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**NORM ATTAINING OPERATORS ON  $L^1[0, 1]$  AND THE  
RADON-NIKODÝM PROPERTY**

J. JERRY UHL, JR.

## NORM ATTAINING OPERATORS ON $L^1[0, 1]$ AND THE RADON-NIKODÝM PROPERTY

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**Let  $Y$  be a strictly convex Banach space. Then norm attaining operators mapping  $L^1[0, 1]$  to  $Y$  are dense in the space of all linear operators from  $L^1[0, 1]$  to  $Y$  if and only if  $Y$  has the Radon-Nikodým property.**

Bishop and Phelps [1] have asked the general question—For which Banach spaces  $X$  and  $Y$  is the collection of norm attaining operators from  $X$  to  $Y$  dense in the space  $B(X, Y)$  of all bounded (linear) operators from  $X$  to  $Y$ . Lindenstrauss in [8] investigated this question and related this question to existence of extreme points and exposed points in the closed unit ball of  $X$ . In the course of his paper Lindenstrauss showed that for some space  $Y$  the norm attaining operators in  $B(L^1[0, 1], Y)$  are not dense in  $B(L^1[0, 1], Y)$  due to the lack of extreme points in the closed unit ball of  $L^1[0, 1]$ . Left open is the following question: For which Banach spaces  $Y$  are the norm attaining operators dense in  $B(L^1[0, 1], Y)$ ? Based on Lindenstrauss's work, one is led to believe that if the closed unit ball of  $Y$  has a rich extreme point or exposed point structure, then the norm attaining operators may be dense in  $B(L^1[0, 1], Y)$ . On the other hand the Radon-Nikodým property is intimately connected with extreme point structure (Rieffel [12], Maynard [10], Huff [6], Davis and Phelps [2], Phelps [11], Huff and Morris [7]). So there is some *prima facie* evidence to support the belief that the norm attaining operators are dense in  $B(L^1[0, 1], Y)$  if and only if  $Y$  has the Radon-Nikodým property. The purpose of this paper is to verify this for strictly convex Banach spaces  $Y$ .

First a few well known results will be collected.

LEMMA A [4, 5]. *If  $(\Omega, \Sigma, \mu)$  is a finite measure space and  $g: \Omega \rightarrow Y$  is  $\mu$ -essentially bounded Bochner integrable function, then*

$$T(f) = \text{Bochner} - \int fg d\mu$$

*defines a member  $T$  of  $B(L^1(\mu), Y)$  with  $\|T\| = \text{ess sup} \|g\|_Y$ .*

LEMMA B [3]. *Any one of the following statements about  $Y$  implies all the others.*

- (i)  *$Y$  has the Radon-Nikodým property.*
- (ii) *If  $(\Omega, \Sigma, \mu)$  is a finite measure space and  $G: \Sigma \rightarrow Y$  is a*

$\mu$ -continuous countably additive measure of bounded variation, then there exists a  $\mu$ -Bochner integrable

$$g: \Omega \longrightarrow Y \text{ with } G(E) = \int_E g d\mu \text{ for all } E \in \Sigma .$$

(iii) If  $\mu$  is Lebesgue measure on  $[0, 1]$ , then for each  $T \in B(L^1[0, 1], Y)$  there is a  $\mu$ -essentially bounded  $g: [0, 1] \rightarrow Y$  with

$$T(f) = \int_{[0,1]} f g d\mu \text{ for all } f \in L^1([0, 1], Y)$$

Moreover, if  $Y$  has the Radon-Nikodým property statement (iii) is true for any finite measure space.

The first theorem is a straight forward observation that is based on the definition of a measurable function.

**THEOREM 1.** *If  $Y$  has the Radon-Nikodým property and if  $(\Omega, \Sigma, \mu)$  is a finite measure space, then the norm attaining operators are dense in  $B(L^1(\mu), Y)$ .*

*Proof.* Let  $T \in B(L^1(\mu), Y)$  and  $\varepsilon > 0$ . Then there exists an essentially bounded Bochner integrable  $g: \Omega \rightarrow Y$  such that  $T(f) = \int_{\Omega} f g d\mu$  for all  $f \in L^1(\mu)$  and there exists a countably valued function

$$h: \Omega \longrightarrow X, \quad h = \sum_{i=1}^{\infty} x_i \chi_{E_i}, \quad x_i \in X,$$

$$E_i \in \Sigma, \quad \mu(E_i) > 0, \quad E_i \cap E_j = \emptyset$$

for  $i \neq j$ , such that  $\text{ess sup } \|g - h\| < \varepsilon/2$ . Define  $T_1: L^1(\mu) \rightarrow Y$  by  $T_1(f) = \int_{\Omega} f h d\mu, f \in L^1(\mu)$ . Then  $\|T - T_1\| < (\varepsilon/2)$ .

Now  $T_1$  will be approximated within  $\varepsilon/2$  by an operator which attains its norm. If  $T_1 = 0$ , there is nothing to prove. Otherwise  $\beta = \sup \|y_i\| > 0$ . Choose  $i_0$  such that  $\beta - \|y_{i_0}\| < \varepsilon/2$  and  $\alpha > 1$  such that  $\varepsilon/4 < (\alpha - 1) \|y_{i_0}\| < \varepsilon/2$  and define

$$T_2(f) = \int_{\bigcup_{i \neq i_0} E_i} f h d\mu + \alpha y_{i_0} \int_{E_{i_0}} f d\mu .$$

It is easy to verify that  $\|T_1 - T_2\| < \varepsilon/2$  and that  $\|T_2\| = \alpha \|y_{i_0}\| = \|T_2(x_{E_{i_0}}/\mu(E_{i_0}))\|$ . Hence  $T_2$  attains its norm and  $\|T - T_2\| < \varepsilon$ , as required.

The operator  $T_2$  constructed in the proof of Theorem 1 has two important properties. First it attains its norm and second there

exists  $E \in \Sigma$ ,  $\mu(E) > 0$  and  $y_0 \in Y$  with  $\|y_0\| = \|T\|$  and  $T_2(f\chi_E) = \int_E f d\mu y_0$  for all  $f \in L^1(\mu)$ . If  $Y$  is strictly convex and real, this property is shared by all norm attaining operators in  $B(L^1(\mu), Y)$ .

LEMMA 2. *Let  $(\Omega, \Sigma, \mu)$  be a finite measure space and  $Y$  be a strictly convex Banach space. If  $T \in B(L^1(\mu), Y)$  attains its norm then there exists a set  $E_0 \in \Sigma$  with  $\mu(E_0) > 0$ ,  $g \in L^\infty(\mu)$  with  $|g| = 1$  on  $E_0$ , and  $y_0 \in Y$  with  $\|y_0\| = \|T\|$  such that*

$$T(f\chi_{E_0}) = \int_{E_0} fgd\mu y_0$$

for all  $f \in L^1(\mu)$ .

If  $Y$  is a real Banach space,  $g$  may be taken as the constant function 1.

*Proof.* If  $\|T\| = 0$ , there is nothing to prove.

Otherwise, choose  $f_0 \in L^1(\mu)$  with  $\|T(f_0)\| = \|T\|$  and  $\|f_0\| = 1$ . With the help of the Hahn-Banach theorem, choose  $y_0^* \in Y^*$  with  $\|y_0^*\| = 1$  and

$$y_0^* T(f_0) = \|T(f_0)\| = \|T\|.$$

Next choose  $h \in L^\infty(\mu)$  with  $\|h\|_\infty = \|T\|$  such that

$$y_0^* T(f) = \int_\Omega fhd\mu$$

for all  $f \in L^1(\mu)$ . A simple computation reveals that  $h = \overline{\text{sgn}} f_0 \|T\|$  on the support of  $f_0$ . (Here  $\text{sgn} f_0 = f_0/|f_0|$ .) Let  $E_0$  be the support of  $f_0$ . Thus if  $f \in L^1(\mu)$ ,

$$y_0^* T(f\chi_{E_0}) = \int_{E_0} f \overline{\text{sgn}} f_0 \|T\| d\mu.$$

Next suppose  $E \subset E_0$ ,  $E \in \Sigma$  and  $\mu(E)$ ,  $\mu(E_0 - E) > 0$ . (The rest of the proof is trivial if  $E_0$  is an atom of  $\mu$ .) Then

$$y_0^* T\left(\frac{\chi_E}{\mu(E)} \text{sgn} f_0\right) = \int_{E_0} \frac{\chi_E}{\mu(E)} \|T\| d\mu = \|T\|,$$

$$y_0^* T\left(\frac{\chi_{E_0-E}}{\mu(E_0 - E)} \text{sgn} f_0\right) = \int_{E_0} \frac{\chi_{E_0-E}}{\mu(E_0 - E)} \|T\| d\mu = \|T\|,$$

and

$$y_0^* T\left(\frac{\chi_{E_0}}{\mu(E_0)} \text{sgn} f_0\right) = \int_{E_0} \frac{\chi_{E_0}}{\mu(E_0)} \|T\| d\mu = \|T\|.$$

From these equalities, one obtains

$$\begin{aligned} \|T\| \mu(E_0) &= \|T(\chi_{E_0} \operatorname{sgn} f_0)\| = \|T(\chi_E \operatorname{sgn} f_0) + T(\chi_{E_0-E} \operatorname{sgn} f_0)\| \\ &\leq \|T(\chi_{E_1} \operatorname{sgn} f_0)\| + \|T(\chi_{E_0-E} \operatorname{sgn} f_0)\| \\ &= \|T\| \mu(E) + \|T\| \mu(E_0 - E) = \|T\| \mu(E_0). \end{aligned}$$

This combined with the fact that  $Y$  is strictly convex shows that  $T(\chi_E \operatorname{sgn} f_0)$  and  $T(\chi_{E_0-E} \operatorname{sgn} f_0)$  are multiples of each other. Since  $T(\chi_{E_0} \operatorname{sgn} f_0) = T(\chi_E \operatorname{sgn} f_0) + T(\chi_{E_0-E} \operatorname{sgn} f_0)$ ,  $T(\chi_E \operatorname{sgn} f_0)$  is a scalar multiple of  $T(\chi_{E_0} \operatorname{sgn} f_0)$ ; i.e.,  $T(\chi_E \operatorname{sgn} f_0) = \gamma T(\chi_{E_0} \operatorname{sgn} f_0)$  for some scalar  $\gamma$ . On the other hand

$$\|T\| \mu(E) = y_0^* T(\chi_E \operatorname{sgn} f_0) = \gamma y_0^* (\chi_{E_0} \operatorname{sgn} f_0) = \gamma \|T\| \mu(E_0);$$

thus  $\gamma = \mu(E)/\mu(E_0)$ . Therefore if  $E \subset E_0$  and  $\mu(E) > 0$ ,

$$\frac{T(\chi_E \operatorname{sgn} f_0)}{\mu(E)} = \frac{T(\chi_{E_0} \operatorname{sgn} f_0)}{\mu(E_0)} = y_0.$$

Now suppose  $f \in L^1(\mu)$  is a simple function. Let  $\varepsilon > 0$  and choose a simple function  $\varphi \in L^1(\mu)$  such that  $\|\overline{\operatorname{sgn} f_0} - \varphi\|_\infty < \varepsilon$ . (Here  $\overline{\operatorname{sgn} f_0}$  is the complex conjugate of  $\operatorname{sgn} f_0$ .) Then  $T(f) = T(f \overline{\operatorname{sgn} f_0} \operatorname{sgn} f_0)$  and  $\|T(f) - T(f\varphi \operatorname{sgn} f_0)\| \leq \|T\| \|\overline{\operatorname{sgn} f_0} \operatorname{sgn} f_0 - \varphi \operatorname{sgn} f_0\|_1 < \varepsilon \|T\| \mu\Omega$ . Now select sets  $A_1, \dots, A_n \in \mathcal{S}$  such that

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \quad \text{and} \quad \varphi = \sum_{i=1}^n \beta_i \chi_{A_i}.$$

Then

$$\begin{aligned} T(f\varphi \operatorname{sgn} f_0 \chi_{E_0}) &= \sum_{i=1}^n \alpha_i \beta_i \frac{T(\chi_{A_i} \cap E_0 \operatorname{sgn} f_0)}{\mu(A_i \cap E_0)} \mu(A_i \cap E_0) \\ &= \sum_{i=1}^n \alpha_i \beta_i \mu(A_i \cap E_0) y_0 = \int_{E_0} f\varphi d\mu y_0. \end{aligned}$$

Letting  $\varepsilon$  go to zero reveals that

$$T(f\chi_{E_0}) = \int_{E_0} f \overline{\operatorname{sgn} f_0} d\mu y_0.$$

Since simple functions are dense in  $L^1(\mu)$ , the equality

$$T(f\chi_{E_0}) = \int_{E_0} f \overline{\operatorname{sgn} f_0} d\mu y_0$$

obtains for all  $f \in L^1(\mu)$ . This proves the first statement.

To prove the second statement, note that if  $Y$  is real, then  $\operatorname{sgn} f_0$  takes on only the values  $+1$  or  $-1$ . If  $\operatorname{sgn} f_0 = 1$  on a set of positive measure  $E$ , in the support of  $f_0$ , take  $E_0 = E$  and proceed

as above. If  $\text{sgn } f_0 = -1$  almost everywhere in the support of  $f_0$ , multiply  $f_0$  and  $y_0^*$  by  $-1$  and proceed as in the last sentence.

With the help of Lemma 2, the main result becomes nothing but a straightforward exhaustion argument.

**THEOREM 3.** *Let  $Y$  be a strictly convex Banach space. If the norm attaining members of  $B(L^1[0, 1], Y)$  are dense in  $B(L^1[0, 1], Y)$ , then  $Y$  has the Radon-Nikodým property.*

*Proof.* Let  $T \in B(L^1[0, 1], Y)$  and  $\varepsilon > 0$  be given. Define a class of Lebesgue measurable sets  $\mathcal{M}$  by agreeing that  $E \in \mathcal{M}$  if there exists an essentially bounded Bochner integrable  $g (= g(E, \varepsilon)): [0, 1] \rightarrow Y$  such that

$$\left\| T(f\chi_E) - \int_E fg d\mu \right\| \leq \varepsilon \|f\chi_E\|_1.$$

Note that if  $A$  is Lebesgue measurable and  $A \subset E \in \mathcal{M}$  then

$$\begin{aligned} \left\| T(f\chi_A) - \int_A fg((E, \varepsilon)d\mu) \right\| &= \left\| T((f\chi_A)\chi_E) - \int_E (f\chi_A)gd\mu \right\| \\ &\leq \|f\chi_A\chi_E\|_1 = \varepsilon \|f\chi_A\|_1. \end{aligned}$$

Therefore, if  $E \in \mathcal{M}$ , every measurable subset of  $E$  belongs to  $\mathcal{M}$ . Now let  $\alpha = \sup \{\mu(E) : E \in \mathcal{M}\}$  and let  $(E_n) \subset \mathcal{M}$  be a sequence such that  $\lim_n \mu(E_n) = \alpha$ . Write  $A_1 = E_1, A_2 = E_2 - E_1, \dots, A_n = E_n - \bigcup_{i=1}^{n-1} E_i$ . Then the  $A_n$ 's are disjoint,  $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty E_n$  and  $\mu(\bigcup_{n=1}^\infty A_n) \geq \alpha$ .  $A_n \subset E_n$  and  $E_n \in \mathcal{M}, A_n \in \mathcal{M}$  and there exists a sequence of essentially bounded functions  $g_n: [0, 1] \rightarrow Y, n = 1, 2, \dots$ , such that for all  $f \in L^1[0, 1]$ ,

$$\left\| T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu \right\| \leq \varepsilon \|f\chi_{A_n}\|_1.$$

Accordingly,

$$\left\| \int_{A_n} fg_n d\mu \right\| \leq \|T(f\chi_{A_n})\| + \varepsilon \|f\chi_{A_n}\|_1 \leq (\|T\| + \varepsilon) \|f\|_1.$$

By Lemma A,

$$\text{ess sup } \|g_n\chi_{A_n}\| - \sup_{\|f\|_1 \leq 1} \left\| \int_{A_n} fg_n d\mu \right\| \leq \|T\| + \varepsilon.$$

Therefore  $\sup_n \text{ess sup } \|g_n\| \leq \|T\| + \varepsilon$ . Now define  $g: [0, 1] \rightarrow Y$  by

$$g(t) = \begin{cases} g_n(t) & \text{for } t \in A_n \\ \hat{0} & \text{for } t \notin \bigcup_{n=1}^\infty A_n. \end{cases}$$

Then  $\text{ess sup } \|g\| \leq \|T\| + \varepsilon$  and if  $f \in L^1[0, 1]$ ,

$$\begin{aligned} & \left\| T(f\chi_{\bigcup_n A_n}) - \int_{\bigcup_n A_n} fg d\mu \right\| \\ & \leq \sum_{n=1}^{\infty} \left\| T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu \right\| \\ & \leq \sum_{n=1}^{\infty} \varepsilon \|f\chi_{A_n}\|_1 \leq \varepsilon \|f\|_1. \end{aligned}$$

Therefore  $\bigcup_n A_n \in \mathcal{M}$ . Next we shall see that  $\mu(\bigcup_n A_n) = 1$ . For, if  $\mu(\bigcup_n A_n) < 1$ , then  $\mu(\bigcup_n E_n) \leq 1$  and  $\alpha < 1$ . Set  $B_0 = [0, 1] - \bigcup_m A_m$  and recall that  $L^1(B_0)$  (Lebesgue integrable functions supported on  $B_0$ ) is isometric to  $L^1[0, 1]$ . Define  $T_1: L^1(B_0) \rightarrow Y$  by  $T_1(f) = T(f\chi_{B_0})$  for  $f \in L^1(B_0)$ . Since  $L^1(B_0)$  is isometric to  $L^1[0, 1]$ , there exists an operator  $T_2: L^1(B_0) \rightarrow Y$  that attains its norm such that  $\|T_1 - T_2\| \leq \varepsilon$ .

An appeal to Lemma 2 produces a  $y_1 \in Y$  and set  $B_1 \subset B_0$  with  $\mu(B_1) > 0$  such that

$$T_2(f) = \int_{B_1} fd\mu y_1$$

for all  $f \in L^1(B_0)$ . Set  $g' = y_1\chi_{B_1}$ . Then

$$\begin{aligned} & \left\| T(f\chi_{B_1}) - \int_{B_1} fg' d\mu \right\| = \|T_1(f\chi_{B_1}) - T_2(f\chi_{B_1})\| \\ & \leq \|T_1 - T_2\| \|f\chi_{B_1}\|_1 \leq \varepsilon \|f\chi_{B_1}\|_1. \end{aligned}$$

Therefore  $B_1 \in \mathcal{M}$ . Now set  $\tilde{g} = g + g'$ . If  $f \in L^1([0, 1])$ ,

$$\begin{aligned} & \left\| T(f\chi_{\bigcup_{n=1}^{\infty} A_n \cup B_1}) - \int_{\bigcup_{n=1}^{\infty} A_n \cup B_1} f\tilde{g} d\mu \right\| \\ & \leq \sum_{n=1}^{\infty} \left\| T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu \right\| + \left\| T(f\chi_{B_1}) - \int_{B_1} fg' d\mu \right\| \\ & \leq \varepsilon \sum_{n=1}^{\infty} \|f\chi_{A_n}\|_1 + \varepsilon \|f\chi_{B_1}\|_1 = \|f\chi_{\bigcup_{n=1}^{\infty} A_n \cup B_1}\|_1. \end{aligned}$$

Therefore  $\bigcup_n A_n \cup B_1 = \bigcup_n E_n \cup B_1 \in \mathcal{M}$ . But

$$\begin{aligned} \mu\left(\bigcup_n E_n \cup B_1\right) &= \mu\left(\bigcup_n E_n\right) + \mu(B_1) \\ &\geq \lim_n \mu(E_n) + \mu(B_1) = \alpha + \mu(B_1) > \alpha \end{aligned}$$

contradicting the definition of  $\alpha$ . Thus  $\mu(\bigcup_n A_n) = 1$  and

$$\left\| T(f) - \int_{[0,1]} fg d\mu \right\| \leq \varepsilon \|f\|_1 \text{ for all } f \in L^1[0, 1].$$

Finally, to check that  $Y$  has the Radon-Nikodým property, let

$g_n: [0, 1] \rightarrow Y$  be a sequence of Bochner integrable essentially bounded functions such that for all  $f \in L^1[0, 1]$

$$\left\| T(f) - \int_{[0,1]} f g_n d\mu \right\| \leq 1/n \|f\|_1$$

for all  $n$ . An appeal to Lemma 1 shows that  $\lim_{n,m} \text{ess sup} \|g_n - g_m\|_1 = 0$ . Hence there exists a Bochner integrable essentially bounded  $g: [0, 1] \rightarrow Y$  with  $\lim_n \text{ess sup} \|g_n - g\| = 0$ . If  $f \in L^1[0, 1]$ , the dominated convergence theorem guarantees that

$$T(f) - \lim_n \int_{[0,1]} f g_n d\mu = \int_{[0,1]} f g d\mu.$$

Thus  $Y$  has the Radon-Nikodým property by Lemma B.

The role of strict convexity seems to be crucial in Theorem 3: for by perturbing co-ordinate functions it is seen easily that norm attaining operators are dense in  $B(L^1[0, 1], c_0)$ ,  $B(L^1[0, 1], l^\infty)$  or for that matter  $B(X, l^\infty)$  for any Banach space  $X$ . See [8, Prop. 3].

On the other hand, the role of strict convexity could be made even more palatable by an affirmative answer to an old question of Diestel's: Does every Banach space with the Radon-Nikodým property have an equivalent strictly convex norm?

**COROLLARY 4.** *If  $X$  is a strictly convex renorming of  $L^1[0, 1]$ , then the norm attaining operators are not dense in  $B(L^1[0, 1], X)$ .*

*Proof.* Evidently  $X$  lacks the Radon-Nikodým property.

This leaves unsolved the question of whether the norm attaining operators are dense in  $B(L^1[0, 1], L^1[0, 1])$ .

Finally say that a Banach space  $X$  has property  $B$  if for every Banach space  $Y$  the norm attaining operators are dense in  $B(Y, X)$ . Lindenstrauss [8, Proposition 4] has observed that if there is a non-compact operator in  $B(c_0, X)$  and  $X$  is strictly convex, then  $X$  lacks property  $B$ . It is not difficult to see that if  $X$  has the Radon-Nikodým property, then every operator in  $B(c_0, X)$  is compact and that the converse is false. Thus Theorem 3 is a better test for Property  $B$  than [8, Proposition 4]. Of course this brings up a question that is well beyond the scope of this note. If  $X$  is a strictly convex Banach space, does  $X$  have property  $B$  if and only if  $X$  has the Radon-Nikodým property?

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