WEAKLY COMPACT SETS IN $H^1$.

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Suppose that $A$ is a uniform algebra on a compact set $X$ and that $\phi: A \to C$ is a nonzero multiplicative linear functional on $A$. Let $M_\phi$ be the set of positive representing measures for $\phi$. If $M_\phi$ is finite dimensional, let $m$ be a core measure of $M_\phi$. The space $H^1$ is the closure of $A$ in $L^1(m)$. The space $H^\infty$ is the weak* (i.e. $\sigma(L^\infty, L^1)$) closure of $A$ in $L^\infty(m)$. The weakly compact sets $R$ in $H^1$ are then those sets such that for all $\varepsilon > 0$ there is a bounded set in $H^\infty$ which approximates $R$ up to $\varepsilon$.

It is well known (see Gamelin [1] for all details) that if $m$ is a core measure in the finite dimensional set $M_\phi$, then the annihilator $N$ of $A$ (or $\Re A$) in the real Banach space $L^1$ is finite dimensional, and is in fact a subspace of $L^\infty$ (see Gamelin [1] p. 108). Since $N$ is finite dimensional there is a constant $K$, such that $\|g\|_\infty \leq K\|g\|$, for all $g \in N$. There also exists a linear projection $P$ of $L^1$ onto $N$, the kernel of $P$ being precisely $\Re A$.

I am very grateful to the referee who pointed out an error in the first draft of this paper and gave a simplification of the proof.

2. Weakly compact sets in $H^1$. The notation used in the proof of the following theorem is the same as in the introduction.

**Theorem.** If $R \subset H^1$ then the following are equivalent

1. $R$ is relatively weakly compact in $H^1$
2. $\forall \varepsilon > 0 \exists M$ such that $\forall f \in R \exists g \in H^\infty$ with $\|g\|_\infty \leq M$ and $\|f - g\| \leq \varepsilon$
3. $\forall \varepsilon > 0 \exists M$ such that $\forall f \in R \exists g \in A$ with $\|g\| \leq M$ and $\|f - g\| \leq \varepsilon$.

**Proof.** (3) $\Rightarrow$ (2) obvious, (2) $\Rightarrow$ (1) follows from general arguments due to Grothendieck ([2] p. 296); (1) $\Rightarrow$ (2) is less trivial. Without loss of generality we may suppose that for all $f \in R$ we have $\|f\|_1 \leq 1$. From now on all calculations are made with fixed $f$. It is clear that all bounds only depend on $\|P\|$ and $K$. Since $\log^+|f| \leq |f|$ it is obvious that $\|\log^+|f|| \leq \|f\|_1 \leq 1$. Since $L^1 = \Re A \oplus N$ we also have uniquely determined elements $u \in \Re A$ and $v \in N$ such that $\log^+|f| = u + v$. Since $v$ is the image of $\log^+|f|$ by the operator $P$ we have
The conjugation operator $*$ is defined on $\overline{\text{Re } A}$ and takes values in $L^p(0 < p < 1)$, hence $\exists K_3$ such that $\|u\|_{L^p} \leq K_3\|u\|$. The function $e^{u+*u}$ is well defined and $f e^{-u-*u} \in H^\infty$. Indeed:

$$|f| e^{-u-*u} = e^{\log |f|} e^{-u} \leq e^{\log |f|} e^{-u} = e^u \leq e^{K_2} = K_4.$$

Hence $f = F e^{v+*u}$ with $\|F\|_{L^\infty} \leq K_4$. The next step is the approximation of $e^{u+*u}$ by functions in $H^\infty$. First remark that $u = \log + |f| - v \geq - K_3$. Put $u_n = \min(u, n) \geq - K_3$ and $u_n = w_n + v_n$ where $w \in \text{Re } A$ and $v \in \mathbb{N}$. We first prove that:

(i) $\|e^{w_n+*w_n}\|_{L^\infty} \leq M_n$ where $M_n$ is independent of $u$

(ii) $\|e^{w_n+*w_n} - e^{u+*u}\|_1 \to 0$ uniformly in $u$ as $n \to \infty$.

**Proof of (i):** Since $\log + |f| = u + v$ we have

$$|u| \leq |v|_{L^\infty} + \log + |f| \leq \log + |f| + K_1.$$

Hence $e^u \leq K_4 |f|$ and so the family $e^u$ is equally integrable (Here it is used that relatively weakly compact sets in $L^1$ are equally integrable (see [2] p. 295).) Consequently $e^{u_n} \to e^u$ uniformly in $u$. Since $v_n = P(u_n - u)$ we also have $\|v_n\|_{L^\infty} \leq K_1 |v_n|_1 \leq K_1 |P| |u_n - u|_1 \leq K_2$ for $n$ large enough. Indeed since $-K_2 \leq u_n - u \leq \log + |f| + K_2 \leq |f| + K_2$ we have that the functions $u$ form an equally integrable family and hence $u_n - u$ uniformly in $u$. All this implies

$$|e^{w_n+*w_n}| = e^{u_n} = e^{u_n - v_n} \leq K_1 e^{u_n} \leq K_1 e^u = M_n.$$

**Proof of (ii)**

$$|e^{u+*u} - e^{w_n+*w_n}| \leq |e^{w_n+*w_n} - e^{u+*u}| + |e^{w_n+*w_n} - e^{u_n+*w_n}|$$

$$+ |e^{u_n+*w_n} - e^{w_n+*w_n}|$$

$$\leq e^v |e^{*u} - e^{*w_n}| + |e^u - e^{u_n}| + |e^{u_n} - e^{w_n}|$$

$$\leq A_n + B_n + C_n.$$

Here is

$$A_n = e^v |e^{*u} - e^{*w_n}|$$

$$B_n = |e^u - e^{u_n}|$$

$$C_n = |e^{u_n} - e^{w_n}|.$$

In the proof of (i) it was already observed that $\|B_n\|_1 \to 0$ uniformly in $u$. For $n$ large enough one has

$$|e^{u_n} - e^{w_n}| = |e^{u_n+v_n} - e^{w_n}| = e^{u_n} |e^v - 1| \leq K_4 e^u |e^v - 1|.$$

Since $\|v_n\|_{L^\infty} \leq K_2 |u_n - u| \to 0$ uniformly in $u$ one has $\|C_n\|_1 \leq$
Remains to show that $\int A_n \to 0$.

Put $E_n = \{ x \mid |u(x) - w_n(x)| \geq \delta \}$ where $\delta > 0$ will be conveniently chosen.

$$\int A_n = \int_{E_n} A_n + \int_{E_n^c} A_n \leq \int_{E_n} 2e^u + \int_{E_n^c} e^u |e^{i*u} - e^{i*w_n}|.$$ 

Since

$$K_3 \int |u - w_n| dm \geq \left( \int |u - w_n|^{1/2} dm \right)^2 \geq \left( \int_{E_n} |u - w_n|^{1/2} dm \right)^2 \geq \delta m(E_n)^2$$

one has $m(E_n) \to 0$ uniformly in $u$, hence by equally integrability of $e^u$ it follows that $\int_{E_n} 2e^u \to 0$ uniformly in $u$. Also

$$\int_{E_n^c} e^u |e^{i* u} - e^{i*w_n}| \leq \int \delta e^u \leq \delta \int K_4 |f| \leq \delta K_4$$

and hence $\| A_n \|_1 \leq \delta K_4 + 2 \int_{E_n^c} e^u$.

The first term is made small by choosing $\delta$, afterwards we choose $n$ to be sure that the second term is also small enough, since this can be done uniformly in $u$ the proof of (ii) is complete.

Fix now $\epsilon > 0$ and let $n$ be large enough to assure $\| e^{w_n + i*w_n} - e^{u + i*u} \|_1 \leq \epsilon / K_4$. It then follows that

$$\| f - Fe^{w_n + i*w_n} \|_1 \leq \| F \|_1 \| e^{u + i*u} - e^{w_n + i*w_n} \|_1 \leq \epsilon .$$

Taking $M = M_n \cdot K_4 = K_2 e^n$ will do the job.

To prove that $(2) \Rightarrow (3)$ we only have to observe that the unit ball of $A$ is dense in the unit ball of $H^\infty$ for the $L^1$ norm. Since $m$ is a core point, $m$ is dominant and we can apply the Arens-Singer result ([1], p. 152, 153).

**References**

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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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