ \| T \| \mu(E) = \gamma \| T(\chi_{E} \text{sgn } f_0) \| = \gamma \| T(\chi_{E_0} \text{sgn } f_0) \| \]
thus \( \gamma = \mu(E) / \mu(E_0). \) Therefore if \( E \subset E_0 \) and \( \mu(E) > 0, \)
\[ \frac{T(\chi_{E} \text{sgn } f_0)}{\mu(E)} = \frac{T(\chi_{E_0} \text{sgn } f_0)}{\mu(E_0)} = \gamma. \]

Now suppose \( f \in L^1(\mu) \) is a simple function. Let \( \varepsilon > 0 \) and choose a simple function \( \varphi \in L^1(\mu) \) such that \( \| \text{sgn } f_0 - \varphi \|_{\infty} < \varepsilon. \) (Here \( \text{sgn } f_0 \)
is the complex conjugate of \( \text{sgn } f_0. \) Then \( T(f) = T(f \text{sgn } f_0) \) and \( \| T(f) - T(f \varphi \text{sgn } f_0) \| \leq \| T \| \| \text{sgn } f_0 - \varphi \text{sgn } f_0 \| < \varepsilon \| T \| \mu \). Now select sets \( A_i, \ldots, A_n \in \Sigma \) such that
\[ f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \quad \text{and} \quad \varphi = \sum_{i=1}^{n} \beta_i \chi_{A_i}. \]
Then
\[ T(f \varphi \text{sgn } f_0 \chi_{E_0}) = \sum_{i=1}^{n} \alpha_i \beta_i \frac{T(\chi_{A_i} \cap E_0 \text{sgn } f_0)}{\mu(A_i \cap E_0)} \mu(A_i \cap E_0) = \sum_{i=1}^{n} \alpha_i \beta_i \mu(A_i \cap E_0) y_0 = \int_{E_0} f \varphi \text{sgn } f_0 \mu y_0. \]
Letting \( \varepsilon \) go to zero reveals that
\[ T(f \chi_{E_0}) = \int_{E_0} f \text{sgn } f_0 \mu y_0. \]
Since simple functions are dense in \( L^1(\mu) \), the equality
\[ T(f \chi_{E_0}) = \int_{E_0} f \text{sgn } f_0 \mu y_0 \]
obtains for all \( f \in L^1(\mu). \) This proves the first statement.

To prove the second statement, note that if \( Y \) is real, then \( \text{sgn } f_0 \) takes on only the values \(+1\) or \(-1\). If \( \text{sgn } f_0 = 1 \) on a set of positive measure \( E, \) in the support of \( f_0, \) take \( E_0 = E \) and proceed
as above. If \( \text{sgn } f_0 = -1 \) almost everywhere in the support of \( f_0 \), multiply \( f_0 \) and \( y_0^* \) by \(-1\) and proceed as in the last sentence.

With the help of Lemma 2, the main result becomes nothing but a straightforward exhaustion argument.

**Theorem 3.** Let \( Y \) be a strictly convex Banach space. If the norm attaining members of \( \mathcal{B}(L'[0, 1], Y) \) are dense in \( \mathcal{B}(L'[0, 1], Y) \), then \( Y \) has the Radon-Nikodým property.

**Proof.** Let \( T \in \mathcal{B}(L'[0, 1], Y) \) and \( \varepsilon > 0 \) be given. Define a class of Lebesgue measurable sets \( \mathcal{M} \) by agreeing that \( E \in \mathcal{M} \) if there exists an essentially bounded Bochner integrable \( g = g(E, \varepsilon): [0, 1] \to Y \) such that

\[
\left\| T(f\chi_E) - \int_E f g d\mu \right\| \leq \varepsilon \left\| f\chi_E \right\|_1.
\]

Note that if \( A \) is Lebesgue measurable and \( A \subset E \in \mathcal{M} \) then

\[
\left\| T(f\chi_A) - \int_A f g((E, \varepsilon)d\mu) \right\| = \left\| T((f\chi_A)\chi_E) - \int_E (f\chi_A)g d\mu \right\|
\leq \left\| f\chi_A \chi_E \right\|_1 = \varepsilon \left\| f\chi_A \right\|_1.
\]

Therefore, if \( E \in \mathcal{M} \), every measurable subset of \( E \) belongs to \( \mathcal{M} \). Now let \( \alpha = \sup \{ \mu(E): E \in \mathcal{M} \} \) and let \( (E_n) \subset \mathcal{M} \) be a sequence such that \( \lim_n \mu(E_n) = \alpha \). Write \( A_1 = E_1, A_2 = E_2 - E_1, \ldots, A_n = E_n - \bigcup_{i=1}^{n-1} E_i \). Then the \( A_i \)'s are disjoint, \( \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n \) and \( \mu(\bigcup_{n=1}^{\infty} A_n) \geq \alpha \). \( A_n \subset E_n \) and \( E_n \in \mathcal{M}, A_n \in \mathcal{M} \) and there exists a sequence of essentially bounded functions \( g_n: [0, 1] \to Y, n = 1, 2, \ldots \), such that for all \( f \in L'[0, 1] \),

\[
\left\| T(f\chi_{A_n}) - \int_{A_n} f g_n d\mu \right\| \leq \varepsilon \left\| f\chi_{A_n} \right\|_1.
\]

Accordingly,

\[
\left\| \int_{A_n} f g_n d\mu \right\| \leq \left\| T(f\chi_{A_n}) \right\| + \varepsilon \left\| f\chi_{A_n} \right\|_1 \leq (\| T \| + \varepsilon) \| f \|_1.
\]

By Lemma A,

\[
\text{ess sup } \| g_n\chi_{A_n} \| - \sup_{\| f \|_1 \leq 1} \left\| \int_{A_n} f g_n d\mu \right\| \leq \| T \| + \varepsilon.
\]

Therefore \( \sup_n \text{ess sup } \| g_n \| \leq \| T \| + \varepsilon \). Now define \( g: [0, 1] \to Y \) by

\[
g(t) = \begin{cases} g_n(t) & \text{for } t \in A_n \\ 0 & \text{for } t \in \bigcup_{n=1}^{\infty} A_n \end{cases}
\]
Then \( \text{ess sup} \| g \| \leq \| T \| + \varepsilon \) and if \( f \in L'[0, 1] \),
\[
\left\| T(f\chi_{A_n}) - \int A_n f g d\mu \right\| \\
\leq \sum_{n=1}^{\infty} \left\| T(f\chi_{A_n}) - \int A_n f g d\mu \right\| \\
\leq \sum_{n=1}^{\infty} \varepsilon \left\| f\chi_{A_n} \right\|_1 \leq \varepsilon \left\| f \right\|_1 .
\]
Therefore \( \bigcup_n A_n \in \mathscr{A} \). Next we shall see that \( \mu \left( \bigcup_n A_n \right) = 1 \). For, if \( \mu \left( \bigcup_n A_n \right) < 1 \), then \( \mu \left( \bigcup_n E_n \right) \leq 1 \) and \( \alpha < 1 \). Set \( B_0 = [0, 1] - \bigcup_n A_n \) and recall that \( L'(B_0) \) (Lebesgue integrable functions supported on \( B_0 \)) is isometric to \( L'[0, 1] \). Define \( T_1 : L'(B_0) \to Y \) by \( T_1(f) = T(f\chi_{B_0}) \) for \( f \in L'(E) \). Since \( L'(B_0) \) is isometric to \( L'[0, 1] \), there exists an operator \( T_2 : L'(B_0) \to Y \) that attains its norm such that \( \| T_1 - T_2 \| \leq \varepsilon \).

An appeal to Lemma 2 produces a \( y \in Y \) and set \( B_1 \subset B_0 \) with \( \mu(B_1) > 0 \) such that
\[
T_2(f) = \int_{B_1} f d\mu y_1
\]
for all \( f \in L'(B_0) \). Set \( g' = y \chi_{B_1} \). Then
\[
\left\| T(f\chi_{B_1}) - \int_{B_1} f g' d\mu \right\| = \left\| T_1(f\chi_{B_1}) - T_2(f\chi_{B_1}) \right\| \\
\leq \left\| T_1 - T_2 \right\| \left\| f\chi_{B_1} \right\|_1 \leq \varepsilon \left\| f\chi_{B_1} \right\|_1 .
\]
Therefore \( B_1 \in \mathscr{A} \). Now set \( \bar{g} = g + g' \). If \( f \in L'(\Omega) \),
\[
\left\| T(f\chi_{E_n \cup B_1}) - \int_{E_n \cup B_1} f \bar{g} d\mu \right\| \\
\leq \sum_{n=1}^{\infty} \left\| T(f\chi_{A_n}) - \int A_n f g d\mu \right\| + \left\| T(f\chi_{B_1}) - \int_{B_1} f g' d\mu \right\| \\
\leq \varepsilon \sum_{n=1}^{\infty} \left\| f\chi_{A_n} \right\| + \varepsilon \left\| f\chi_{B_1} \right\| = \left\| f\chi_{E_n \cup B_1} \right\| .
\]
Therefore \( \bigcup_n A_n \cup B_1 = \bigcup_n E_n \cup B_1 \in \mathscr{A} \). But
\[
\mu \left( \bigcup_n E_n \cup B_1 \right) = \mu \left( \bigcup_n E_n \right) + \mu(B_1) \\
\geq \lim_n \mu(E_n) + \mu(B_1) = \alpha + \mu(B_1) > \alpha
\]
contradicting the definition of \( \alpha \). Thus \( \mu \left( \bigcup_n A_n \right) = 1 \) and
\[
\left\| T(f) - \int_{[0,1]} f g d\mu \right\| \leq \varepsilon \left\| f \right\|_1 \text{ for all } f \in L'[0, 1] .
\]
Finally, to check that \( Y \) has the Radon-Nikodym property, let
$g_n: [0, 1] \rightarrow Y$ be a sequence of Bochner integrable essentially bounded functions such that for all $f \in L^i[0, 1]$

$$\left\| T(f) - \int_{[0,1]} fg_n d\mu \right\| \leq 1/n \| f \|,$$

for all $n$. An appeal to Lemma 1 shows that $\lim_{n,m} \text{ess sup} \| g_n - g_m \|$. Hence there exists a Bochner integrable essentially bounded $g: [0, 1] \rightarrow Y$ with $\lim_n \text{ess sup} \| g_n - g \| = 0$. If $f \in L^i[0, 1]$, the dominated convergence theorem guarantees that

$$T(f) - \lim_n \int_{[0,1]} fg_n d\mu = \int_{[0,1]} fg d\mu.$$

Thus $Y$ has the Radon-Nikodým property by Lemma B.

The role of strict convexity seems to be crucial in Theorem 3: for by perturbing co-ordinate functions it is seen easily that norm attaining operators are dense in $B(L^i[0, 1], c_0)$, $B(L^i[0, 1], l^\infty)$ or for that matter $B(X, l^\infty)$ for any Banach space $X$. See [8, Prop. 3].

On the other hand, the role of strict convexity could be made even more palatable by an affirmative answer to an old question of Diestel’s: Does every Banach space with the Radon-Nikodým property have an equivalent strictly convex norm?

**Corollary 4.** If $X$ is a strictly convex renorming of $L^i[0, 1]$, then the norm attaining operators are not dense in $B(L^i[0, 1], X)$.

**Proof.** Evidently $X$ lacks the Radon-Nikodým property.

This leaves unsolved the question of whether the norm attaining operators are dense in $B(L^i[0, 1], L^i[0, 1])$.

Finally say that a Banach space $X$ has property $B$ if for every Banach space $Y$ the norm attaining operators are dense in $B(Y, X)$. Lindenstrauss [8, Proposition 4] has observed that if there is a non-compact operator in $B(c_0, X)$ and $X$ is strictly convex, then $X$ lacks property $B$. It is not difficult to see that if $X$ has the Radon-Nikodým property, then every operator in $B(c_0, X)$ is compact and that the converse in false. Thus Theorem 3 is a better test for Property $B$ than [8, Proposition 4]. Of course this brings up a question that is well beyond the scope of this note. If $X$ is a strictly convex Banach space, does $X$ have property $B$ if and only if $X$ has the Radon-Nikodým property?

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REFERENCES


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