ON EXTENDING HIGHER DERIVATIONS GENERATED BY CUP PRODUCTS TO THE INTEGRAL CLOSURE

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Let $A = k[x_1, \ldots, x_d]$ be a finitely generated integral domain over a field $k$ of characteristic zero. Let $\bar{A}$ denote the integral closure of $A$ in its quotient field. A well known result due to A. Seidenberg says that any first order $k$-derivation of $A$ can be extended to $\bar{A}$. This result is known to be false for higher order derivations. In this paper, the authors investigate what types of higher derivations on $A$ can be extended to $\bar{A}$. The main results are for higher derivations which are cup products. Set $\text{Der}^1(A) = \text{Der}^1(A)_0$ and inductively define $\text{Der}^n(A)_0$ as follows:

$$\text{Der}^n(A)_0 = \{\phi \in \text{Der}^n(A) | \Delta \phi \in \sum_{i=1}^{n-1} \text{Der}^i(A)_0 \cup \text{Der}^{n-i}(A)_0\}.$$  

The authors show that if $\phi \in \text{Der}^n(A)_0$, then $\phi(A) \subseteq \bar{A}$. Various examples are given which indicate that the above mentioned result is about as good as possible.

Introduction. Throughout this paper, $A = k[x_1, \ldots, x_d]$ will denote a finitely generated integral domain over a field $k$ of characteristic zero. We shall let $Q$ denote the quotient field of $A$ and $\bar{A}$ the integral closure of $A$ in $Q$. For each $n = 1, 2, \ldots$, we shall let $\text{Der}^n(A)$ denote the $A$-module of all $n$th order $k$-derivations of $A$ to $A$. Thus, $\phi \in \text{Der}^n(A)$ if and only if $\phi \in \text{Hom}_k(A, A)$, and for all $a_0, \ldots, a_n \in A$ we have

$$\phi(a_0a_1 \cdots a_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq i_1 < \cdots < i_n} a_{i_1} \cdots a_{i_n} \phi(a_{i_1} \cdots \hat{a}_{i_i} \cdots \hat{a}_{i_n} \cdots a_n).$$

The authors refer the reader to [3] for the various facts about $\text{Der}^n(A)$ used in this paper. Of particular importance is the fact that any $n$th order derivation $\phi \in \text{Der}^n(A)$ can naturally be extended to an $n$th order derivation of any localization of $A$ [Thm 15; 3].

We shall need the Hochschild coboundary operator $\Delta$ which is defined as follows: If $\phi \in \text{Hom}_k(A, A)$, then $\Delta \phi: A \times A \to A$ is the $k$-bilinear mapping defined by $\Delta \phi(a_1, a_2) = \phi(a_1a_2) - a_1\phi(a_2) - a_2\phi(a_1)$. We shall also need the cup product $\phi \cup \psi$ of two $k$-linear mappings $\phi$ and $\psi$ of $A$. $\phi \cup \psi: A \times A \to A$ is the $k$-bilinear mapping defined by $\phi \cup \psi(a_1, a_2) = \phi(a_1)\psi(a_2)$. If $P$ and $P$ are two $A$-submodules of $\text{Hom}_k(A, A)$, then $P \cup P$ will denote the set of all $k$-bilinear mappings of $A \times A$ into $A$ which are finite $A$-linear combinations of mappings.
of the form \( \varphi \cup \psi \) for \( \varphi \in P, \psi \in P' \). Thus, if \( \varphi \) is an \( n \)th order \( k \)-derivation of \( A \) such that \( \Delta \varphi \in \sum_{i=1}^{\infty} \text{Der}_i(A) \cup \text{Der}_{i-1}(A) \), then there exist constants \( e_{ij} \in A \) and \( k \)-derivations \( \varphi^{(j)}_i, \lambda^{(j)}_i \in \text{Der}_i(A) \) such that for all \( a \) and \( b \) in \( A \), we have

\[
\varphi(ab) = a\varphi(b) + b\varphi(a) + \sum e_{ij} \varphi^{(j)}_i(a) \lambda^{(j)}_i(b) + \cdots
\]

(2)

Now the purpose of this paper is to study which \( n \)th order \( k \)-derivations \( \varphi: A \to A \) can be extended to \( \overline{A} \). In [4], A. Seidenberg showed that any 1st order derivation of \( A \) must map \( A \) to \( A \). In [1], an example was given which shows that 2nd order derivations \( \varphi \in \text{Der}_2(A) \) need not have the property that \( \varphi(A) \subseteq \overline{A} \). Since we shall have use of this example latter, we present it here.

**Example 1.** Consider the curve \( X^2 = Y^3 \) over the rational numbers \( \mathbb{Q} \). Let \( A \) be the coordinate ring of this curve i.e. \( A = \mathbb{Q}[x, y] = \mathbb{Q}[X, Y]/(X^2 - Y^3) \). One can easily check that \( A \) is a domain whose integral closure is given by \( \overline{A} = A[\sqrt[3]{x}/y] \). Since the quotient field of \( A \) is a finite separable extension of \( \mathbb{Q}(y) \), it follows that any 2nd order derivation \( \varphi \in \text{Der}_2(A) \) is determined by its values on \( y \) and \( y^2 \). A simple calculation shows that if \( \varphi(y) = a \), and \( \varphi(y^2) = b \) (where \( a \) and \( b \) lie in the quotient field of \( A \)), then

\[
\varphi(x) = \frac{3y}{8} \left( \frac{2ya + b}{x} \right), \quad \varphi(x^2) = 3yb - 3y^2a
\]

and

\[
\varphi(xy) = \frac{5y^2}{8} \left( \frac{3b - 2ya}{x} \right).
\]

If we set \( a = 1 \) and \( b = -2y \), then \( \varphi \in \text{Der}_3^2(A) \), and one easily checks that \( \varphi(x/y) = x/y^2 \not\in \overline{A} \).

Thus, higher derivations on \( A \) need not extend to \( \overline{A} \). At the end of [1], the author conjectured that any \( \varphi \in \text{Der}_2^2(A) \) such that \( \Delta \varphi \in \text{Der}_2(A) \cup \text{Der}_1(A) \) must map \( A \) to \( \overline{A} \). In this paper, we shall show that this conjecture is correct. We shall also formulate sufficient conditions on \( \varphi \in \text{Der}_2^2(A) \) in order that \( \varphi(\overline{A}) \subseteq \overline{A} \). We assume the reader is familiar with [1].

**Main results.**

**Theorem 1.** Let \( A = k[x_1, \ldots, x_n] \) be a finitely generated integral domain over a field \( k \) of characteristic zero. Let \( \overline{A} \) denote the integral closure of \( A \) in its quotient field \( Q \). Let \( \varphi \in \text{Der}_1(A) \) and
assume \( \Delta \phi \in \text{Der}^1_k(A) \cup \text{Der}^1(A) \). Then \( \phi(\mathcal{A}) \subseteq \mathcal{A} \).

**Proof.** Let \( \text{Min}(\mathcal{A}) \) denote the collection of height one primes in \( \mathcal{A} \). Since \( \mathcal{A} \) is a Krull domain, we have \( \mathcal{A} = \bigcap \{ \mathcal{A}_q \mid q \in \text{Min}(\mathcal{A}) \} \). Here as usual \( \mathcal{A}_q \) means \( \mathcal{A} \) localized at the prime \( q \). Let \( q \in \text{Min}(\mathcal{A}) \). Then \( p = q \cap A \in \text{Min}(A) \). Let us set \( R = A_p \) and \( \bar{R} = (\mathcal{A}_p) = \mathcal{A}_p \) the integral closure of \( R \) in \( Q \). Let \( \mathcal{A}_q \) denote the extended prime ideal \( qR \) in \( R \). Then \( \mathcal{A}_q = \mathcal{A}_p \). Now since \( R \) is a localization of \( A \), we see that \( \phi \in \text{Der}^1_k(\mathcal{R}) \). Suppose we could show that \( \phi(\mathcal{R}) \subseteq \mathcal{R} \). Then \( \phi(\mathcal{A}_q) \subseteq \mathcal{A}_q \) or equivalently \( \phi(\mathcal{A}_p) \subseteq \mathcal{A}_p \). Since \( \mathcal{A} \) is the intersection of the \( \mathcal{A}_p \), the theorem would be proven. Thus to prove Theorem 1, it suffices to prove the following assertion:

"Under the same hypotheses as Theorem 1, let \( p \in \text{Min}(A) \), \( R = A_p \) and \( \bar{R} = A_q \). Then \( \phi(\mathcal{R}) \subseteq \mathcal{R} \)."

So fix a minimal prime \( p \in \text{Min}(A) \), and set \( R = A_p \), \( \bar{R} = A_q \). We have already noted that \( \phi \in \text{Der}^1(\mathcal{R}) \), and one easily sees that \( \Delta \phi \in \text{Der}^1_k(\mathcal{R}) \cup \text{Der}^1(\mathcal{R}) \). Now if \( A = \mathcal{A} \), there is nothing to prove. Hence, we may assume \( A \neq A \). Then the conductor \( C \) of \( A \) in \( A \) is a proper ideal in \( A \). If \( C \not\subset p \), then \( R = \bar{R} \) and again there is nothing to prove. Hence we may assume \( C \subset p \). In this case, \( CR \) is the conductor of \( R \) in \( \bar{R} \).

We now follow the proof of Theorem 3 in [1]. Let the transcendence degree of \( A \) over \( k \) be \( r \), and let \( m \) denote the maximal ideal in \( R \). Then \( R/m \) is the quotient field of \( A/p \) and hence has transcendence degree \( r - 1 \) over \( k \). Let \( \{\alpha_1, \ldots, \alpha_r\} \) be a transcendence basis of \( R/m \) over \( k \). Pull these \( \alpha_i \) back to elements \( \alpha_i \) in \( R - m \). Then \( F = k(\alpha_1, \ldots, \alpha_r) \) is a field of transcendence degree \( r - 1 \) over \( k \), and \( F \subset R \).

We know that \( \bar{R} \) is a semilocal ring with maximal ideals \( m_1, \ldots, m_t \) lying over \( m \) in \( R \). Set \( J = \bigcap_{i=1}^t m_i \), the Jacobson radical of \( \bar{R} \). Each local ring \( V_i = \bar{R}_{m_i}, i = 1, \ldots, t \), is a discrete rank one valuation ring dominating \( R \). By [Thm 18, p. 45; 6], we can find an element \( \beta \in J \) such that \( \beta \) generates the maximal ideal in each \( V_i \). Since the Krull dimension of \( \bar{R} \) is one, we see that \( J \) is the radical of the ideal \( CR \) in \( \bar{R} \). Thus, some power of \( \beta \), say \( \beta^n \), lies in \( CR \). We shall have use of this remark later.

It was shown in [1], that \( \text{Der}^1(\bar{R}) \) is a free \( \bar{R} \)-module with basis \{\( \delta_0, \delta_1, \ldots, \delta_{r-1} \)\}. The derivations \( \delta_i \) satisfy the following relations:

(3) \( \delta_0(\beta) = 1, \delta_0(\alpha_i) = 0 = \delta_i(\beta) \) for \( i = 1, \ldots, r - 1 \)

and
We observe that the derivations $\delta_i$ commute on the field $F(\beta)$. Since $\beta$ is a uniformizing parameter for $V_\alpha$ and $\beta$ is transcendental over $\mathbb{F}$, $Q$ is a separable algebraic extension of $F(\beta)$. Therefore the derivations on $F(\beta)$ have a unique extension to $Q$. It follows that the $\delta_i$ commute on $Q$. It follows from [2; Thm 16, 11. 2] that the union $\bigcup_{i=1}^r \text{Der}_i^*(Q)$ is a free $Q$-algebra generated by $\delta_0, \ldots, \delta_{r-1}$. In particular, $\varphi$ can be written as a unique polynomial of degree two in $\delta_0, \ldots, \delta_{r-1}$. The coefficients of this polynomial lie in $Q$. Let us write $\varphi$ as follows:

(4) $\varphi = \sum_{i=0}^{r-1} a_i \delta_i + \sum_{0 \leq i < j \leq r-1} a_{ij} \delta_i \delta_j + \sum_{i=0}^{r-1} a_{ii} \delta_i^2$. 

Since $\delta \varphi \in \text{Der}_i^*(R) \cup \text{Der}_i^*(R)$, we can write for all $a$ and $b$ in $R$:

(5) $\varphi(ab) = a \varphi(b) + b \varphi(a) + \sum_{t \in R} e_t \psi_t(a) \lambda_t(b)$

where $e_t \in R$ and $\psi_t, \lambda_t \in \text{Der}_i^*(R)$. One easily checks that equation (5) continues to hold for all $a$ and $b$ in $Q$. Now by [Thm 1; 4], each $\psi_t$ and $\lambda_t$ extends to $R$. It then easily follows that $CR$ is differential under $\psi_t$ and $\lambda_t$, i.e. $\psi_t(CR) \subset CR$ and $\lambda_t(CR) \subset CR$. Thus, $CR$ remains differential under $\psi_t$ and $\lambda_t$ when considered as an ideal in $R$. Hence, [Thm 1; 5] implies that each $m_t$ in $R$ is differential under $\psi_t$ and $\lambda_t$. Write each $\psi_t$ and $\lambda_t$ as a linear combination of $\delta_0, \delta_1, \ldots, \delta_{r-1}$:

(6) $\psi_t = \sum_{i=0}^{r-1} \mu_{it} \delta_i \quad \lambda_t = \sum_{i=0}^{r-1} \gamma_{it} \delta_i$.

Here the coefficients $\mu_{it}$ and $\gamma_{it}$ lie in $R$. Then $\psi_t(J) \subset J$ and $\lambda_t(J) \subset J$ imply that $\mu_{10}$ and $\gamma_{10}$ lie in $J$. If we now substitute the expressions in equations (6) and (4) into equation (5) and then make various substitutions of the form $a, b = \alpha_0, \ldots, \alpha_{r-1}, \beta$, we see that all the coefficients, except possibly $a_0$, appearing in (4) lie in $R$. We further get that $a_0 \in J$ for $i = 1, \ldots, r - 1$, and $a_0 \in J^*$.

Thus, to complete the proof of the assertion $\varphi(R) \subseteq R$, we must show that $a_0$ in (4) lies in $R$. We shall show this by arguing that $a_0 \in V_i$ for every $i = 1, \ldots, t$.

So fix an $i = 1, \ldots, t$, and let $v_i: V_i \rightarrow Z$ be the valuation of $V_i$ given by $v_i(\alpha) = 1$. We wish to show that $v_i(a_0) \geq 0$. Let us assume $v_i(a_0) < 0$. We need the following lemma:

**Lemma 1.** There exist two elements $x$ and $y$ in $R$ such that

(a) The value $N = v_i(x)$ of $x$ is the smallest positive value of
any element in \( R \).

(b) The value \( v(\gamma) \) of \( \gamma \) is not a multiple of \( N \).

Proof. Since \( R \subset V_1 \), we have \( v_i(z) \geq 0 \) for every element \( z \) in \( R \). So we can certainly find an element \( x \) in \( R \) which satisfies (a). As pointed out earlier, \( \beta^* \in CR \subset R \). Thus, \( \beta^{*-1} \in R \) for any nonnegative integer \( l \).

Now suppose no \( y \in R \) can be found satisfying (b). Then for every nonnegative integer \( l \), we must have \( n + l = v_i(\beta^{*-1}) \) is a multiple of \( N \). This can only happen if \( N = 1 \). We shall show this is impossible.

If \( N = 1 \), then \( x = \gamma \beta \) for some unit \( \gamma \) in \( V_1 \). We want to consider

\[
\varphi(x) = \sum_{i=0}^{r-1} a_i \delta_i(x) + \sum_{0 \leq i < j \leq r-1} a_{ij} \delta_i \delta_j(x) + \sum_{i=0}^{r-1} a_i \delta_i^2(x)
\]

which is an element of \( R \). Now we have

\[
\delta_i(x) = \beta \delta_i(\gamma) + \gamma
\]

\[
\delta_i(x) = \beta \delta_i(\gamma) \quad i = 1, \ldots, r - 1
\]

\[
\delta_i \delta_j(x) = \beta \delta_j(\gamma) + \delta_i(\gamma) \quad i = 1, \ldots, r - 1
\]

\[
\delta_i \delta_j(x) = \beta \delta_j(\gamma) \quad 0 < i < j \leq r - 1
\]

and

\[
\delta \delta(\gamma) = \beta \delta(\gamma) + 2 \delta \delta(\gamma).
\]

Since the \( \delta_i \) are derivations on \( \bar{R} \), they naturally extend to \( V_1 \). Thus, the elements in equation (7) are all elements of \( V_1 \), and clearly \( \delta(\gamma) \) is a unit in \( V_1 \). If we now use the facts that \( a_i, \ldots, a_{r-1}, a_{ij} \in \bar{R} \), \( a_0 \in J \) and \( a_{00} \in J^2 \), we see that

\[
v_i \left[ \sum_{i=1}^{r-1} a_i \delta_i(x) + \sum_{0 \leq i < j \leq r-1} a_{ij} \delta_i \delta_j(x) + \sum_{i=0}^{r-1} a_i \delta_i^2(x) \right] \geq 0
\]

Thus, \( v_i(\varphi(x)) = v_i(a_0) + v_i(\delta(\gamma)) = v_i(a_0) = 0 \). But, \( \varphi(x) \in R \) means the value of \( \varphi(x) \) must be nonnegative. Thus, we have reached a contradiction and the proof of Lemma 1 is complete.

Now among all the elements \( z \) of \( R \) such that \( v_i(z) \) is not a multiple of \( N \) pick one, say \( y \), of smallest value \( M \). Lemma 1 guarantees that such an element \( y \in R \) exists. Then \( M - N > 0 \), and \( M - N \) is not the value of any element of \( R \). Since \( v_i(x) = N, x = \gamma \beta^N \) for some unit \( \gamma \in V_1 \). An argument similar to that in Lemma 1 shows that \( v_i(\varphi(x)) = v_i(a_0) + N - 1 \). Now there are two cases to consider. Either \( \varphi(x) \) is a unit in \( R \) or it is not. If \( \varphi(x) \) is a nonunit, then \( v_i(\varphi(x)) \geq N \). But this implies \( v_i(a_0) \geq 1 \) which is contrary to
our assumption. Thus, \( \varphi(x) \) is a unit. So \( v_i(a_0) = 1 - N \). But now a similar computation applied to \( y \) gives us that \( v_i(\varphi(y)) = v_i(a_0) + M - 1 = M - N \). Since \( \varphi(y) \in R \), and \( M - N \) is not the value of anything in \( R \), we have reached a contradiction.

Thus, \( v_i(a_0) \geq 0 \) and the proof of Theorem 1 is complete.

In our proof of Theorem 2 below, we shall need the fact that the coefficient \( a_0 \) in equation (4) actually lies in \( J \). The proof of Theorem 1 shows that \( a_0 \in R \). To see that \( a_0 \in J \), we proceed as follows: Since \( \varphi(R) \subseteq \tilde{R} \), equation (5) immediately implies that \( \varphi(CR) \subseteq CR \). In the notation of Theorem 1, we wish to argue that \( v_i(a_0) \geq 1 \). Suppose \( v_i(a_0) = 0 \). Let \( N \) be the minimum positive value of any element in \( CR \), and let \( x \in CR \) have value \( N \). Then as in Lemma 1, \( v_i(\varphi(x)) = v_i(a_0) + N - 1 = N - 1 \). Since \( \varphi(x) \in CR \) this is impossible. Thus \( v_i(a_0) \geq 1 \).

For Theorem 2, we shall need the following definition:

**DEFINITION.** Set \( \text{Der}_1^R(A)_0 = \text{Der}_1^R(A) \) and inductively define \( \text{Der}_i^R(A)_0 \) as follows:

\[
\text{Der}_i^R(A)_0 = \left\{ \phi \in \text{Der}_i^R(A) \mid \Delta \phi \in \sum_{i=1}^{n-1} \text{Der}_i^R(A)_0 \cup \text{Der}_{i-1}^R(A)_0 \right\}
\]

Thus, Theorem 1 states that if \( \phi \in \text{Der}_1^R(A)_0 \), then \( \phi(A) \subseteq \bar{A} \). We can now prove the general result.

**Theorem 2.** Let \( A = k[x_1, \ldots, x_n] \) be a finitely generated integral domain over a field \( k \) of characteristic zero. Let \( \bar{A} \) denote the integral closure of \( A \) in its quotient field \( Q \). Let \( \phi \in \text{Der}_1^R(A)_0 \). Then \( \phi(\bar{A}) \subseteq \bar{A} \).

**Proof.** The proof proceeds along the same lines as in Theorem 1. It suffices to show that for every prime \( p \) of height one in \( A \), \( \phi(R) \subseteq \tilde{R} \). Here, as in Theorem 1, \( \tilde{R} \) denotes the integral closure of \( R = A_p \) in \( Q \). One easily checks that \( \phi \in \text{Der}_1^R(R)_0 \). We shall adopt all the notation used in Theorem 1. Thus, \( CR \) is the conductor of \( R \) in \( \tilde{R} \).

For the purposes of this proof, let us define \( \text{Der}_i^R(R) \) inductively as follows:

\[
\text{Der}_i^R(R) = \text{Der}_1^R(R)
\]

\[
\text{Der}_i^R(R) = \left\{ \phi \in \text{Der}_i^R(R) \mid \Delta \phi \in \sum_{i=1}^{n-1} \text{Der}_i^R(R) \cup \text{Der}_{i-1}^R(R) \right\}
\]

and \( \phi(R) \subseteq \tilde{R} \).
Then we have already proven that \( \Der^1(R)_o = \Der^1(R)_{\bar{R}} \) in Theorem 1, and we shall show that \( \Der^n(R)_o = \Der^n(R)_{\bar{R}} \) for all \( n \).

Now we know that \( \bigcup_n \Der^n(Q) \) is a free \( Q \)-algebra generated by \( \delta_0, \ldots, \delta_{r-1} \). Thus if \( \varphi \in \Der^n(R) \), then \( \varphi = g(\delta_0, \ldots, \delta_{r-1}) \) for some polynomial \( g(X_0, \ldots, X_{r-1}) \in Q[X_0, \ldots, X_{r-1}] \) of degree less than or equal to \( n \). We further know this polynomial is unique. We now need the following lemma:

**Lemma 2.** Let \( \varphi \in \Der^n(R)_{\bar{R}}, \) and write \( \varphi = g(\delta_0, \ldots, \delta_{r-1}) \). Then the coefficients of any monomials of \( g \) which contain \( \delta_j(1 \leq j \leq n) \) lie in \( J^j \).

**Proof.** We proceed by induction on \( n \). The case \( n = 1 \) was proven in Theorem 1. The case \( n = 2 \) was proven in Theorem 1 and the remarks following Theorem 1. Thus, we may assume Lemma 2 has been proven for all elements of \( \Der^n(R)_{\bar{R}} \) with \( m < n \).

Let \( \varphi \in \Der^n(R)_{\bar{R}} \). Then there exist constants \( c_{i_j} \in R \) and derivations \( \psi^{(j)}, \lambda^{(j)} \in \Der^j(R)_{\bar{R}}, j = 1, \ldots, n-1, \) such that for all \( a \) and \( b \) in \( Q \) equation (2) is satisfied. Our induction hypothesis applies to the derivations \( \psi^{(j)} \) and \( \lambda^{(j)} \). So we can write:

\[
\psi^{(j)} = \sum c_{i_{j-1}}^{(j)} \delta_t + \sum c_{i_{j-2}i_{j}}^{(j)} \delta_t \delta_{t_2} + \cdots + \sum c_{i_{j-1} \cdots i_{j}}^{(j)} \delta_t \cdots \delta_{t_j},
\]

\[
\lambda^{(j)} = \sum d_{i_{j-1}}^{(j)} \delta_t + \sum d_{i_{j-2}i_{j}}^{(j)} \delta_t \delta_{t_2} + \cdots + \sum d_{i_{j-1} \cdots i_{j}}^{(j)} \delta_t \cdots \delta_{t_j}.
\]

In (10), the coefficient of any monomial in either expression which contains \( \delta^j \) will lie in \( J^j \). We note that since \( \psi^{(j)}, \lambda^{(j)} : R \to \bar{R} \), all the coefficients of (10) lie in \( \bar{R} \).

Now write out the polynomial \( g(\delta_0, \ldots, \delta_{r-1}) \) which gives us \( \varphi \) as follows:

\[
\varphi = \sum a_t \delta_t + \sum a_{t_1} \delta_{t_1} \delta_{t_2} + \cdots + \sum a_{t_1 \cdots t_n} \delta_{t_1} \cdots \delta_{t_n}.
\]

Since \( \varphi(\bar{R}) \subset \bar{R} \), one easily checks that all the coefficients \( a_t, a_{t_1t_2}, \ldots, a_{t_1 \cdots t_n} \) of (11) lie in \( \bar{R} \). We now substitute equations (10) and (11) into (2) and get:

\[
\sum a_t \delta_t(ab) + \sum a_{t_1t_2} \delta_{t_1} \delta_{t_2}(ab) + \cdots + \sum a_{t_1 \cdots t_n} \delta_{t_1} \cdots \delta_{t_n}(ab) = a(\sum a_t \delta_t(b) + \cdots + \sum a_{t_1 \cdots t_n} \delta_{t_1} \cdots \delta_{t_n}(b))
\]

\[
+ b(\sum a_t \delta_t(a) + \cdots + \sum a_{t_1 \cdots t_n} \delta_{t_1} \cdots \delta_{t_n}(a))
\]

\[
+ \sum e_{i_{j-1}} \left( \sum c_{i_{j-1}}^{(j)} \delta_t(a) \right) \left( \sum d_{i_{j-1} \cdots i_{j}}^{(j)} \delta_t(b) \right) + \cdots
\]

\[
+ \sum e_{i_{j-1}} \left( \sum c_{i_{j-1} \cdots i_{j}}^{(j)} \delta_t(a) \right) \left( \sum d_{i_{j-1} \cdots i_{j}}^{(j)} \delta_t(b) \right)
\]

\[
\times \left( \sum d_{i_{j-1} \cdots i_{j}}^{(j)} \delta_t(b) \right).
\]
After simplifying (12) and comparing coefficients, we see that any coefficient of (11) (except possibly for $a_0$) in a monomial containing $\delta^i_j$ lies in $J_i$. Thus, the lemma will be complete if we show $a_0 \in J$.

Since $\varphi(R) \subset R$, one easily sees using (2) that $\varphi(CR) \subset CR$. Thus, to argue $a_0 \in J$, one can proceed exactly as in the remarks following Theorem 1. Pick an element $x \in CR$ of minimum value $N = \nu_t(x)$. If $\nu_t(a_0) = 0$, then $\nu_t(\varphi(x)) = N - 1$ which is a contradiction. This completes the proof of Lemma 2.

We now proceed to prove Theorem 2 by induction on $n$. A. Seidenberg's original result [Thm; 4], and Theorem 1 give us the case $n = 1$ and $n = 2$. Thus, assume Theorem 2 is correct for all $m < n$, and let $\varphi \in \text{Der}^* (R)$. We can expand $\varphi$ as in equation (2) for some choice of constants $e_{ij} \in R$ and derivations $\psi_{i}^{[j]}, \lambda_{i}^{[j]} \in \text{Der}^*_i (R)$. By our induction hypothesis, $\text{Der}^*_i (R) = \text{Der}^*_i (R)$. So by Lemma 2, each $\psi_{i}^{[j]}$ and $\lambda_{i}^{[j]}$ can be written as in equation (10) with the coefficients of any monomials containing $\delta^i_j$ lying in $J_i$. Now write $\varphi$ as in equation (11). Following the same substitutions as in Lemma 2, we see that all the coefficients $a_{i_1}, \ldots, a_{i_{t-1}}, a_{i_{t+1}}, \ldots, a_{i_{t+\ldots+n}}$ lie in $R$. Further, the coefficients appearing in terms containing $\delta^i_0$ lie in $J_i$, except possibly for $a_0$. Thus, as in Theorem 1, we have to argue that $\nu_t(a_0) \geq 0$ for all $i = 1, \ldots, t$. But this argument is exactly the same as in Theorem 1. Assume $\nu_t(a_0) < 0$. The coefficients of (11) lying in the right powers of $J$ exactly mean that $\nu_t(\varphi(z)) = \nu_t(a_0) + \nu_t(z) - 1$ for any nonunit $z$ of $R$. Thus we proceed exactly as before to argue that $\nu_t(a_0) < 0$ is impossible. This completes the proof of Theorem 2.

The reader may be wondering if a slightly weaker hypothesis on $\varphi \in \text{Der}^*_i (A)$ will imply $\varphi(\bar{A}) \subset \bar{A}$. In particular, it is natural to ask the following question: Suppose $\varphi \in \text{Der}^*_i (A)$ such that

$$\Delta \in \sum_{i=1}^{n-1} \text{Der}^*_i (A) \cup \text{Der}^{n-i}_i (A).$$

Then is $\varphi(\bar{A}) \subseteq \bar{A}$? Theorem 1 implies this is true if $n = 2$. We shall give an example which shows that for $n > 2$ the answer to the above question is in general negative.

**Example 2.** We return to Example 1 at the beginning of this paper. We may equally well describe the ring $A$ as $A = Q[t', t^2]$. Set $\delta = \partial / \partial t$, a first order derivation on the quotient field of $A$. One can easily check that $t\delta, t^2\delta, \delta^3 - (2/t)\delta, t\delta^2 - \delta$ and $\delta^3 - (3/t)\delta^2 + (3/t^2)\delta$ are all derivations on $A$. Set
\( \varphi = t^2 \partial \left( \delta^3 - \frac{3}{t} \delta^2 + \frac{3}{t^2} \delta \right) - \frac{9}{2} t^2 \partial \left( \delta^2 - \frac{2}{t} \delta \right) + \frac{3}{2} \left( \delta^2 - \frac{2}{t} \delta \right) (t \delta) . \)

Then \( \varphi \in \text{Der}_t^1 (A) \). If we expand \( \varphi \) out, we get
\[
\varphi = t^2 \delta^4 - 6t \delta^3 + 15 \delta^2 - \frac{18}{t} \delta.
\]

Now the integral closure \( \overline{A} \) of \( A \) is just \( Q[t] \), and thus \( \varphi(\overline{A}) \not\subset \overline{A} \). However one can easily check that
\[
\Delta \varphi = 4 \left( \delta^3 - \frac{3}{t} \delta^2 + \frac{3}{t^2} \delta \right) \cup (t^2 \delta) + 6(t \delta^2 - \delta) \cup (t \delta^2 - \delta)
\]
\[
+ (t^2 \delta) \cup \left( \delta^3 - \frac{3}{t} \delta^2 + \frac{3}{t^2} \delta \right).
\]

Thus
\[
\Delta \varphi \in \text{Der}_t^2 (A) \cup \text{Der}_t^2 (A) + \text{Der}_t^2 (A) \cup \text{Der}_t^2 (A) + \text{Der}_t^2 (A) \cup \text{Der}_t^2 (A),
\]

but \( \varphi(\overline{A}) \not\subset \overline{A} \).

This example shows that we really need the stronger statement \( \varphi \in \text{Der}_t^2 (A) \) in order to conclude the \( \varphi(\overline{A}) \subset \overline{A} \).

Finally, we note that the methods used in Theorems 1 and 2 give a new proof of A. Seidenberg's original theorem for finitely generated domains:

**Theorem (A. Seidenberg).** Let \( A = k[x_1, \ldots, x_t] \) be a finitely generated integral domain over a field \( k \) of characteristic zero. Let \( \overline{A} \) denote the integral closure of \( A \) in its quotient field \( Q \). Let \( \delta \in \text{Der}_t^1 (A) \). Then \( \delta(\overline{A}) \subset \overline{A} \).

**Proof.** Using the same notation as in Theorem 1, we see that it suffices to prove \( \delta(\overline{R}) \subset \overline{R} \). Write \( \delta = a_0 \delta + \cdots + a_{r-1} \delta_{r-1} \) with the \( a_i \in Q \). Since \( \delta(a_i) \in R \), we see \( a_i, \cdots, a_{r-1} \in R \). As before, it remains to argue that \( v_i(a_i) \geq 0 \) for all \( i = 1, \cdots, t \). So fix an \( i = 1, \cdots, t \) and assume \( v_i(a_i) < 0 \). Pick \( x \in R \) such that \( N = v_i(x) \) is the minimum positive value of any element of \( R \). Then \( v_i(\delta(x)) = v_i(a_i) + N - 1 \). Since \( \delta(x) \in R \), we conclude that \( v_i(a_i) = 1 - N \). By an argument similar to that in Lemma 1, we can find an element \( y \in R \) such that \( M = v_i(y) \) is the minimum positive value of anything in \( R \) which is not a multiple of \( N \). Then \( v_i(\delta(y)) = M - N \) which is impossible.

**References**


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