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ANALYTIC EXTENSIONS OF VECTOR-VALUED FUNCTIONS

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J. GLOBEVNIK

Let Δ be the open unit disc in C , $\partial\Delta$ its boundary and $B \subset \partial\Delta$ a relatively open set. Let X be a complex Banach space. Denote by $H_B(\Delta, X)$ the set of all continuous functions from $\Delta \cup B$ to X which are analytic on Δ . A set $P \subset X$ is said to have the analytic extension property with respect to $H_B(\Delta, X)$ if for each relatively closed set $F \subset B$ of Lebesgue measure 0 and for each continuous function $f: F \rightarrow P$ there exists $g \in H_B(\Delta, X)$ with $g|_F = f$ and $g(\Delta \cup B) \subset P$.

THEOREM. Let $P \subset X$ be an open set. Then P has the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$ if and only if P is connected.

By a result of E. A. Heard and J. H. Wells any closed disc in C has the analytic extension property with respect to $H_B(\Delta, C)$ for every relatively open $B \subset \partial\Delta$ (see [9]). The special case $B = \partial\Delta$ is the well known Rudin-Carleson theorem (see [4], [10], [12]). This result was generalized to the vector case by proving that every closed ball in X has the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$ (see [6]), the special case $B = \partial\Delta$ is the Rudin-Carleson theorem for vector-valued functions (see [5], [11], [14]).

It is a natural question whether the balls above can be replaced by some other sets:

Problem (see [8]). Obtain a (geometrical, topological) characterization of the sets having the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$.

It seems that this problem is not solved even for the subsets of C .

Taking $B = \partial\Delta$, $F = \{-1, 1\}$ it is trivial to see that every set having the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$, is pathwise connected. The converse is not true in general as shown by taking $P = \{t: 0 \leq t \leq 1\}$. However, the converse turns out to be true for open sets and this is the main result of the present paper.

Throughout, we denote by $\bar{\Delta}$ the closure of Δ . Given $r > 0$ we denote by $B_r(X)$ the open ball in X of radius r , centered at the origin. If K is a compact Hausdorff space we denote by $C(K, X)$

the space of all continuous functions from K to X . By $A(\Delta, X)$ we denote the Banach space of all continuous functions from $\bar{\Delta}$ to X , analytic on Δ , with sup norm, and we write $A = A(\Delta, C)$. We write $I = \{t: 0 \leq t \leq 1\}$ and we denote the set of all positive integers by N .

For the proof of theorem we shall need four lemmas.

LEMMA 1. *Suppose that G is a closed subset of $\partial\Delta$ of Lebesgue measure 0 and let $U(G) \subset \bar{\Delta}$ be a neighbourhood of G . Let $p: I \rightarrow X$ be a path in a complex Banach space X and let $\varepsilon > 0$ be arbitrary. There exists $\phi \in A(\Delta, X)$ having the following properties:*

- (i) $\|\phi(z) - p(1)\| < \varepsilon$ ($z \in G$)
- (ii) $\|\phi(z) - p(0)\| < \varepsilon$ ($z \in \bar{\Delta} - U(G)$)
- (iii) $\phi(\bar{\Delta}) \subset p(I) + B_\varepsilon(X)$.

Proof. By the Mergelyan theorem for analytic functions into a Banach space (see [3]) there exists a polynomial $f: C \rightarrow X$ satisfying $\|f(z) - p(z)\| < \varepsilon$ ($z \in I$). By the continuity of f there exists an open neighbourhood V of I such that $f(V) \subset p(I) + B_\varepsilon(X)$. Let $W \subset V$ be an open set, bounded by a Jordan curve, containing the point 1 in its boundary and satisfying $I - \{1\} \subset W$, $\bar{W} \subset V$. Let $T \subset W$ be a neighbourhood of the point 0 in W such that $\|f(z) - p(0)\| < \varepsilon$ ($z \in T$). Assume for a moment that $\alpha \in A$ satisfies $\alpha(\bar{\Delta}) \subset \bar{W}$, $\alpha(G) = \{1\}$ and $\alpha(\bar{\Delta} - U(G)) \subset T$. Then it is easy to check that $\phi = f \circ \alpha$ has all the required properties. It remains to prove the existence of such an α . By the Riemann mapping theorem (see [13]) there exists a homeomorphism β from $\bar{\Delta}$ onto \bar{W} , analytic on Δ and satisfying $\beta(0) = 0$, $\beta(1) = 1$. Let $S \subset \Delta$ be a neighbourhood of 0 such that $\beta(S) \subset T$. By the Rudin-Carleson theorem (see [12]) there exists $\gamma \in A$ satisfying $\gamma(\bar{\Delta}) \subset \bar{\Delta}$, $\gamma(G) = \{1\}$. Also (see [15], p. 205) there exists $\psi \in A$ satisfying $\psi(\bar{\Delta}) \subset \bar{\Delta}$, $\psi(G) = \{1\}$, $|\psi(z)| < 1$ ($z \in \bar{\Delta} - G$). Let $U_1 \subset U(G)$ be an open subset of $\bar{\Delta}$ containing G . Now $\bar{\Delta} - U_1$ is a compact set disjoint from G and it follows that for sufficiently large $n \in N$ we have $\psi^n(z) \cdot \gamma(z) \in S$ ($z \in \bar{\Delta} - U_1$). Now putting $\alpha(z) = \beta[\psi^n(z) \cdot \gamma(z)]$ ($z \in \bar{\Delta}$) it is easy to see that α has all the required properties.

LEMMA 2. *Let X be a complex Banach space and let Q be an open connected subset of X . Given a compact subset K of Q and a point $x \in K$ there exists $\delta_0 > 0$ such that for every $\delta: 0 < \delta < \delta_0$ there exists a path $p: I \rightarrow X$ satisfying*

- (i) $p(0) = x$
- (ii) $K \subset p(I) + B_\delta(X)$
- (iii) $p(I) + B_{\delta_0}(X) \subset Q$.

Proof. By the compactness of K there exists an $\varepsilon > 0$ such that $K + B_{\varepsilon}(X) \subset Q$. Cover K by a finite number of balls, say by B_1, B_2, \dots, B_n of radii ε whose centers lie in K . With no loss of generality assume that the center of B_1 is x . By the connectedness of Q there exists a path $q: I \rightarrow X$, satisfying $q(I) \subset Q$, $q(0) = x$, and connecting the centers of all B_i . By the compactness of $q(I)$ there exists $\delta_0: 0 < \delta_0 < \varepsilon$ such that $q(I) + B_{\delta_0}(X) \subset Q$. Let δ satisfy $0 < \delta < \delta_0$ and cover K by a finite number of balls D_1, D_2, \dots, D_m of radii δ whose centers lie in K . Let $1 \leq i \leq n$. Consider those balls D_k whose centers lie in B_i . Connect all these centers by a path p_i starting and ending at the center of B_i and satisfying $p_i(I) \subset B_i$. Having done this for all i , denote by q_i ($1 \leq i \leq n-1$) the part of the path q between the centers of B_i, B_{i+1} . Now define p as the sum of the paths

$$p = \sum_{i=1}^{n-1} (p_i + q_i) + p_n.$$

If $s \in I$ is such that $p(s)$ is in none of the balls B_i ($1 \leq i \leq n$) then $p(s) \in q(I)$ and consequently $p(s) + B_{\delta_0}(X) \subset q(I) + B_{\delta_0}(X) \subset Q$. If $s \in I$ is such that $p(s)$ is in some B_i then $p(s) + B_{\delta_0}(X) \subset B_i + B_{\delta_0}(X) \subset K + B_{\varepsilon}(X) \subset Q$. On the other hand, if $y \in K$ then $y \in D_k$ for some ball D_k whose center is contained in $p(I)$ which means that $y \in p(I) + B_{\delta}(X)$.

LEMMA 3. *Let $F \subset \partial\Delta$ be a closed set of Lebesgue measure 0 and let $U(F) \subset \bar{\Delta}$ be a neighbourhood of F . Suppose that Q is an open connected set in a complex Banach space X containing the point 0. Let $\varepsilon > 0$ be arbitrary. Given $f \in C(F, X)$ satisfying $f(F) \subset Q$ there exists $\tilde{f} \in A(\Delta, X)$ satisfying*

- (i) $\tilde{f}|_F = f$
- (ii) $\tilde{f}(\bar{\Delta}) \subset Q$
- (iii) $\|\tilde{f}(z)\| < \varepsilon$ ($z \in \bar{\Delta} - U(F)$).

Proof. $f(F) \cup \{0\}$ is a compact set contained in Q . By Lemma 2 there exists $\delta: 0 < \delta < \varepsilon/5$ and a path $p: I \rightarrow X$ satisfying $f(F) \subset p(I) + B_{\delta}(X)$, $p(I) + B_{\delta_0}(X) \subset Q$ and $p(0) = 0$. Since F is a compact set the function f is uniformly continuous on F . By the assumption F is nowhere dense on $\partial\Delta$. It follows that

$$F = \bigcup_{i=1}^n F_i$$

where $F_i \subset \partial\Delta$ are disjoint closed sets such that

$$\|f(\eta) - f(\zeta)\| < \delta \quad (\eta, \zeta \in F_i; 1 \leq i \leq n).$$

Let U_i ($1 \leq i \leq n$) be disjoint open subsets of $\bar{\Delta}$ satisfying $F_i \subset U_i \subset U(F)$ ($1 \leq i \leq n$). Since $f(F) \subset p(I) + B_\delta(X)$ there exist $t_i \in I$ and $z_i \in F_i$ ($1 \leq i \leq n$) such that

$$\|p(t_i) - f(z_i)\| < \delta \quad (1 \leq i \leq n).$$

Applying Lemma 1 to the paths $t \mapsto p(t, t)$ ($1 \leq i \leq n$) there exist functions $\phi_i \in A(\Delta, X)$ ($1 \leq i \leq n$) satisfying

$$\begin{aligned} \|\phi_i(z) - p(t_i)\| &< \delta \quad (z \in F_i) \\ \|\phi_i(z)\| &< \delta/n \quad (z \in \bar{\Delta} - U_i) \\ \phi_i(\bar{\Delta}) &\subset p(I) + B_\delta(X). \end{aligned}$$

Now define $\Psi \in A(\Delta, X)$ by

$$\Psi = \sum_{i=1}^n \phi_i.$$

If $z \in \Delta - \bigcup_{i=1}^n U_i$ then

$$\|\Psi(z)\| \leq \sum_{i=1}^n \|\phi_i(z)\| < n \cdot \delta/n = \delta.$$

If $z \in U_i$ for some i then $z \notin U_j$ ($i \neq j$) and

$$\Psi(z) = \phi_i(z) + \sum_{\substack{j=1 \\ j \neq i}}^n \phi_j(z) \in p(I) + B_\delta(X) + B_\delta(X) \subset p(I) + B_{2\delta}(X).$$

Consequently $\Psi(\bar{\Delta}) \subset p(I) + B_{2\delta}(X)$. Now define $\theta \in C(F, X)$ by $\theta(z) = \Psi(z) - f(z)$ ($z \in F$). If $z \in F$ then $z \in F_i$ for some i and consequently

$$\begin{aligned} \|\theta(z)\| &\leq \|\Psi(z) - p(t_i)\| + \|p(t_i) - f(z_i)\| + \|f(z_i) - f(z)\| \\ &\leq \sum_{\substack{j=1 \\ j \neq i}}^n \|\phi_j(z)\| + \|\phi_i(z) - p(t_i)\| + 2\delta \\ &< 4\delta. \end{aligned}$$

By the Rudin-Carleson theorem for vector valued functions there exists $\tilde{\theta} \in A(\Delta, X)$ satisfying $\|\tilde{\theta}\| < 4\delta$, $\tilde{\theta}|_F = \theta$. Finally, define $\tilde{f}(z) = \Psi(z) - \tilde{\theta}(z)$ ($z \in \bar{\Delta}$). Clearly $\tilde{f} \in A(\Delta, X)$. Further, $\tilde{f}(\bar{\Delta}) \subset p(I) + B_{2\delta}(X) + B_{4\delta}(X) \subset p(I) + B_{6\delta}(X) \subset Q$. Clearly $\tilde{f}|_F = f$. Also, if $z \in \bar{\Delta} - U(F)$ then $z \in \bar{\Delta} - \bigcup_{i=1}^n U_i$ hence $\|\tilde{f}(z)\| \leq \|\Psi(z)\| + \|\tilde{\theta}(z)\| < \delta + 4\delta < \varepsilon$.

LEMMA 4. *Let E be closed subset of $\partial\Delta$ and let $G \subset \partial\Delta - E$ be a relatively closed set of Lebesgue measure 0. Let $H \subset \partial\Delta - E$ be a compact set of Lebesgue measure 0, disjoint from G . Let Q be an open connected set in a complex Banach space X containing the point 0 and suppose that $f \in C(H, X)$ satisfies $f(H) \subset Q$.*

There exists $\delta_0 > 0$ such that for every $\delta: 0 < \delta < \delta_0$ and for

every $\varepsilon: 0 < \varepsilon < \delta$ and for every neighbourhood $U \subset \bar{\Delta} - E$ of H there exists a continuous function $\tilde{f}: \bar{\Delta} - E \rightarrow X$, analytic on Δ and satisfying

- (i) $\tilde{f}|_H = f$
- (ii) $\tilde{f}|_G = 0$
- (iii) $\|\tilde{f}(z)\| < \varepsilon$ ($z \in (\bar{\Delta} - E) - E - U$)
- (iv) $f(\bar{\Delta} - E) + B_\varepsilon(X) \subset Q$.

Proof. With no loss of generality we may assume that $U \cap G = \emptyset$. By Lemma 2 there exists $\delta_0 > 0$ such that for every $\delta: 0 < \delta < \delta_0$ there exists a path $p: I \rightarrow X$ satisfying $p(0) = 0$, $f(H) \subset p(I) + B_\delta(X)$, $p(I) + B_{3\delta}(X) \subset Q$. Let $0 < \delta < \delta_0$ and $0 < \varepsilon < \delta$. Applying Lemma 3 to the function f and to the (open connected) set $p(I) + B_\delta(X)$ there exists $\tilde{f}_1 \in A(\Delta, X)$ satisfying

$$\begin{aligned} \tilde{f}_1|_H &= f \\ \tilde{f}_1(\bar{\Delta}) + B_{3\delta}(X) &\subset Q \\ \|\tilde{f}_1(z)\| &< \varepsilon/2 \quad (z \in \bar{\Delta} - U). \end{aligned}$$

Define

$$f_2(s) = \begin{cases} -\tilde{f}_1(s) & (s \in G) \\ 0 & (s \in H). \end{cases}$$

Then f_2 is continuous on $G \cup H$ and satisfies $\|f_2(s)\| < \varepsilon/2$ ($s \in G \cup H$). By Theorem 2 in [6] there exists a continuous function $\tilde{f}_2: \bar{\Delta} - E \rightarrow X$, analytic on Δ , satisfying $\tilde{f}_2|_{G \cup H} = f_2$ and $\|\tilde{f}_2(z)\| \leq \varepsilon/2$ ($z \in \bar{\Delta} - E$). Put $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. It is easy to check that \tilde{f} has all the required properties.

Proof of theorem. Let Q be an open connected subset of a complex Banach space X . Let $E \subset \partial\Delta$ be a closed set and let $F \subset \partial\Delta - E$ be a relatively closed set of Lebesgue measure 0. Suppose that $f: F \rightarrow X$ is a continuous function satisfying $f(F) \subset Q$. We will prove that there exists a continuous extension $\tilde{f}: \bar{\Delta} - E \rightarrow X$, $\tilde{f}|_F = f$, which is analytic on Δ and which satisfies $f(\bar{\Delta} - E) \subset Q$.

If E is empty then the statement of the theorem is proved by Lemma 3. So assume that E is not empty. With no loss of generality assume that $0 \in Q$. As in [6] write $F = \bigcup_{n=1}^{\infty} F_n$ where $F_n \subset \bar{\Delta} - E$ are compact sets such that there exist disjoint open sets $U_n \subset \bar{\Delta} - E$ satisfying $F_n \subset U_n$ for all n .

Now we define inductively a sequence $\{D_n\}$ of open subsets of $\bar{\Delta} - E$ satisfying $F_n \subset D_n \subset U_n$ for all n , a decreasing sequence $\{\delta_n\}$ of positive numbers and a sequence $\{\phi_n\}$ of functions from $\bar{\Delta} - E$ to X having the following properties:

(i) for each $i \in N$, ϕ_i is continuous on $\bar{A} - E$ and analytic on A

(ii) $\phi_i | F_j = 0$ ($i \neq j$; $i, j \in N$)

(iii) $\phi_i | F_i = f | F_i$ ($i \in N$)

(iv) $\phi_i(\bar{A} - E) + B_{\delta_i}(X) \subset Q$ ($i \in N$)

(v) $\|\phi_i(z)\| < \delta_i/2^{i+1}$ ($z \in (\bar{A} - E) - D_i$; $i \in N$)

(vi) $\|\sum_{j=1}^i \phi_j(z)\| < \delta_{i+1}/2$ ($z \in D_{i+1}$; $i \in N$).

If $i = 1$, put $D_1 = U_1$ and apply Lemma 4 to the function $f | F_1$ to obtain δ_1 satisfying $B_{\delta_1}(X) \subset Q$ and ϕ_1 which satisfies (i)-(v) above for $i = 1$. Now assume that δ_i, D_i, ϕ_i ($1 \leq i \leq n$) are given satisfying (i)-(v) for $1 \leq i \leq n$ and (vi) $1 \leq i \leq n - 1$. Applying Lemma 4 to the function $f | F_{n+1}$ there exists δ_{n+1} : $0 < \delta_{n+1} < \delta_n$ such that Lemma 4 holds for $\delta = \delta_{n+1}$. Since the function

$$z \longmapsto \sum_{j=1}^n \phi_j(z)$$

is continuous on $\bar{A} - E$ and equal 0 on F_{n+1} there exists a neighbourhood $D_{n+1} \subset \bar{A} - E$ of F_{n+1} satisfying $D_{n+1} \subset U_{n+1}$ and such that (vi) is satisfied for $i = n$. Now, by Lemma 4 there exists ϕ_{n+1} satisfying (i)-(v) for $i = n + 1$.

Define

$$\tilde{f}(z) = \sum_{i=1}^{\infty} \phi_i(z) \quad (z \in \bar{A} - E).$$

If $z \in (\bar{A} - E) - \bigcup_{j=1}^{\infty} D_j$ then $\|\phi_i(z)\| < \delta_i/2^{i+1} < \delta_1/2^{i+1}$. Consequently the series converges uniformly for all such z . By

$$\sum_{i=1}^{\infty} \|\phi_i(z)\| < \delta_1/2$$

and by $B_{\delta_1}(X) \subset Q$ we have $f(z) \in Q$ for all such z . Suppose that $z \in D_k$ for some k . Then $z \notin D_j$ for $j \neq k$ and by the above argument the series converges uniformly on D_k . Further, by (v) and (vi) we have

$$\left\| \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \phi_j(z) \right\| \leq \left\| \sum_{j=1}^{k-1} \phi_j(z) \right\| + \sum_{j=k+1}^{\infty} \|\phi_j(z)\| < \delta_k/2 + \delta_k/2 = \delta_k.$$

Consequently by (iv) $f(z) \in \phi_k(\bar{A} - E) + B_{\delta_k}(X) \subset Q$. Since each compact subset of $\bar{A} - E$ misses all but a finite number of the sets D_i the series converges uniformly on compact subsets of $\bar{A} - E$. Consequently \tilde{f} is continuous on $\bar{A} - E$, analytic on A and, as shown above, satisfies $\tilde{f}(\bar{A} - E) \subset Q$. By the properties of ϕ_i we have also $\tilde{f} | F = f$.

COROLLARY (see [7]). *Given any open connected subset Q of a*

separable complex Banach space X there exists an analytic function $\tilde{f}: \Delta \rightarrow X$ whose range is contained and dense in Q .

Proof. Put $E = \{1\}$ and let $F = \{z_n\} \subset \partial\Delta - \{1\}$ be an injective sequence converging to 1. Let $f(z_n) = w_n$ where $\{w_n\} \subset Q$ is a sequence dense in Q and then apply theorem.

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