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**NORM ATTAINING OPERATORS ON $L^1[0, 1]$ AND THE
RADON-NIKODÝM PROPERTY**

J. JERRY UHL, JR.

NORM ATTAINING OPERATORS ON $L^1[0, 1]$ AND THE RADON-NIKODÝM PROPERTY

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Let Y be a strictly convex Banach space. Then norm attaining operators mapping $L^1[0, 1]$ to Y are dense in the space of all linear operators from $L^1[0, 1]$ to Y if and only if Y has the Radon-Nikodým property.

Bishop and Phelps [1] have asked the general question—For which Banach spaces X and Y is the collection of norm attaining operators from X to Y dense in the space $B(X, Y)$ of all bounded (linear) operators from X to Y . Lindenstrauss in [8] investigated this question and related this question to existence of extreme points and exposed points in the closed unit ball of X . In the course of his paper Lindenstrauss showed that for some space Y the norm attaining operators in $B(L^1[0, 1], Y)$ are not dense in $B(L^1[0, 1], Y)$ due to the lack of extreme points in the closed unit ball of $L^1[0, 1]$. Left open is the following question: For which Banach spaces Y are the norm attaining operators dense in $B(L^1[0, 1], Y)$? Based on Lindenstrauss's work, one is led to believe that if the closed unit ball of Y has a rich extreme point or exposed point structure, then the norm attaining operators may be dense in $B(L^1[0, 1], Y)$. On the other hand the Radon-Nikodým property is intimately connected with extreme point structure (Rieffel [12], Maynard [10], Huff [6], Davis and Phelps [2], Phelps [11], Huff and Morris [7]). So there is some *prima facie* evidence to support the belief that the norm attaining operators are dense in $B(L^1[0, 1], Y)$ if and only if Y has the Radon-Nikodým property. The purpose of this paper is to verify this for strictly convex Banach spaces Y .

First a few well known results will be collected.

LEMMA A [4, 5]. *If (Ω, Σ, μ) is a finite measure space and $g: \Omega \rightarrow Y$ is μ -essentially bounded Bochner integrable function, then*

$$T(f) = \text{Bochner} - \int fg d\mu$$

defines a member T of $B(L^1(\mu), Y)$ with $\|T\| = \text{ess sup } \|g\|_Y$.

LEMMA B [3]. *Any one of the following statements about Y implies all the others.*

- (i) *Y has the Radon-Nikodým property.*
- (ii) *If (Ω, Σ, μ) is a finite measure space and $G: \Sigma \rightarrow Y$ is a*

μ -continuous countably additive measure of bounded variation, then there exists a μ -Bochner integrable

$$g: \Omega \longrightarrow Y \text{ with } G(E) = \int_E g d\mu \text{ for all } E \in \Sigma .$$

(iii) If μ is Lebesgue measure on $[0, 1]$, then for each $T \in B(L^1[0, 1], Y)$ there is a μ -essentially bounded $g: [0, 1] \rightarrow Y$ with

$$T(f) = \int_{[0,1]} f g d\mu \text{ for all } f \in L^1([0, 1], Y)$$

Moreover, if Y has the Radon-Nikodým property statement (iii) is true for any finite measure space.

The first theorem is a straight forward observation that is based on the definition of a measurable function.

THEOREM 1. *If Y has the Radon-Nikodým property and if (Ω, Σ, μ) is a finite measure space, then the norm attaining operators are dense in $B(L^1(\mu), Y)$.*

Proof. Let $T \in B(L^1(\mu), Y)$ and $\varepsilon > 0$. Then there exists an essentially bounded Bochner integrable $g: \Omega \rightarrow Y$ such that $T(f) = \int_{\Omega} f g d\mu$ for all $f \in L^1(\mu)$ and there exists a countably valued function

$$h: \Omega \longrightarrow X, \quad h = \sum_{i=1}^{\infty} x_i \chi_{E_i}, \quad x_i \in X,$$

$$E_i \in \Sigma, \quad \mu(E_i) > 0, \quad E_i \cap E_j = \emptyset$$

for $i \neq j$, such that $\text{ess sup} \|g - h\| < \varepsilon/2$. Define $T_1: L^1(\mu) \rightarrow Y$ by $T_1(f) = \int_{\Omega} f h d\mu$, $f \in L^1(\mu)$. Then $\|T - T_1\| < (\varepsilon/2)$.

Now T_1 will be approximated within $\varepsilon/2$ by an operator which attains its norm. If $T_1 = 0$, there is nothing to prove. Otherwise $\beta = \sup \|y_i\| > 0$. Choose i_0 such that $\beta - \|y_{i_0}\| < \varepsilon/2$ and $\alpha > 1$ such that $\varepsilon/4 < (\alpha - 1) \|y_{i_0}\| < \varepsilon/2$ and define

$$T_2(f) = \int_{\bigcup_{i \neq i_0} E_i} f h d\mu + \alpha y_{i_0} \int_{E_{i_0}} f d\mu .$$

It is easy to verify that $\|T_1 - T_2\| < \varepsilon/2$ and that $\|T_2\| = \alpha \|y_{i_0}\| = \|T_2(x_{E_{i_0}}/\mu(E_{i_0}))\|$. Hence T_2 attains its norm and $\|T - T_2\| < \varepsilon$, as required.

The operator T_2 constructed in the proof of Theorem 1 has two important properties. First it attains its norm and second there

exists $E \in \Sigma$, $\mu(E) > 0$ and $y_0 \in Y$ with $\|y_0\| = \|T\|$ and $T_2(f\chi_E) = \int_E f d\mu y_0$ for all $f \in L^1(\mu)$. If Y is strictly convex and real, this property is shared by all norm attaining operators in $B(L^1(\mu), Y)$.

LEMMA 2. *Let (Ω, Σ, μ) be a finite measure space and Y be a strictly convex Banach space. If $T \in B(L^1(\mu), Y)$ attains its norm then there exists a set $E_0 \in \Sigma$ with $\mu(E_0) > 0$, $g \in L^\infty(\mu)$ with $|g| = 1$ on E_0 , and $y_0 \in Y$ with $\|y_0\| = \|T\|$ such that*

$$T(f\chi_{E_0}) = \int_{E_0} fg d\mu y_0$$

for all $f \in L^1(\mu)$.

If Y is a real Banach space, g may be taken as the constant function 1.

Proof. If $\|T\| = 0$, there is nothing to prove.

Otherwise, choose $f_0 \in L^1(\mu)$ with $\|T(f_0)\| = \|T\|$ and $\|f_0\| = 1$. With the help of the Hahn-Banach theorem, choose $y_0^* \in Y^*$ with $\|y_0^*\| = 1$ and

$$y_0^* T(f_0) = \|T(f_0)\| = \|T\|.$$

Next choose $h \in L^\infty(\mu)$ with $\|h\|_\infty = \|T\|$ such that

$$y_0^* T(f) = \int_\Omega f h d\mu$$

for all $f \in L^1(\mu)$. A simple computation reveals that $h = \overline{\text{sgn } f_0} \|T\|$ on the support of f_0 . (Here $\text{sgn } f_0 = f_0/|f_0|$.) Let E_0 be the support of f_0 . Thus if $f \in L^1(\mu)$,

$$y_0^* T(f\chi_{E_0}) = \int_{E_0} f \overline{\text{sgn } f_0} \|T\| d\mu.$$

Next suppose $E \subset E_0$, $E \in \Sigma$ and $\mu(E), \mu(E_0 - E) > 0$. (The rest of the proof is trivial if E_0 is an atom of μ .) Then

$$y_0^* T\left(\frac{\chi_E}{\mu(E)} \text{sgn } f_0\right) = \int_{E_0} \frac{\chi_E}{\mu(E)} \|T\| d\mu = \|T\|,$$

$$y_0^* T\left(\frac{\chi_{E_0-E}}{\mu(E_0 - E)} \text{sgn } f_0\right) = \int_{E_0} \frac{\chi_{E_0-E}}{\mu(E_0 - E)} \|T\| d\mu = \|T\|,$$

and

$$y_0^* T\left(\frac{\chi_{E_0}}{\mu(E_0)} \text{sgn } f_0\right) = \int_{E_0} \frac{\chi_{E_0}}{\mu(E_0)} \|T\| d\mu = \|T\|.$$

From these equalities, one obtains

$$\begin{aligned} \|T\| \mu(E_0) &= \|T(\chi_{E_0} \operatorname{sgn} f_0)\| = \|T(\chi_E \operatorname{sgn} f_0) + T(\chi_{E_0-E} \operatorname{sgn} f_0)\| \\ &\leq \|T(\chi_E \operatorname{sgn} f_0)\| + \|T(\chi_{E_0-E} \operatorname{sgn} f_0)\| \\ &= \|T\| \mu(E) + \|T\| \mu(E_0 - E) = \|T\| \mu(E_0). \end{aligned}$$

This combined with the fact that Y is strictly convex shows that $T(\chi_E \operatorname{sgn} f_0)$ and $T(\chi_{E_0-E} \operatorname{sgn} f_0)$ are multiples of each other. Since $T(\chi_{E_0} \operatorname{sgn} f_0) = T(\chi_E \operatorname{sgn} f_0) + T(\chi_{E_0-E} \operatorname{sgn} f_0)$, $T(\chi_E \operatorname{sgn} f_0)$ is a scalar multiple of $T(\chi_{E_0} \operatorname{sgn} f_0)$; i.e., $T(\chi_E \operatorname{sgn} f_0) = \gamma T(\chi_{E_0} \operatorname{sgn} f_0)$ for some scalar γ . On the other hand

$$\|T\| \mu(E) = y_0^* T(\chi_E \operatorname{sgn} f_0) = \gamma y_0^* T(\chi_{E_0} \operatorname{sgn} f_0) = \gamma \|T\| \mu(E_0);$$

thus $\gamma = \mu(E)/\mu(E_0)$. Therefore if $E \subset E_0$ and $\mu(E) > 0$,

$$\frac{T(\chi_E \operatorname{sgn} f_0)}{\mu(E)} = \frac{T(\chi_{E_0} \operatorname{sgn} f_0)}{\mu(E_0)} = y_0.$$

Now suppose $f \in L^1(\mu)$ is a simple function. Let $\varepsilon > 0$ and choose a simple function $\varphi \in L^1(\mu)$ such that $\|\overline{\operatorname{sgn} f_0} - \varphi\|_\infty < \varepsilon$. (Here $\overline{\operatorname{sgn} f_0}$ is the complex conjugate of $\operatorname{sgn} f_0$.) Then $T(f) = T(f \overline{\operatorname{sgn} f_0} \operatorname{sgn} f_0)$ and $\|T(f) - T(f\varphi \operatorname{sgn} f_0)\| \leq \|T\| \|\overline{\operatorname{sgn} f_0} \operatorname{sgn} f_0 - \varphi \operatorname{sgn} f_0\|_1 < \varepsilon \|T\| \mu\Omega$. Now select sets $A_1, \dots, A_n \in \Sigma$ such that

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \quad \text{and} \quad \varphi = \sum_{i=1}^n \beta_i \chi_{A_i}.$$

Then

$$\begin{aligned} T(f\varphi \operatorname{sgn} f_0 \chi_{E_0}) &= \sum_{i=1}^n \alpha_i \beta_i \frac{T(\chi_{A_i} \cap E_0 \operatorname{sgn} f_0)}{\mu(A_i \cap E_0)} \mu(A_i \cap E_0) \\ &= \sum_{i=1}^n \alpha_i \beta_i \mu(A_i \cap E_0) y_0 = \int_{E_0} f \varphi d\mu y_0. \end{aligned}$$

Letting ε go to zero reveals that

$$T(f\chi_{E_0}) = \int_{E_0} f \overline{\operatorname{sgn} f_0} d\mu y_0.$$

Since simple functions are dense in $L^1(\mu)$, the equality

$$T(f\chi_{E_0}) = \int_{E_0} f \overline{\operatorname{sgn} f_0} d\mu y_0$$

obtains for all $f \in L^1(\mu)$. This proves the first statement.

To prove the second statement, note that if Y is real, then $\operatorname{sgn} f_0$ takes on only the values $+1$ or -1 . If $\operatorname{sgn} f_0 = 1$ on a set of positive measure E , in the support of f_0 , take $E_0 = E$ and proceed

as above. If $\text{sgn } f_0 = -1$ almost everywhere in the support of f_0 , multiply f_0 and y_0^* by -1 and proceed as in the last sentence.

With the help of Lemma 2, the main result becomes nothing but a straightforward exhaustion argument.

THEOREM 3. *Let Y be a strictly convex Banach space. If the norm attaining members of $B(L^1[0, 1], Y)$ are dense in $B(L^1[0, 1], Y)$, then Y has the Radon-Nikodým property.*

Proof. Let $T \in B(L^1[0, 1], Y)$ and $\varepsilon > 0$ be given. Define a class of Lebesgue measurable sets \mathcal{M} by agreeing that $E \in \mathcal{M}$ if there exists an essentially bounded Bochner integrable $g(=g(E, \varepsilon)): [0, 1] \rightarrow Y$ such that

$$\left\| T(f\chi_E) - \int_E fg d\mu \right\| \leq \varepsilon \|f\chi_E\|_1.$$

Note that if A is Lebesgue measurable and $A \subset E \in \mathcal{M}$ then

$$\begin{aligned} \left\| T(f\chi_A) - \int_A fg((E, \varepsilon)d\mu) \right\| &= \left\| T((f\chi_A)\chi_E) - \int_E (f\chi_A)gd\mu \right\| \\ &\leq \|f\chi_A\chi_E\|_1 = \varepsilon \|f\chi_A\|_1. \end{aligned}$$

Therefore, if $E \in \mathcal{M}$, every measurable subset of E belongs to \mathcal{M} . Now let $\alpha = \sup \{\mu(E) : E \in \mathcal{M}\}$ and let $(E_n) \subset \mathcal{M}$ be a sequence such that $\lim_n \mu(E_n) = \alpha$. Write $A_1 = E_1$, $A_2 = E_2 - E_1$, \dots , $A_n = E_n - \bigcup_{i=1}^{n-1} E_i$. Then the A_n 's are disjoint, $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty E_n$ and $\mu(\bigcup_{n=1}^\infty A_n) \geq \alpha$. $A_n \subset E_n$ and $E_n \in \mathcal{M}$, $A_n \in \mathcal{M}$ and there exists a sequence of essentially bounded functions $g_n: [0, 1] \rightarrow Y$, $n = 1, 2, \dots$, such that for all $f \in L^1[0, 1]$,

$$\left\| T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu \right\| \leq \varepsilon \|f\chi_{A_n}\|_1.$$

Accordingly,

$$\left\| \int_{A_n} fg_n d\mu \right\| \leq \|T(f\chi_{A_n})\| + \varepsilon \|f\chi_{A_n}\|_1 \leq (\|T\| + \varepsilon) \|f\|_1.$$

By Lemma A,

$$\text{ess sup } \|g_n\chi_{A_n}\| - \sup_{\|f\|_1 \leq 1} \left\| \int_{A_n} fg_n d\mu \right\| \leq \|T\| + \varepsilon.$$

Therefore $\sup_n \text{ess sup } \|g_n\| \leq \|T\| + \varepsilon$. Now define $g: [0, 1] \rightarrow Y$ by

$$g(t) = \begin{cases} g_n(t) & \text{for } t \in A_n \\ \bar{0} & \text{for } t \notin \bigcup_{n=1}^\infty A_n. \end{cases}$$

Then $\text{ess sup } \|g\| \leq \|T\| + \varepsilon$ and if $f \in L^1[0, 1]$,

$$\begin{aligned} & \left\| T(f\chi_{\bigcup_n A_n}) - \int_{\bigcup_n A_n} fg d\mu \right\| \\ & \leq \sum_{n=1}^{\infty} \left\| T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu \right\| \\ & \leq \sum_{n=1}^{\infty} \varepsilon \|f\chi_{A_n}\|_1 \leq \varepsilon \|f\|_1. \end{aligned}$$

Therefore $\bigcup_n A_n \in \mathcal{M}$. Next we shall see that $\mu(\bigcup_n A_n) = 1$. For, if $\mu(\bigcup_n A_n) < 1$, then $\mu(\bigcup_n E_n) \leq 1$ and $\alpha < 1$. Set $B_0 = [0, 1] - \bigcup_n A_n$ and recall that $L^1(B_0)$ (Lebesgue integrable functions supported on B_0) is isometric to $L^1[0, 1]$. Define $T_1: L^1(B_0) \rightarrow Y$ by $T_1(f) = T(f\chi_{B_0})$ for $f \in L^1(B_0)$. Since $L^1(B_0)$ is isometric to $L^1[0, 1]$, there exists an operator $T_2: L^1(B_0) \rightarrow Y$ that attains its norm such that $\|T_1 - T_2\| \leq \varepsilon$.

An appeal to Lemma 2 produces a $y_1 \in Y$ and set $B_1 \subset B_0$ with $\mu(B_1) > 0$ such that

$$T_2(f) = \int_{B_1} f d\mu y_1$$

for all $f \in L^1(B_0)$. Set $g' = y_1\chi_{B_1}$. Then

$$\begin{aligned} & \left\| T(f\chi_{B_1}) - \int_{B_1} fg' d\mu \right\| = \|T_1(f\chi_{B_1}) - T_2(f\chi_{B_1})\| \\ & \leq \|T_1 - T_2\| \|f\chi_{B_1}\|_1 \leq \varepsilon \|f\chi_{B_1}\|_1. \end{aligned}$$

Therefore $B_1 \in \mathcal{M}$. Now set $\tilde{g} = g + g'$. If $f \in L^1([0, 1])$,

$$\begin{aligned} & \left\| T(f\chi_{\bigcup_{n=1}^{\infty} A_n \cup B_1}) - \int_{\bigcup_{n=1}^{\infty} A_n \cup B_1} f\tilde{g} d\mu \right\| \\ & \leq \sum_{n=1}^{\infty} \left\| T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu \right\| + \left\| T(f\chi_{B_1}) - \int_{B_1} fg' d\mu \right\| \\ & \leq \varepsilon \sum_{n=1}^{\infty} \|f\chi_{A_n}\|_1 + \varepsilon \|f\chi_{B_1}\|_1 = \|f\chi_{\bigcup_{n=1}^{\infty} A_n \cup B_1}\|_1. \end{aligned}$$

Therefore $\bigcup_n A_n \cup B_1 = \bigcup_n E_n \cup B_1 \in \mathcal{M}$. But

$$\begin{aligned} \mu\left(\bigcup_n E_n \cup B_1\right) &= \mu\left(\bigcup_n E_n\right) + \mu(B_1) \\ &\geq \lim_n \mu(E_n) + \mu(B_1) = \alpha + \mu(B_1) > \alpha \end{aligned}$$

contradicting the definition of α . Thus $\mu(\bigcup_n A_n) = 1$ and

$$\left\| T(f) - \int_{[0,1]} fg d\mu \right\| \leq \varepsilon \|f\|_1 \text{ for all } f \in L^1[0, 1].$$

Finally, to check that Y has the Radon-Nikodým property, let

$g_n: [0, 1] \rightarrow Y$ be a sequence of Bochner integrable essentially bounded functions such that for all $f \in L^1[0, 1]$

$$\left\| T(f) - \int_{[0,1]} fg_n d\mu \right\| \leq 1/n \|f\|_1$$

for all n . An appeal to Lemma 1 shows that $\lim_{n,m} \text{ess sup} \|g_n - g_m\|_1 = 0$. Hence there exists a Bochner integrable essentially bounded $g: [0, 1] \rightarrow Y$ with $\lim_n \text{ess sup} \|g_n - g\| = 0$. If $f \in L^1[0, 1]$, the dominated convergence theorem guarantees that

$$T(f) - \lim_n \int_{[0,1]} fg_n d\mu = \int_{[0,1]} fg d\mu .$$

Thus Y has the Radon-Nikodým property by Lemma B.

The role of strict convexity seems to be crucial in Theorem 3: for by perturbing co-ordinate functions it is seen easily that norm attaining operators are dense in $B(L^1[0, 1], c_0)$, $B(L^1[0, 1], l^\infty)$ or for that matter $B(X, l^\infty)$ for any Banach space X . See [8, Prop. 3].

On the other hand, the role of strict convexity could be made even more palatable by an affirmative answer to an old question of Diestel's: Does every Banach space with the Radon-Nikodým property have an equivalent strictly convex norm?

COROLLARY 4. *If X is a strictly convex renorming of $L^1[0, 1]$, then the norm attaining operators are not dense in $B(L^1[0, 1], X)$.*

Proof. Evidently X lacks the Radon-Nikodým property.

This leaves unsolved the question of whether the norm attaining operators are dense in $B(L^1[0, 1], L^1[0, 1])$.

Finally say that a Banach space X has property B if for every Banach space Y the norm attaining operators are dense in $B(Y, X)$. Lindenstrauss [8, Proposition 4] has observed that if there is a non-compact operator in $B(c_0, X)$ and X is strictly convex, then X lacks property B . It is not difficult to see that if X has the Radon-Nikodým property, then every operator in $B(c_0, X)$ is compact and that the converse is false. Thus Theorem 3 is a better test for Property B than [8, Proposition 4]. Of course this brings up a question that is well beyond the scope of this note. If X is a strictly convex Banach space, does X have property B if and only if X has the Radon-Nikodým property?

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Vol. 63, No. 1

March, 1976

Ralph Artino, <i>Gevrey classes and hypoelliptic boundary value problems</i>	1
B. Aupetit, <i>Caractérisation spectrale des algèbres de Banach commutatives</i>	23
Leon Bernstein, <i>Fundamental units and cycles in the period of real quadratic number fields. I</i>	37
Leon Bernstein, <i>Fundamental units and cycles in the period of real quadratic number fields. II</i>	63
Robert F. Brown, <i>Fixed points of automorphisms of compact Lie groups</i>	79
Thomas Ashland Chapman, <i>Concordances of noncompact Hilbert cube manifolds</i>	89
William C. Connett, V and Alan Schwartz, <i>Weak type multipliers for Hankel transforms</i>	125
John Wayne Davenport, <i>Multipliers on a Banach algebra with a bounded approximate identity</i>	131
Gustave Adam Efroymsen, <i>Substitution in Nash functions</i>	137
John Sollion Hsia, <i>Representations by spinor genera</i>	147
William George Kitto and Daniel Eliot Wulbert, <i>Korovkin approximations in L_p-spaces</i>	153
Eric P. Kronstadt, <i>Interpolating sequences for functions satisfying a Lipschitz condition</i>	169
Gary Douglas Jones and Samuel Murray Rankin, III, <i>Oscillation properties of certain self-adjoint differential equations of the fourth order</i>	179
Takaši Kusano and Hiroshi Onose, <i>Nonoscillation theorems for differential equations with deviating argument</i>	185
David C. Lantz, <i>Preservation of local properties and chain conditions in commutative group rings</i>	193
Charles W. Neville, <i>Banach spaces with a restricted Hahn-Banach extension property</i>	201
Norman Oler, <i>Spaces of discrete subsets of a locally compact group</i>	213
Robert Olin, <i>Functional relationships between a subnormal operator and its minimal normal extension</i>	221
Thomas Thornton Read, <i>Bounds and quantitative comparison theorems for nonoscillatory second order differential equations</i>	231
Robert Horace Redfield, <i>Archimedean and basic elements in completely distributive lattice-ordered groups</i>	247
Jeffery William Sanders, <i>Weighted Sidon sets</i>	255
Aaron R. Todd, <i>Continuous linear images of pseudo-complete linear topological spaces</i>	281
J. Jerry Uhl, Jr., <i>Norm attaining operators on $L^1[0, 1]$ and the Radon-Nikodým property</i>	293
William Jennings Wickless, <i>Abelian groups in which every endomorphism is a left multiplication</i>	301