

Pacific Journal of Mathematics

WEAKLY COMPACT SETS IN H^1 .

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Suppose that A is a uniform algebra on a compact set X and that $\phi: A \rightarrow C$ is a nonzero multiplicative linear functional on A . Let M_ϕ be the set of positive representing measures for ϕ . If M_ϕ is finite dimensional, let m be a core measure of M_ϕ . The space H^1 is the closure of A in $L^1(m)$. The space H^∞ is the weak* (i.e. $\sigma(L^\infty, L^1)$) closure of A in $L^\infty(m)$. The weakly compact sets R in H^1 are then those sets such that for all $\varepsilon > 0$ there is a bounded set in H^∞ which approximates R up to ε .

It is well known (see Gamelin [1] for all details) that if m is a core measure in the finite dimensional set M_ϕ , then the annihilator N of A (or $\text{Re } A$) in the real Banach space L^1_R is finite dimensional, and is in fact a subspace of L^∞_R (see Gamelin [1] p. 108). Since N is finite dimensional there is a constant K_1 such that $\|g\|_1 \leq \|g\|_\infty \leq K_1 \|g\|_1$ for all $g \in N$. There also exists a linear projection P of L^1_R onto N , the kernel of P being precisely $\overline{\text{Re } A}$.

I am very grateful to the referee who pointed out an error in the first draft of this paper and gave a simplification of the proof.

2. Weakly compact sets in H^1 . The notation used in the proof of the following theorem is the same as in the introduction.

THEOREM. *If $R \subset H^1$ then the following are equivalent*

- (1) R is relatively weakly compact in H^1
- (2) $\forall \varepsilon > 0 \exists M$ such that $\forall f \in R \exists g \in H^\infty$ with $\|g\|_\infty \leq M$ and $\|f - g\| \leq \varepsilon$
- (3) $\forall \varepsilon > 0 \exists M$ such that $\forall f \in R \exists g \in A$ with $\|g\| \leq M$ and $\|f - g\| \leq \varepsilon$.

Proof. (3) \Rightarrow (2) obvious, (2) \Rightarrow (1) follows from general arguments due to Grothendieck ([2] p. 296); (1) \Rightarrow (2) is less trivial. Without loss of generality we may suppose that for all $f \in R$ we have $\|f\|_1 \leq 1$. From now on all calculations are made with fixed f . It is clear that all bounds only depend on $\|P\|$ and K_1 . Since $\log^+ |f| \leq |f|$ it is obvious that $\|\log^+ |f|\|_1 \leq \|f\|_1 \leq 1$. Since $L^1_R = \overline{\text{Re } A} \oplus N$ we also have uniquely determined elements $u \in \overline{\text{Re } A}$ and $v \in N$ such that $\log^+ |f| = u + v$. Since v is the image of $\log^+ |f|$ by the operator P we have

$$\|v\|_\infty \leq K_1 \|v\|_1 \leq K_1 \cdot \|P\| \|\log^+ |f|\| \leq K_1 \cdot \|P\| = K_2.$$

The conjugation operator* is defined on $\overline{\operatorname{Re} A}$ and takes values in L^p ($0 < p < 1$), hence $\exists K_3$ such that $\|*u\|_{1/2} \leq K_3 \|u\|_1$. The function e^{u+i^*u} is well defined and $f e^{-u-i^*u} \in H^\infty$. Indeed:

$$|f| \cdot |e^{-u-i^*u}| = e^{\log|f|} e^{-u} \leq e^{\log^+|f|} e^{-u} = e^v \leq e^{K_2} = K_4.$$

Hence $f = F \cdot e^{u+i^*u}$ with $\|F\|_\infty \leq K_4$. The next step is the approximation of e^{u+i^*u} by functions in H^∞ . First remark that $u = \log^+ |f| - v \geq -K_2$. Put $u_n = \min(u, n) \geq -K_2$ and $u_n = w_n + v_n$ where $w_n \in \overline{\operatorname{Re} A}$ and $v_n \in N$. We first prove that:

- (i) $\|e^{w_n+i^*w_n}\|_\infty \leq M_n$ where M_n is independent of u
- (ii) $\|e^{w_n+i^*w_n} - e^{u+i^*u}\|_1 \rightarrow 0$ uniformly in u as $n \rightarrow \infty$.

Proof of (i): Since $\log^+ |f| = u + v$ we have

$$|u| \leq \|v\|_\infty + \log^+ |f| \leq \log^+ |f| + K_2.$$

Hence $e^u \leq K_4 \cdot |f|$ and so the family e^u is equally integrable (Here it is used that relatively weakly compact sets in L^1 are equally integrable (see [2] p. 295).) Consequently $e^{u_n} \rightarrow e^u$ uniformly in u . Since $v_n = P(u_n - u)$ we also have $\|v_n\|_\infty \leq K_1 \|v_n\|_1 \leq K_1 \cdot \|P\| \|u_n - u\|_1 \leq K_2$ for n large enough. Indeed since $-K_2 \leq u_n \leq u \leq \log^+ |f| + K_2 \leq |f| + K_2$ we have that the functions u form an equally integrable family and hence $u_n \rightarrow u$ uniformly in u . All this implies

$$|e^{w_n+i^*w_n}| = e^{w_n} = e^{u_n - v_n} \leq K_4 e^{u_n} \leq K_4 e^n = M_n.$$

Proof of (ii)

$$\begin{aligned} |e^{u+i^*u} - e^{w_n+i^*w_n}| &\leq |e^{u+i^*u} - e^{u+i^*w_n}| + |e^{u+i^*w_n} - e^{w_n+i^*w_n}| \\ &\quad + |e^{u_n+i^*w_n} - e^{w_n+i^*w_n}| \\ &\leq e^u |e^{i^*u} - e^{i^*w_n}| + |e^u - e^{u_n}| + |e^{u_n} - e^{w_n}| \\ &\leq A_n + B_n + C_n. \end{aligned}$$

Here is

$$\begin{aligned} A_n &= e^u |e^{i^*u} - e^{i^*w_n}| \\ B_n &= |e^u - e^{u_n}| \\ C_n &= |e^{u_n} - e^{w_n}|. \end{aligned}$$

In the proof of (i) it was already observed that $\|B_n\|_1 \rightarrow 0$ uniformly in u . For n large enough one has

$$|e^{u_n} - e^{w_n}| = |e^{w_n+v_n} - e^{w_n}| = e^{w_n} |e^{v_n} - 1| \leq K_4 e^u |e^{v_n} - 1|.$$

Since $\|v_n\|_\infty \leq K_2 \|u_n - u\| \rightarrow 0$ uniformly in u one has $\|C_n\|_1 \leq$

$K_4 \|e^u\|_1 \cdot \|e^{v_n} - 1\|_\infty \rightarrow 0$ uniformly in u . Remains to show that $\int A_n \rightarrow 0$.

Put $E_n = \{x \mid |*u(x) - *w_n(x)| \geq \delta\}$ where $\delta > 0$ will be conveniently chosen.

$$\int A_n = \int_{E_n} A_n + \int_{E_n^c} A_n \leq \int_{E_n} 2e^u + \int_{E_n^c} e^u |e^{i^*u} - e^{i^*w_n}|.$$

Since

$$\begin{aligned} K_3 \int |u - w_n| dm &\geq \left(\int |*u - *w_n|^{1/2} dm \right)^2 \geq \left(\int_{E_n} |*u - *w_n|^{1/2} dm \right)^2 \\ &\geq \delta m(E_n)^2 \end{aligned}$$

one has $m(E_n) \rightarrow 0$ uniformly in u , hence by equally integrability of e^u it follows that $\int_{E_n} 2e^u \rightarrow 0$ uniformly in u . Also

$$\int_{E_n^c} e^u |e^{i^*u} - e^{i^*w_n}| \leq \int \delta e^u \leq \delta \int K_4 |f| \leq \delta K_4$$

and hence $\|A_n\|_1 \leq \delta K_4 + 2 \int_{E_n} e^u$.

The first term is made small by choosing δ , afterwards we choose n to be sure that the second term is also small enough, since this can be done uniformly in u the proof of (ii) is complete.

Fix now $\varepsilon > 0$ and let n be large enough to assure $\|e^{w_n + i^*w_n} - e^{u + i^*u}\|_1 \leq \varepsilon/K_4$. It then follows that

$$\|f - F e^{w_n + i^*w_n}\|_1 \leq \|F\|_\infty \|e^{u + i^*u} - e^{w_n + i^*w_n}\|_1 \leq \varepsilon.$$

Taking $M = M_n \cdot K_4 = K_4^2 e^n$ will do the job.

To prove that (2) \Rightarrow (3) we only have to observe that the unit ball of A is dense in the unit ball of H^∞ for the L^1 norm. Since m is a core point, m is dominant and we can apply the Arens-Singer result ([1], p. 152, 153).

REFERENCES

1. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, 1969.
2. A. Grothendieck, *Espaces Vectoriels Topologiques*, Sao Paulo, 1964.

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